Always assume that $N$ includes 0.

A. Are the following propositions? Write yes or no. (1 point each).

1. $(A \cup B) \land (B \cup C)$. \textbf{NO}
2. $\forall x, y \in \mathbb{Z}(x \land y)$. \textbf{YES}
3. $\forall x \in A \land x \notin B$. \textbf{NO}
4. If $x$ is in $A \land x \in B$, then $x$ is in $A \land B$. \textbf{NO}
5. $\forall x[(x \notin A \land x \notin B) \rightarrow (x \in A \land B)]$. \textbf{YES}
6. $x \in (A \subseteq B)$. \textbf{NO}
7. Does $A$ imply $B$? \textbf{NO}

B. Are the following propositions true or false? (1 point each).

1. $\forall a, b \in \mathbb{R}(a \leq b \land b \leq a \rightarrow a = b)$. \textbf{TRUE}
2. $|a + b| < |a| + |b|$. \textbf{FALSE}
3. $\exists c \in \mathbb{Z} \forall a \in \mathbb{Z}(a^c)$. \textbf{TRUE}
4. The $\subseteq$ relation on $P(U)$ is reflexive, symmetric, and transitive. \textbf{FALSE}
5. $(A \land B \land C) = (A \land B = \emptyset)$. \textbf{TRUE}
6. $(A \land B \land C) \rightarrow \exists x[x \in A \land x \notin B]$. \textbf{FALSE}
7. $\forall n \in \mathbb{N}[(\cap_{i=0}^n \{x \in \mathbb{Z}|(x|i)\}) = \{1\}]$. \textbf{FALSE}

C. Use definitions to explain the meaning of the following, each).

1. Set $R$ is a relation on set $A = R \subseteq A \times A$
2. Integer $x$ is odd $\equiv \exists k \in \mathbb{Z}(x = 2k + 1)$
3. Relation $R$ on set $A$ is transitive $\equiv \forall a, b, c \in A[aRb \land bRc \rightarrow aRc]$
4. Relation $R$ on set $A$ is anti-symmetric $\equiv \forall a, b \in A[aRb \land bRa \rightarrow a = b]$
5. For sets $A$ and $B$, $A \cap B = \{x \mid x \in A \land x \in B\}$
6. For set $A$, $P(A) = \{x \mid x \subseteq A\}$
D. Provide counterexamples disproving the following propositions (explain why they are counterexamples). (4 points each).

1. The square of a real number is always positive.
   \[ 0 \in \mathbb{R}, \text{ but } 0^2 = 0 \text{ is not positive.} \]

2. \( A \rightarrow (B \rightarrow C) \equiv (A \rightarrow B) \rightarrow C. \)
   If \( A = F, B = T, \) and \( C = F, \) then \( A \rightarrow (B \rightarrow C) \equiv F \rightarrow (T \rightarrow F) \equiv T, \)
   but \( (A \rightarrow B) \rightarrow C \equiv (F \rightarrow T) \rightarrow F \equiv T \rightarrow F \equiv F. \) This is a counterexample because the truth values are different.
   The only other counterexample is \( A = F, B = F, \) and \( C = F. \)

3. Relation \( R \) on \( \mathbb{R}^+ \), defined as \( a R b \iff a < (1/b) \), is transitive.
   \((1, 0.1) \in R \) because \( 1 < 10 = 1/0.1 \)
   \((0.1, 1) \in R \) because \( 0.1 < 1 = 1/1 \)
   But \( (1, 1) \notin R \) because \( 1 < 1/1 \) is not true.
   \[ \therefore 1R0.1 \land 0.1R1 \text{ does not imply } 1R1 \]

4. \( \forall x, y \in \mathbb{R} \left[ x \neq y \rightarrow (\exists b \in \mathbb{R} \mid x b = y) \right] \).
   Pick \( x = 0, y = 5 \). Then there does not exist any \( b \) such that \( x \cdot b = y \), because \( x b = 0 \cdot b = 0 \neq 5 = y \), no matter what \( b \) is.
E. Make truth tables for the following formulas. You can write compact truth tables if you prefer, in which case you should circle the final column that you fill in. (3 points each).

1. \((P \land Q) \rightarrow Q\)

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2. \((\neg A \lor B) \land (C \land A)\)

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F. Enumerate every element of the following sets. Assume \(A = \{a, b\}\) and \(B = \{b, c\}\). (3 points each).

1. \(P(A) = \{\emptyset, a, b, ab, a^2, b^2\}\)

2. \(B \times (A \cap B) = \{(b, b), (c, b)\}\)

3. \(P(A \cup B) = \{\emptyset, a, b, ab, a^2, b^2, a^2b, a^2b^2, ab^2, ab^2c, ab^2c^2\}\)

4. \(\{x \in \mathbb{R}^+ | x < 5 \land 0.75x \in \mathbb{Z}\} = \{\frac{4}{3}, \frac{8}{3}, \frac{12}{3}\}\)

5. \(\bigcap_{i=1}^{10} \{x \in \mathbb{N} | x < 10 \land (i|x)\} = \{0\}\)
G. Prove the following theorems. (6 points each).

