33. Adapt the proof in Example 4 in Section 1.7 to prove that if \( n = abc \), where \( a, b, \) and \( c \) are positive integers, then \( a \leq \sqrt[3]{n}, b \leq \sqrt[4]{n}, \) or \( c \leq \sqrt{\sqrt[n]{n}}. \)

34. Prove that \( \sqrt{2} \) is irrational.

35. Prove that between every two rational numbers there is an irrational number.

36. Prove that between every rational number and every irrational number there is an irrational number.

*37. Let \( S = x_1y_1 + x_2y_2 + \cdots + x_ny_n \), where \( x_1, x_2, \ldots, x_n \) and \( y_1, y_2, \ldots, y_n \) are orderings of two different sequences of positive real numbers, each containing \( n \) elements.

a) Show that \( S \) takes its maximum value over all orderings of the two sequences when both sequences are sorted (so that the elements in each sequence are in nondecreasing order).

b) Show that \( S \) takes its minimum value over all orderings of the two sequences when one sequence is sorted into nondecreasing order and the other is sorted into nonincreasing order.

38. Prove or disprove that if you have an 8-gallon jug of water and two empty jugs with capacities of 5 gallons and 3 gallons, respectively, then you can measure 4 gallons by successively pouring some of or all of the water in a jug into another jug.

39. Verify the 3\( x + 1 \) conjecture for these integers.

a) 6  b) 7  c) 17  d) 21

40. Verify the 3\( x + 1 \) conjecture for these integers.

a) 16  b) 11  c) 35  d) 113

41. Prove or disprove that you can use dominoes to tile the standard checkerboard with two adjacent corners removed (that is, corners that are not opposite).

42. Prove or disprove that you can use dominoes to tile a standard checkerboard with all four corners removed.

43. Prove that you can use dominoes to tile a rectangular checkerboard with an even number of squares.

44. Prove or disprove that you can use dominoes to tile a 5 \( \times \) 5 checkerboard with three corners removed.

45. Use a proof by exhaustion to show that a tiling using dominoes of a 4 \( \times \) 4 checkerboard with opposite corners removed does not exist. [Hint: First show that you can assume that the squares in the upper left and lower right corners are removed. Number the squares of the original checkerboard from 1 to 16, starting in the first row, moving right in this row, then starting in the leftmost square in the second row and moving right, and so on. Remove squares 1 and 16. To begin the proof, note that square 2 is covered either by a domino laid horizontally, which covers squares 2 and 3, or vertically, which covers squares 2 and 6. Consider each of these cases separately, and work through all the subcases that arise.]

*46. Prove that when a white square and a black square are removed from an 8 \( \times \) 8 checkerboard (colored as in the text) you can tile the remaining squares of the checkerboard using dominoes. [Hint: Show that when one black and one white square are removed, each part of the partition of the remaining cells formed by inserting the barriers shown in the figure can be covered by dominoes.]

47. Show that by removing two white squares and two black squares from an 8 \( \times \) 8 checkerboard (colored as in the text) you can make it impossible to tile the remaining squares using dominoes.

*48. Find all squares, if they exist, on an 8 \( \times \) 8 checkerboard such that the board obtained by removing one of these squares can be tiled using straight triominoes. [Hint: First use arguments based on coloring and rotations to eliminate as many squares as possible from consideration.]

*49. a) Draw each of the five different tetrominoes, where a tetromino is a polyomino consisting of four squares.

b) For each of the five different tetrominoes, prove or disprove that you can tile a standard checkerboard using these tetrominoes.

*50. Prove or disprove that you can tile a 10 \( \times \) 10 checkerboard using straight tetrominoes.

Key Terms and Results

**TERMS**

- **proposition**: a statement that is true or false
- **propositional variable**: a variable that represents a proposition
- **truth value**: true or false
- \( \neg p \) (negation of \( p \)): the proposition with truth value opposite to the truth value of \( p \)
- \( p \lor q \) (disjunction of \( p \) and \( q \)): the proposition "\( p \) or \( q \)" which is true if and only if at least one of \( p \) and \( q \) is true

**logical operators**: operators used to combine propositions

- **compound proposition**: a proposition constructed by combining propositions using logical operators

- **truth table**: a table displaying all possible truth values of propositions