Announcements

- Homework 2 is graded, scores on Blackboard
- Graded HW and sample solutions given at end of this lecture
  - Make sure score matches the one on Blackboard
  - If not, let us know within one week
- Mean on homework 2: 66/90 (73%)
- Many of you made mistakes on questions 2, 3
- Similar questions will be on midterm – review the sample solutions!

Midterm

- Midterm next Tuesday, Oct. 2, in lecture
- Midterm will cover all topics up to today’s lecture:
  - propositional logic, first order logic, inference rules, proof techniques, sets, functions
  - Good way to prepare is to go over lectures and homeworks
  - To help you prepare, will give solutions for HW3 on Thursday
- Also, Thursday’s lecture will be a review session
  - Weilin will go over difficult homework problems

More on Midterm

- Midterm is closed-book, closed-notes, closed-phones, closed-laptops, closed-tablets etc.
- But you are allowed to bring three sheets of hand-written or typed notes prepared by you
- I’m out of town until next Wednesday, therefore Weilin will be proctoring the midterm

Sequences

- A sequence is a discrete structure to represent an ordered list.
- Example: 1, 2, 3, 5, 8 is a finite sequence with five terms
- Example: 1, 2, 3, 5, 8, 13, 21, . . . is an infinite sequence
  (Fibonacci numbers)
- Formally, sequence is a function from a subset of $\mathbb{Z}$ to a set $S$.
- Notation $a_n$ represents $n$’th term in the sequence
- For Fibonacci sequence, $a_0 = 1, a_1 = 2, a_2 = 3, a_3 = 5$ etc.

Sequence Examples

- What is the sequence defined by $a_n = \frac{1}{n}$ for $n \geq 1$?
- What is the sequence defined by $a_n = n^2$ for $n \geq 1$?
- What is the sequence defined by $a_n = (-1)^n$ for $n \geq 0$?
Arithmetic Progression

- Some kinds of sequences come up a lot in discrete math.
- An arithmetic progression is a sequence of the form:
  \[ a, a + d, a + 2d, a + 3d, \ldots \]
  Here, the real number \( a \) is called the initial term.
- Also, \( d \) is called the common difference.
- Example: \( 2, 5, 8, 11, \ldots \).
  What is the common difference?
  What is the initial term?

Arithmetic Progression, cont.

- Arithmetic progressions can always be written as
  \[ a_n = a_0 + d \cdot n \]
  where \( a_0 \) is the initial term and \( d \) is the common difference.
- Example: \( a_n = -1 + 4n \) for \( n \geq 0 \) is an arithmetic progression
  with elements:
  \[-1, 3, 7, 11, 15, \ldots \]
- Example: What is the closed-form definition for the sequence \( 2, 5, 8, 11, \ldots ? \)

Geometric Progression

- Another class of sequences that come up often are geometric progressions.
- A geometric progression is a sequence of the form:
  \[ a, ar, ar^2, ar^3, \ldots \]
  Here, \( a \) is called the initial term, and \( r \) is the common ratio.
- Example: \( 1, -3, 9, -27, \ldots \).
  What is the initial term?
  What is the common ratio?

Geometric Progression, cont.

- Geometric progressions can always be written as
  \[ a_n = a_0 \cdot r^n \]
  where \( a_0 \) is the initial term and \( r \) is the common ratio.
- Example: The sequence defined by \( a_n = 6 \cdot \left( \frac{1}{2} \right)^n \) for \( n \geq 0 \) is
  an geometric progression with elements:
  \[ 6, 2, 2, \frac{2}{3}, \frac{2}{3}, \ldots \]
- Example: What is the closed-form definition for the sequence
  \( 1, -3, 9, -27, \ldots ? \)

Summations

- Given a sequence, one common operation is to sum up all the terms in that sequence.
- For this purpose, we use the summation notation:
  \[ \sum_{j=m}^{n} a_j = a_m + a_{m+1} + \ldots + a_n \]
- Example:
  \[ \sum_{j=1}^{3} j = 1 + 2 + 3 = 6 \]
  The variable \( j \) in this notation is called index of summation.
  Also, \( m \) and \( n \) are called the lower and upper limits of the summation.