1. The relation $R$ on $\mathbb{R}$, defined as $xRy \leftrightarrow |x| \leq |y|$, is reflexive and transitive.
   1. Prove $R$ is reflexive, i.e. $\forall x \in \mathbb{R} \ (xRx)$.
   2. $|x| \leq |x|$ is true for all $x \in \mathbb{R}$
   3. $xRx$ by def. of $R$
   4. $R$ is reflexive.
   5. Prove $R$ is transitive, i.e. $\forall x,y,z \in \mathbb{R} (xRy \land yRz \rightarrow xRz)$
   6. Assume $xRy \land yRz$
   7. By def of $R$: $|x| \leq |y| \land |y| \leq |z|$
   8. The $\leq$ relation is transitive, so $|x| \leq |z|$
   9. $xRz$ by def of $R$
   10. $R$ is transitive.

2. For sets $A$, $B$, $C$, and $D$, if $A \times B$ and $C \times D$ are disjoint, then either $A$ and $C$ are disjoint, or $B$ and $D$ are disjoint.
   1. Assume $A \times B$ and $C \times D$ are disjoint, prove $A$ and $C$ are disjoint, or $B$ and $D$ are disjoint.
   2. If $A$ and $C$ are disjoint, the proof is done, so assume $A$ and $C$ are not disjoint and prove $B$ and $D$ are disjoint.
   3. Assume $B$ and $D$ are not disjoint and seek a contradiction.
   4. From line 2, there exists an $x$ in $A$ and $C$.
   5. From line 3, there exists a $y$ in $B$ and $D$.
   6. By def. of Cartesian product, $(x,y) \in A \times B$ and $(x,y) \in C \times D$.
   7. But $A \times B$ and $C \times D$ are disjoint, so line 6 is a contradiction.
   8. $\therefore B$ and $D$ are disjoint.
   9. Thus if $A \times B$ and $C \times D$ are disjoint, either $A$ and $C$ are disjoint, or $B$ and $D$ are disjoint.
G. Prove the following theorems. (6 points each).

3. For every real number \( x \), where \( x \neq 0 \) and \( x \neq 1 \), there exists a real number \( y \) such that \( y/x = y-x \).

1. Let \( y = x^2/(x-1) \), which is defined because \( x \neq 1 \).
2. Then \( y/x = (x^2/(x-1))/x \) \( \xi \) substitute \( y \), defined because \( x \neq 0 \)
3. \( = (x^2/x(x-1)) \) \( \xi \) multiply by \( 1 = (x-1)/(x-1) \)
4. \( = (x/(x-1)) \) \( \xi \) divide \( \bar{2} \)
5. \( = ((x^2-x^2+x)/(x-1)) \) \( \xi \) add \( o = x^2 - x^2 \)
6. \( = ((x^2-x(x-1))/(x-1)) \) \( \xi \) factor out \( -x \)
7. \( = (x^2/(x-1)) - (x(x-1)/(x-1)) \) \( \xi \) split fraction \( \bar{3} \)
8. \( = y - x \) \( \xi \) substitute \( y \), divide \( \bar{3} \)
9. \( \therefore \exists y \in \mathbb{R} \ [y/x = y-x] \)

4. \( \forall n \in \mathbb{N} [(n^2+n) \text{ is even}] \). (Hint: Break into cases for \( n \) being even and odd).

1. Case 1: \( n \) is odd
2. \( \exists k \in \mathbb{Z} \ [n = 2k+1] \) \( \xi \) def. odd \( \bar{3} \)
3. Then \( n^2+n = (2k+1)^2 + (2k+1) \) \( \xi \) substitute \( n \)
4. \( = 4k^2+4k+1+2k+1 \) \( \xi \) multiply \( \bar{3} \)
5. \( = 2(2k^2+3k+1) \) \( \xi \) simplify \( \bar{3} \)
6. Define \( q = 2k^2+3k+1 \), \( q \in \mathbb{Z} \) \( \xi \) \( \mathbb{Z} \) closed under mult/ADD \( \bar{3} \)
7. \( \exists q \in \mathbb{Z} \ [n^2+n = 2q] \) \( \xi \) \( \mathbb{Z} \) gen. \( \bar{3} \)
8. \( n^2+n \) is even \( \xi \) def even \( \bar{3} \)

9. Case 2: \( n \) is even
10. \( \exists k \in \mathbb{Z} \ [n = 2k] \) \( \xi \) def. even \( \bar{3} \)
11. Then \( n^2+n = (2k)^2 + 2k \) \( \xi \) substitute \( n \)
12. \( = 4k^2+2k \) \( \xi \) simplify \( \bar{3} \)
13. Define \( q = 2k^2+k \), \( q \in \mathbb{Z} \) \( \xi \) \( \mathbb{Z} \) closed under mult/ADD \( \bar{3} \)
14. \( \exists q \in \mathbb{Z} \ [n^2+n = 2q] \) \( \xi \) \( \mathbb{Z} \) gen \( \bar{3} \)
15. \( n^2+n \) is even \( \xi \) def even \( \bar{3} \)
G. Prove the following theorems. (6 points each).

