Announcements

- First homework assignment out today!
- Due in one week, i.e., before lecture next Tuesday 09/11
- Weilin’s Tuesday office hours are 9-10 AM, not 10-11 AM

Review

- Propositional logic is simplest kind of logic
- Building blocks are propositions, i.e., statements that are true or false
- Formulas in propositional logic are formed using propositional variables and boolean connectives
- Connectives: negation \( \neg \), conjunction \( \land \), disjunction \( \lor \), conditional \( \rightarrow \), biconditional \( \leftrightarrow \)
- Truth table shows truth value of formula under all possible assignments to variables

Operator Precedence

- Given a formula \( p \land q \lor r \), do we parse this as \( (p \land q) \lor r \) or \( p \land (q \lor r) \)?
- Without settling on a convention, formulas without explicit paranthesization are ambiguous.
- To avoid ambiguity, we will specify precedence for logical connectives.
- Operator precedence is a convention that tells us how to parse formulas if they are not explicitly paranthesized.

Operator Precedence, cont.

- Negation \( (\neg) \) has higher precedence than all other connectives.
- Question: Does \( \neg p \land q \) mean (i) \( \neg(p \land q) \) or (ii) \( (\neg p) \land q \)?
- Conjunction \( (\land) \) has next highest precedence.
- Question: Does \( p \land q \lor r \) mean (i) \( (p \land q) \lor r \) or (ii) \( p \land (q \lor r) \)?
- Disjunction \( (\lor) \) has third highest precedence.
- Next highest is precedence is \( \rightarrow \), and lowest precedence is \( \leftrightarrow \)

Operator Precedence Example

- Which is the correct interpretation of the formula \( p \lor q \land r \leftrightarrow q \rightarrow \neg r \)

(A) \( ((p \lor (q \land r)) \leftrightarrow q) \rightarrow (\neg r) \)
(B) \( ((p \lor q) \land r) \leftrightarrow (q \rightarrow (\neg r)) \)
(C) \( (p \lor (q \land r)) \leftrightarrow (q \rightarrow (\neg r)) \)
(D) \( (p \lor ((q \land r) \leftrightarrow q)) \rightarrow (\neg r) \)
Validity, Unsatisfiability

- The truth value of a propositional formula depends on truth assignments to variables
- **Example:** $\neg p$ evaluates to true under the assignment $p = F$ and to false under $p = T$
- Some formulas evaluate to true for every assignment, e.g., $p \lor \neg p$
- Such formulas are called tautologies or valid formulas
- Some formulas evaluate to false for every assignment, e.g., $p \land \neg p$
- Such formulas are called unsatisfiable formulas or contradictions

Interpretations

- Concepts of validity, satisfiability are very important in logic!
- To make them precise, we'll define interpretation of formula
- An interpretation $I$ for a formula $F$ is a mapping from each propositional variable in $F$ to exactly one truth value
  
  $I : \{ p \mapsto \text{true}, q \mapsto \text{false}, \ldots \}$
  
  - In general, for formula with $n$ propositional variables, there are $2^n$ interpretations
  - Each interpretation corresponds to one row in the truth table

Examples

- Consider the formula $F : p \land q \rightarrow \neg p \lor \neg q$
- Let $I_1$ be the interpretation such that $[p \mapsto \text{true}, q \mapsto \text{false}]$
- What does $F$ evaluate to under $I_1$?
  - Thus, $I_1 \models F$
- Let $I_2$ be the interpretation such that $[p \mapsto \text{true}, q \mapsto \text{true}]$
- What does $F$ evaluate to under $I_2$?
  - Thus, $I_2 \not\models F$

Entailment

- Under an interpretation, every propositional formula evaluates to $T$ or $F$
  
  Formula $F$ + Interpretation $I = \text{Truth value}$
  
  - We write $I \models F$ if $F$ evaluates to true under $I$
  - Similarly, $I \not\models F$ if $F$ evaluates to false under $I$
  - **Theorem:** $I \models F$ if and only if $I \not\models \neg F$

Another Example

- Let $F_1$ and $F_2$ be two propositional formulas
  
  - Suppose $F_1$ evaluates to true under interpretation $I$
  - What does $F_2 \land \neg F_1$ evaluate to under $I$?