Summation Examples

- Consider the sequence \( a_n = n^2 \). What is the value of this summation?
  \[ \sum_{j=1}^{4} a_j \]
- What is the value of this summation?
  \[ \sum_{j=1}^{6} (-1)^j \]
Nested Summations

- It is also common to nest summations within one another.
- What is the value of the following summation?
  \[ \sum_{i=1}^{2} \sum_{j=1}^{3} ij \]
- Answer:
  \[ \sum_{i=1}^{2} \sum_{j=1}^{3} ij = \sum_{i=1}^{2} (i + 2i + 3i) = 2 \cdot 6 + 12 = 18 \]

Closed Form of Summations

- Some summations arise all the time in discrete mathematics
- Example: Sum of all numbers from 1 to n:
  \[ \sum_{i=1}^{n} i = \frac{n(n + 1)}{2} \]
- For such common summations, it is often useful to derive a closed form
- The closed form expresses the value of the summation as a formula without summations
- The closed form of above summation is:
  \[ \sum_{i=1}^{n} i = \frac{n(n + 1)}{2} \]

Example

- Compute the value of the summation:
  \[ \sum_{i=21}^{50} i = 1275 - 210 = 1065 \]
- We can rewrite this summation as:
  \[ \sum_{i=1}^{50} i - \sum_{i=1}^{20} i \]
- By previous definition, first summation is:
- Second summation is:

Useful Property Summations

- Given a summation consisting of addition and multiplication terms, we can decompose it as follows:
  \[ \sum_{j=m}^{n} (ax_j + by_j) = a \sum_{j=m}^{n} x_j + b \sum_{j=m}^{n} y_j \]
- Example: Compute the value of
  \[ \sum_{i=1}^{10} 3i + 2 \]
- This can be written as
  \[ 3 \sum_{i=1}^{10} i + \sum_{i=1}^{10} 2 \]

Arithmetic Series

- The sum of the terms in an arithmetic progression \( a, a + d, a + 2d, \ldots \) is called an arithmetic series.
- Let’s derive a closed form for arithmetic series:
  \[ \sum_{i=1}^{n} (a + di) \]
- By the earlier property, we can write this as:
  \[ \sum_{i=1}^{n} a + d \sum_{i=1}^{n} i \]

Closed Form for Arithmetic Series

- What is \( \sum_{i=1}^{n} a \)?
- By earlier closed form, we have:
  \[ \sum_{i=1}^{n} i = \frac{n \cdot (n + 1)}{2} \]
- Thus, we can write entire arithmetic series in closed form as:
  \[ an + dn \cdot \frac{n(n + 1)}{2} \]
Example

- What is the closed form for the following summation?
  \[ \sum_{j=3}^{n} (2 + 3j) \]

- Trick: Write this as the difference of two summations:
  \[ \sum_{j=1}^{n} (2 + 3j) - \sum_{j=1}^{2} (2 + 3j) \]

- Expand the second term:
  \[ \sum_{j=1}^{n} (2 + 3j) - 13 \]

Example, cont.

- Now, compute the closed form for first term:
  \[ \sum_{j=1}^{n} 2 + 3 \sum_{j=1}^{n} j - 13 \]

- Using known closed forms, this can be rewritten as:
  \[ 2n + 3\frac{n(n+1)}{2} - 13 \]

Geometric Series

- The sum of the terms in a geometric progression \(a, ar, ar^2, \ldots\) is called a geometric series.

- Theorem: Closed form of geometric series (\(r \neq 1\)):
  \[ \sum_{j=0}^{n} (ar^j) = a \cdot \frac{r^{n+1} - 1}{r - 1} \]

- This is very useful to know—memorize it!

