Some examples of strong induction

Template: $P(n_0) \land (((n_0 \leq i \leq n) \Rightarrow P(i)) \Rightarrow P(n + 1))$

1. Using strong induction, I will prove that every positive integer can be written as a sum of distinct powers of 2. Thus for $n \geq 1$, $P(n) = \text{“} n \text{ can be written as a sum of distinct powers of } 2 \text{”}$. $P(1)$ is true since $1 = 2^0$. Now consider any $n \geq 1$. There exists an integer $k$ so that $2^k \leq n + 1 < 2^{k+1}$.

If $2^k = n + 1$, then $n + 1$ can be written as a sum of distinct powers of 2. If $2^k \neq n + 1$, then

$2^k < n + 1 < 2^{k+1}$

so

$0 < n + 1 - 2^k < 2^{k+1} - 2^k = 2^k \leq n$.

Since the value of $n + 1 - 2^k$ is positive but less than $n$, the inductive hypothesis guarantees that $n + 1 - 2^k$ can be written as a sum of distinct powers of 2 and the powers are less than $k$.

Thus $n + 1 = 2^k + \text{ a sum of distinct powers of 2 }$ and the powers are distinct.

2. Using strong induction, I will prove that the Fibonacci sequence:

$a_0 = 1, 
\quad a_1 = 1, 
\quad a_{k+1} = a_k + a_{k+1}, \text{ for } k \geq 1.$

satisfies for $k \geq 1$,

$a_k \geq \left(\frac{3}{2}\right)^{k-2}.$

Thus for $k \geq 1$, $P(k) = \text{“} a_k \geq \left(\frac{3}{2}\right)^{k-2} \text{”}$. $P(1)$ is true since $a_i = 1 \geq \frac{2}{3} = \left(\frac{3}{2}\right)^{1-2}$. Now consider any $k \geq 1$. If we assume $a_{k-1} \geq \left(\frac{3}{2}\right)^{k-3}$ and $a_k \geq \left(\frac{3}{2}\right)^{k-2}$, then

$a_{k+1} = a_k + a_{k-1} \geq \left(\frac{3}{2}\right)^{k-2} + \left(\frac{3}{2}\right)^{k-3}
\geq \left(\frac{3}{2} + 1\right)\left(\frac{3}{2}\right)^{k-3}
\geq \left(\frac{5}{2}\right)\left(\frac{3}{2}\right)^{k-3}
\geq \left(\frac{9}{4}\right)\left(\frac{3}{2}\right)^{k-3}
\geq \left(\frac{3}{2}\right)^2\left(\frac{3}{2}\right)^{k-3} = \left(\frac{3}{2}\right)^{(k+1)-2}$
3. Using strong induction, I will prove that integer larger than one has a prime factor. Thus for 
\[ n \geq 2, \quad P(n) = \text{“} n \text{ has a prime factor”.} \]  
\[ P(2) \text{ is true since the prime } 2 \text{ divides } 2. \text{ Now consider any } n \geq 2. \text{ The integer } n+1 \text{ is either prime or not. If it is prime then it has a prime factor. If } n+1 \text{ is not prime then it has some factor } k \text{ satisfying } 2 \leq k < n+1. \]

Thus by the inductive hypothesis, \( k \) has a prime factor and so \( n+1 \) must have that same prime factor.

**An example of double induction**

**Template:**
\[
P(m_0, n_0) \land ((n_0 \leq n) \Rightarrow (P(m_0, n) \Rightarrow P(m_0, n + 1)))) \land ((m_0 \leq m \land n_0 \leq n) \Rightarrow (P(m, n) \Rightarrow P(m + 1, n)))
\]

or
\[
P(m_0, n_0) \land ((m_0 \leq m) \Rightarrow (P(m, n_0) \Rightarrow P(m + 1, n_0)))) \land ((m_0 \leq m \land n_0 \leq n) \Rightarrow (P(m, n) \Rightarrow P(m, n + 1)))
\]

Notice the first version does the final induction in the first parameter: \( m \) and the second version does the final induction in the second parameter: \( n \). Thus, the “basis induction step” (i.e. the one in the middle) is also different in the two versions.

By double induction, I will prove that for \( m, n \geq 1 \)
\[
\sum_{i=1}^{m} \left( \sum_{j=1}^{n} ij \right) = \frac{mn(m+1)(n+1)}{4}.
\]

For \( m, n \geq 1 \), let \( P(m, n) = \text{“} \sum_{i=1}^{m} \left( \sum_{j=1}^{n} ij \right) = \frac{mn(m+n+2)}{2} \text{”} \).

First basis step: \( P(1,1) \) is true since \( \sum_{i=1}^{1} \left( \sum_{j=1}^{1} ij \right) = \sum_{i=1}^{1} \left( \sum_{j=1}^{1} 1 \right) = 1 = \frac{1 \cdot 1(1+1)(1+1)}{4} \).

Inductive basis step for \( n = 1 \): For \( m \geq 1 \), \( P(m,1) \Rightarrow P(m+1,1) \), since if
\[
\sum_{i=1}^{m} \left( \sum_{j=1}^{1} ij \right) = \frac{m(m+1) \cdot 2}{4} = \frac{m(m+1)}{2}, \text{ then}
\]
\[
\sum_{i=1}^{m+1} \left( \sum_{j=1}^{1} ij \right) = \sum_{i=1}^{m} \left( \sum_{j=1}^{1} ij \right) + \sum_{j=1}^{1} (m+1) \cdot 1
\]
\[
= \frac{m(m+1)}{2} + (m+1)
\]
\[
= \frac{(m+1)((m+1)+1) \cdot 2}{4}.
\]

Inductive step: For \( m, n \geq 1 \), \( P(m, n) \Rightarrow P(m, n + 1) \), since if \( \sum_{i=1}^{m} \left( \sum_{j=1}^{n} ij \right) = \frac{mn(m+1)(n+1)}{4} \),
then
\[
\sum_{i=1}^{m} \left( \sum_{j=1}^{n+1} ij \right) = \sum_{i=1}^{m} \left( \sum_{j=1}^{n} ij + (i(n+1)) \right) \\
= \frac{mn(m+1)(n+1)}{4} + (n+1)\sum_{i=1}^{m} i \\
= \frac{mn(m+1)(n+1)}{4} + (n+1)\frac{m(m+1)}{2} \\
= \frac{mn(m+1)(n+1)}{4} + \frac{2m(m+1)(n+1)}{4} \\
= \frac{m(n+1)(m+1)((n+1)+1)}{4}
\]