Satisfiability, Validity

- $F$ is **satisfiable** iff there exists interpretation $I$ s.t. $I \models F$
- $F$ is **valid** iff for all interpretations $I$, $I \models F$
- $F$ is **unsatisfiable** iff for all interpretations $I$, $I \not\models F$
- $F$ is **contingent** if it is satisfiable, but not valid.
  
  - Valid formulas also called tautologies
  - Unsatisfiable formulas called contradictions
True/False Questions

Are the following statements true or false?

▶ If a formula is valid, then it is also satisfiable.
▶ If a formula is satisfiable, then its negation is unsatisfiable.
▶ If \( F_1 \) and \( F_2 \) are satisfiable, then \( F_1 \land F_2 \) is also satisfiable.
▶ A formula can be both contingent and unsatisfiable

Duality Between Validity and Unsatisfiability

\( F \) is valid if and only if \( \neg F \) is unsatisfiable

▶ Proof:
▶ By definition, \( F \) is valid iff for all interpretations \( I, I \models F \)
▶ By theorem, \( I \models F \) iff \( I \nvdash \neg F \)
▶ Thus, \( F \) is valid iff for all interpretations \( I, I \nvdash \neg F \)
▶ But if for all interpretations \( I, I \nvdash \neg F \), then \( \neg F \) is unsat
▶ Thus, \( F \) valid iff \( \neg F \) unsat

Proving Validity

▶ Question: How can we prove that a propositional formula is a tautology?
▶ Exercise: Which formulas are tautologies? Prove your answer.
  1. \( (p \to q) \leftrightarrow (\neg q \to \neg p) \)
  2. \( (p \land q) \lor \neg p \)

Proving Satisfiability, Unsatisfiability, Contingency

▶ Similarly, can prove satisfiability, unsatisfiability, contingency using truth tables:
▶ Satisfiable: There exists a row where formula evaluates to true
▶ Unsatisfiable: In all rows, formula evaluates to false
▶ Contingent: Exists a row where formula evaluates to true, and another row where it evaluates to false

Exercises

1. Prove \( \neg (p \land q) \land \neg (\neg p \lor \neg q) \) is unsatisfiable
2. Prove \( (p \to q) \to (q \to p) \) is a contingency

Implication

▶ Formula \( F_1 \) implies \( F_2 \) (written \( F_1 \Rightarrow F_2 \)) iff for all interpretations \( I, I \models F_1 \to F_2 \)
▶ \( F_1 \Rightarrow F_2 \) iff \( F_1 \to F_2 \) is valid
▶ Caveat: \( F_1 \Rightarrow F_2 \) is not a propositional logic formula; \( \Rightarrow \) is not part of PL syntax!
▶ Instead, \( F_1 \Rightarrow F_2 \) is a semantic judgment, like satisfiability!
Syntax vs. Semantics

- **Syntax**: What you are allowed to write
  - \(\land, \rightarrow\) are part of PL syntax, but \(*, \Rightarrow\) are not!
  - \(p_1 \land p_2\) is a syntactically valid PL formula, \(p_1 * p_2\) is not!

- **Semantics**: Concerns meaning of what is written
  - Validity, satisfiability semantic notions b/c they concern meaning of the formula
  - Semantics gives meaning to syntax

- Difference between syntax vs. semantics crucial in CS

Equivalence

- Consider two propositional formulas \(F_1\) and \(F_2\).
- Sometimes \(F_1\) and \(F_2\) always have same truth value for every interpretation, e.g., \(p \lor p\) and \(p \land p\)
- Such formulas \(F_1\) and \(F_2\) called equivalent, written \(F_1 \equiv F_2\)
  - \(F_1 \equiv F_2\) or \(F_1 \leftrightarrow F_2\)
- More precisely, formulas \(F_1\) and \(F_2\) are equivalent iff for all interpretations \(I\), \(I \models F_1 \leftrightarrow F_2\)

\[ F_1 \leftrightarrow F_2 \text{ iff } F_1 \equiv F_2 \text{ is valid} \]

- \(\equiv, \leftrightarrow\) not part of PL syntax; they are semantic judgments!