- Let’s prove why this closed form is correct

Derivation of Geometric Series Closed Form

- Theorem: Closed form of geometric series (\(r \neq 1\)):
  \[ \sum_{j=0}^{n} (ar^j) = a \cdot \frac{r^{n+1} - 1}{r - 1} \]

- First, let’s call the summation on left \(S\):

- Now, let’s multiply \(S\) by \(r\):
  \[ rS = r \sum_{j=0}^{n} ar^j = \sum_{j=0}^{n} ar^{j+1} \]

Derivation continued

- Now, change index of summation from \(j\) to \(k\) where \(k = j + 1\):
  \[ rS = \sum_{j=0}^{n} ar^{j+1} \]

- Now, rewrite this as:
  \[ rS = \sum_{k=1}^{n+1} ar^k - a + ar^{n+1} \]

Derivation, cont.

- Now, observe first term on left hand side is \(S\)!

- Thus, we have:
  \[ rS = S + ar^{n+1} - a \]

- Collecting \(S\) on one side, we get:
  \[ S = a \cdot \frac{r^{n+1} - 1}{r - 1} \]

- This is exactly the closed form from the theorem!
Example 1
\[ \sum_{j=0}^{n} (a r^j) = a \cdot \frac{r^{n+1} - 1}{r - 1} \]

- Compute the value of \( \sum_{i=0}^{5} 3 \cdot 2^i \)
- What is \( a \)?
- What is \( r \)?
- Using closed form, we have:
  \[ \sum_{i=0}^{5} 3 \cdot 2^i = 3 \cdot \frac{2^6 - 1}{2 - 1} = 189 \]

Example 2
- For \( |r| < 1 \), derive a closed form for the summation
  \[ \sum_{n=0}^{\infty} a \cdot r^n \]
- Using closed form for geometric series, this is equivalent to:
  \[ \lim_{n \to \infty} a \cdot \frac{r^{n+1} - 1}{r - 1} \]
- Since \( |r| < 1 \), \( r^{n+1} \) becomes 0 as \( n \) approaches infinity
- Thus, this is equivalent to:
  \[ a \cdot \frac{0 - 1}{r - 1} = a \cdot \frac{1}{1 - r} \]

Example 3
- Compute the value of the summation:
  \[ \sum_{k=0}^{\infty} 3 \cdot \left( \frac{1}{9} \right)^k \]
- Using previous formula, this sum is given by \[ \frac{3}{1 - \frac{1}{9}} = \frac{27}{8} \]

Revisiting Sets
- Earlier we talked about sets and cardinality of sets
- Recall: Cardinality of a set is number of elements in that set
- This definition makes sense for sets with finitely many element, but more involved for infinite sets
- Agenda: Revisit the notion of cardinality for infinite sets

Cardinality of Infinite Sets
- Sets with infinite cardinality are classified into two classes:
  1. Countably infinite sets (e.g., natural numbers)
  2. Uncountably infinite sets (e.g., real numbers)
- A set \( A \) is called countably infinite if there is a bijection between \( A \) and the set of positive integers.
- A set \( A \) is called countable if it is either finite or countably infinite
- Otherwise, the set is called uncountable or uncountably infinite

Example
Prove: The set of odd positive integers is countably infinite.
- Need to find a function \( f \) from \( \mathbb{Z}^+ \) to the set of odd positive integers, and prove that \( f \) is bijective
- Consider \( f(n) = 2n - 1 \) from \( \mathbb{Z}^+ \) to odd positive integers
- We need to show \( f \) is bijective (i.e., one-to-one and onto)
- Let’s first prove injectivity, then surjectivity
Example, cont.

Prove injectivity of \( f(n) = 2n - 1 \)

- **Recall**: Function is injective if \( f(a) = f(b) \rightarrow a = b \)
- Suppose \( f(a) = f(b) \). Then \( 2a - 1 = 2b - 1 \)
- This implies \( a = b \), establishing injectivity

Another Way to Prove Countable-ness

- One way to show a set \( A \) is countably infinite is to give bijection between \( \mathbb{Z}^+ \) and \( A \)
- Another way is by showing members of \( A \) can be written as a sequence \( (a_1, a_2, a_3, \ldots) \)
- Since such a sequence is a bijective function from \( \mathbb{Z}^+ \) to \( A \), writing \( A \) as a sequence \( a_1, a_2, a_3, \ldots \) establishes one-to-one correspondence

Example, cont.

