The concept of a predicate signature (Handout 6) can be generalized as follows. A **signature** is a set of symbols of three kinds—**object constants**, **function constants**, and **predicate constants**—with a positive integer, called the **arity**, assigned to every function constant and to every predicate constant. **Terms** of such a signature are defined recursively:

- every object constant is a term,
- every object variable is a term,
- if \( t_1, \ldots, t_n \) are terms and \( f \) is a function constant of arity \( n \) then \( f(t_1, \ldots, t_n) \) is a term.

The class of atomic formulas includes, in addition to expressions of the form

\[ P(t_1, \ldots, t_n) \]

as in Handout 6, expressions of a second kind: **equalities**

\[ (t_1 = t_2) \]

where \( t_1, t_2 \) are terms. Otherwise, the definition of a formula remains the same. The expression \( t_1 \neq t_2 \) is shorthand for \( \neg(t_1 = t_2) \).

As an example, consider the **signature of first-order arithmetic**

\[ \{a, s, f, g\} \]

where \( a \) is an object constant (intended to represent 0), \( s \) is a unary function constant (for the successor function), and \( f, g \) are binary function constants (for addition and multiplication). Since this signature includes no predicate constants, its only atomic formulas are equalities.

**Problem 9.1** Represent the following English sentences by first-order formulas:

- There exists at most one \( x \) such that \( P(x) \).
- There exists exactly one \( x \) such that \( P(x) \).
- There exist at least two \( x \) such that \( P(x) \).
• There exist at most two \( x \) such that \( P(x) \).

• There exist exactly two \( x \) such that \( P(x) \).

For a signature containing function constants, an interpretation \( I \) consists of

• a non-empty set \( |I| \), called the universe of \( I \),

• for every object constant \( c \) of \( \sigma \), an element \( c^I \) of \( |I| \),

• for every function constant \( f \) of \( \sigma \), a function \( f^I \) from \( |I|^n \) to \( |I| \), where \( n \) is the arity of \( f \),

• for every predicate constant \( P \) of \( \sigma \), a function \( P^I \) from \( |I|^n \) to \( \{f, t\} \), where \( n \) is the arity of \( P \).

For example, the intended interpretation \( I \) of (1) is defined as follows:

\[
\begin{align*}
|I| &= \mathbb{N}, \\
a^I &= 0, \\
s^I(n) &= n + 1, \\
f^I(m, n) &= m + n, \\
g^I(m, n) &= m \cdot n.
\end{align*}
\]  

(2)

In this more general setting, a term \( t \) without variables is not necessarily an object constant; such a term may contain function constants. The notation \( t^I \) is extended to these more complex terms by the recursive equation

\[
f(t_1, \ldots, t_n)^I = f^I(t_1^I, \ldots, t_n^I).
\]

The recursive definition of \( F^I \) is extended by a clause for equalities:

\[
(t_1 = t_2)^I = t \text{ iff } t_1^I = t_2^I.
\]

**Problem 9.2** For each of the following sentences determine whether it is satisfiable:

• \( a = b \),

• \( \forall xy(x = y) \),

• \( \forall xy(x \neq y) \).
In the presence of function symbols, the definition of a substitutable term is stated as follows: A term \( t \) is substitutable for a variable \( v \) in a formula \( F \) if, for each variable \( w \) occurring in \( t \), no part of \( F \) of the form \( KwG \) contains an occurrence of \( v \) which is free in \( F \).

In the presence of equality, the natural deduction system is extended by the axioms
\[ \Rightarrow t = t \]
for any term \( t \), and by two inference rules:
\[
\begin{align*}
(R) & \quad \frac{\Gamma \Rightarrow t_1 = t_2 \quad \Delta \Rightarrow F^{w}_{t_1}}{\Gamma, \Delta \Rightarrow F^{w}_{t_2}} \\
& \quad \frac{\Gamma \Rightarrow t_1 = t_2 \quad \Delta \Rightarrow F^{w}_{t_2}}{\Gamma, \Delta \Rightarrow F^{w}_{t_1}}
\end{align*}
\]
where \( t_1 \) and \( t_2 \) are substitutable for \( v \) in \( F \).

Prove the given formulas in the natural deduction system.

**Problem 9.3** \( x = y \rightarrow f(x, y) = f(y, x) \).

**Problem 9.4** \( \forall x \exists y(y = f(x)) \).

**Problem 9.5** \( \exists y(x = y \land y = z) \rightarrow x = z \).

**Problem 9.6** \( (\exists x P(x) \land \exists x \neg P(x)) \rightarrow \exists x y(x \neq y) \).

**Problem 9.7** \( \forall x(x = a \rightarrow P(x)) \leftrightarrow P(a) \).

**Problem 9.8** \( \exists x(x = a \land P(x)) \leftrightarrow P(a) \).