Important Equivalences

- Some important equivalences are useful to know!
  - Law of double negation: \(\neg \neg p \equiv p\)
  - Identity Laws: \(p \land T \equiv p\) \(p \lor F \equiv p\)
  - Domination Laws: \(p \lor T \equiv T\) \(p \land F \equiv F\)
  - Idempotent Laws: \(p \lor p \equiv p\) \(p \land p \equiv p\)
  - Negation Laws: \(p \land \neg p \equiv F\) \(p \lor \neg p \equiv T\)

Checking Implication

- **Question**: How can we check if \(F_1 \Rightarrow F_2\)?
- **Exercise**: Does \(p \lor q\) imply \(p\)? Prove your answer!

Checking Equivalence

- **Question**: How can we prove \(F_1 \equiv F_2\)?
- **Exercise**: Prove \(p \rightarrow q\) and \(\neg p \lor q\) are equivalent

Commutativity and Distributivity Laws

- **Commutative Laws**: \(p \lor q \equiv q \lor p\) \(p \land q \equiv q \land p\)
- **Distributivity Law #1**: \((p \lor (q \land r)) \equiv (p \lor q) \land (p \lor r)\)
- **Distributivity Law #2**: \((p \land (q \lor r)) \equiv (p \land q) \lor (p \land r)\)
- **Associativity Laws**: \(p \lor (q \lor r) \equiv (p \lor q) \lor r\) \(p \land (q \lor r) \equiv (p \land q) \land r\)
### De Morgan’s Laws

- Let \( cs243 \) be the proposition “John took CS243” and \( cs303 \) be the proposition “John took CS303”
- In simple English what does \( \neg (cs243 \land cs303) \) mean?
- De Morgan’s law expresses exactly this equivalence!
- **De Morgan’s Law #1:** \( \neg (p \land q) \iff \neg p \lor \neg q \)
- **De Morgan’s Law #2:** \( \neg (p \lor q) \iff \neg p \land \neg q \)

When you “push” negations in, \( \land \) becomes \( \lor \) and vice versa

### Using Equivalences

- We saw one way to prove two formulas are equivalent: use truth table
- Another way: use known equivalences to rewrite one formula as the other
- **Examples:** Prove following formulas are equivalent using known equivalences.
  1. \( \neg (p \lor (\neg p \land q)) \) and \( \neg p \land \neg q \)
  2. \( \neg (p \rightarrow q) \) and \( p \land \neg q \)

### Formalizing English Arguments in Logic

- We can use logic to prove correctness of English arguments.
- For example, consider the argument:
  - If Joe drives fast, he gets a speeding ticket.
  - Joe did not get a ticket.
  - Therefore, Joe did not drive fast.
- Let \( f \) be the proposition "Joe drives fast" , and \( t \) be the proposition "Joe gets a ticket"
- How do we encode this argument as a logical formula?

### Example, cont.

"If Joe drives fast, he gets a speeding ticket. Joe did not get a ticket. Therefore, he did not drive fast."

\[ ((f \rightarrow t) \land \neg t) \rightarrow \neg f \]

- How can we prove this argument is valid?
- Can do this in two ways:
  1. Use truth table to show formula is tautology
  2. Use known equivalences to rewrite formula to true
- Let’s use equivalences

### Another Example

- Can also use to logic to prove an argument is not valid.
- Suppose your friend George make the following argument:
  - If Jill carries an umbrella, it is raining.
  - Jill is not carrying an umbrella.
  - Therefore it is not raining.
- Let’s use logic to prove George’s argument doesn’t hold water.
- Let \( u = "\text{Jill is carrying an umbrella}" \), and \( r = "\text{It is raining}" \)
- How do we encode this argument in logic?

### Example, cont.

"If Jill carries an umbrella, it is raining. Jill is not carrying an umbrella. Therefore it is not raining."

\[ ((u \rightarrow r) \land \neg u) \rightarrow \neg r \]

- How can we prove George’s argument is invalid?
Summary

- A formula is **valid** if it is true for all interpretations.
- A formula is **satisfiable** if it is true for at least one interpretation.
- A formula is **unsatisfiable** if it is false for all interpretations.
- A formula is **contingent** if it is true in at least one interpretation, and false in at least one interpretation.
- Two formulas $F_1$ and $F_2$ are **equivalent**, written $F_1 \equiv F_2$, if $F_1 \leftrightarrow F_2$ is valid.