Prove surjectivity of \( f(n) = 2n - 1 \)

- **Recall**: Function is surjective if for every \( b \in B \), there exists some \( a \in A \) such that \( f(a) = b \)
- **Proof by contradiction**: Suppose there is some odd positive integer \( b \) such that \( \forall x \in \mathbb{Z}^+. 2x - 1 \neq b \)
- This implies \( \frac{b+1}{2} \) is not an integer.
- But since \( b \) is an odd positive integer, \( b + 1 \) is even
- Thus, \( b + 1 \) is divisible by 2, yielding a contradiction.
- Since we showed that there is a bijection (namely \( 2n - 1 \)) from positive integers to odd positive integers, the set of odd positive integers is countably infinite

Rational Numbers are Countable

- Not too surprising \( \mathbb{Z} \) and odd \( \mathbb{Z}^+ \) are countably infinite
- **More surprising**: Set of rationals is also countably infinite!
- We’ll prove that the set of positive rational numbers is countable by showing how to enumerate them in a sequence
- **Recall**: Every positive rational number can be written as the quotient \( p/q \) of two positive integers \( p, q \)

Another Example

Prove that the set of all integers is countable

- We can list all integers in a sequence, alternating positive and negative integers:
  \[ a_n = 0, 1, -1, 2, -2, 3, -3, \ldots \]
- Observe that this sequence defines the bijective function:
  \[ f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n-1)/2 & \text{if } n \text{ odd} \end{cases} \]

Rationals in a Table

- Now imagine placing rationals in a table such that:
  1. Rationals with \( p = 1 \) go in first row, \( p = 2 \) in second row, etc.
  2. Rationals with \( q = 1 \) in 1st column, \( q = 2 \) in 2nd column, \( \ldots \)
Enumerating the Rationals

- How to enumerate entries in this table without missing any?
- Trick: First list those with \( p + q = 2 \), then \( p + q = 3 \), …
- Traverse table diagonally from left-to-right, in the order shown by arrows

This allows us to list all rationals in a sequence:

\[
\begin{align*}
1 & 1 \\
2 & 1 \\
3 & 1 \\
4 & 1 \\
5 & 1 \\
& \ldots
\end{align*}
\]

Hence, set of rationals is countable

Uncountability of Real Numbers

- Prime example of uncountably infinite sets is real numbers
- The fact that \( \mathbb{R} \) is uncountably infinite was proven by George Cantor using the famous Cantor’s diagonalization argument
- This was a shocking result in mathematics in the 1800’s
- This argument has inspired many similar famous proofs in the theory of computation
- Brief look at the diagonalization argument

Cantor’s Diagonalization Argument

- For contradiction, assume the set of reals was countable
- Since any subset of a countable set is also countable, this would imply the set of reals between 0 and 1 is also countable
- Now, if reals between 0 and 1 are countable, we can list them in a table in some order:

\[
\begin{align*}
R_0 &= 0.a_{01}a_{02}a_{03}\ldots \\
R_1 &= 0.a_{11}a_{12}a_{13}\ldots \\
R_2 &= 0.a_{21}a_{22}a_{23}\ldots \\
& \ldots
\end{align*}
\]

Clearly, this new number \( R \) differs from each number \( R_i \) in the table in at least one digit (its \( i \)'th digit)

Diagonalization Argument, concluded

- Since \( R \) is not in the table, this is not a complete enumeration of all reals between 0 and 1
- Hence, the set of real between 0 and 1 is not countable
- Since the superset of any uncountable set is also uncountable, set of reals is uncountably infinite

Diagonalization Argument, cont

\[
\begin{align*}
R_0 &= 0.a_{0_1}a_{0_2}a_{0_3}\ldots \\
R_1 &= 0.a_{1_1}a_{1_2}a_{1_3}\ldots \\
R_2 &= 0.a_{2_1}a_{2_2}a_{2_3}\ldots \\
& \ldots
\end{align*}
\]