5. \( \forall n \in \mathbb{N}, \sum_{i=0}^{n} i(i+1) = n(n+1)(n+2)/3 \).

1. Proof by induction
2. Base Case
3. \( \sum_{i=0}^{0} i(i+1) = 0(0+1) = 0 = 0(0+1)(0+2)/3 \)
4. Induction Case
5. Assume \( \sum_{i=0}^{k} i(i+1) = k(k+1)(k+2)/3 \) for \( k \in \mathbb{N} \)
6. Prove \( \sum_{i=0}^{k+1} i(i+1) = (k+1)(k+2)(k+3)/3 \)
7. \( \sum_{i=0}^{k+1} i(i+1) = \sum_{i=0}^{k} i(i+1) + (k+1)(k+2) \) \( \text{Separate term} \)
8. \( = k(k+1)(k+2)/3 + (k+1)(k+2) \) \( \text{Inductive hypothesis} \)
9. \( = k(k+1)(k+2)/3 + 3(k+1)(k+2)/3 \) \( \text{Common denominator} \)
10. \( = (k+1)(k+2)(k+3)/3 \) \( \text{Add} \)
11. \( \sum_{i=0}^{k+1} i(i+1) = (k+1)(k+2)(k+3)/3 \)
12. Combination of Base Case and Inductive Case proves theorem
H. Decide whether or not the following propositions are true, and then either prove or disprove them. (7 points each).

1. If $R$ is a relation from $X$ to $Y$, and $S$ and $T$ are relations from $Y$ to $Z$, then $(R \circ S) \cap (R \circ T) \subseteq R \circ (S \cap T)$.

   \[
   R = \{(1, 2), (1, 3)\} \\
   S = \{(2, 4)\} \\
   T = \{(3, 4)\} \\
   R \circ S = \{(1, 4)\} \\
   R \circ T = \{(1, 4)\} \\
   (R \circ S) \cap (R \circ T) = \{(1, 4)\} \\
   R \circ (S \cap T) = \emptyset \\
   S \cap T = \emptyset
   \]

   For the given definitions of $R$, $S$ and $T$,
   \[
   (R \circ S) \cap (R \circ T) = \{(1, 4)\},
   \]
   which is not a subset of
   \[
   R \circ (S \cap T) = \emptyset.
   \]

   \[
   \text{TRUE} \\
   \]

2. The relation $R$ on $\mathbb{R}$, defined as $xRy \iff (x - y) \in \mathbb{Z}$, is reflexive, symmetric, and transitive.

   1. Proof $R$ is reflexive
   2. $x - x = 0$ for all $x \in \mathbb{R}$
   3. $0 \in \mathbb{Z}$
   4. $xRx$ for all $x \in \mathbb{R}$, by def of $R$. So $R$ is reflexive.
   5. Proof $R$ is symmetric.
   6. Assume $xRy$.
   7. By def of $R$, $(x - y) \in \mathbb{Z}$ as well.
   8. Since $\mathbb{Z}$ is closed under mult., $-(x - y) \in \mathbb{Z}$ as well.
   9. $-(x - y) = (y - x)$, so $(y - x) \in \mathbb{Z}$
   10. Then $yRx$ by def of $R$, so $R$ is symmetric.
   11. Proof $R$ is transitive.
   12. Assume $xRy$ and $yRz$.
   13. Then $(x - y) \in \mathbb{Z}$ and $(y - z) \in \mathbb{Z}$ by def of $R$.
   14. $(x - y) + (y - z) \in \mathbb{Z}$ because $\mathbb{Z}$ is closed under addition.
   15. $(x - y) + (y - z) = x - z$, so $(x - z) \in \mathbb{Z}$
   16. Then $xRz$ by def of $R$, so $R$ is transitive.
H. Decide whether or not the following propositions are true, and then either prove or disprove them. (7 points each).

3. For real numbers \(a\) and \(b\), \((\frac{a+b}{2})^n < a^n + b^n\) for \(n \geq 2\). \textbf{FALSE}

Let \(a = b = 1\), and \(n = 2\). Then
\[
\left(\frac{a+b}{2}\right)^n = \left(\frac{1+1}{2}\right)^2 = 1^2 = 1, \quad \text{and}
\]
\[
\left(\frac{a^n + b^n}{2}\right) = \left(\frac{1^2 + 1^2}{2}\right) = \frac{2}{2} = 1. \quad \text{Since}
\]
\[
\left(\frac{a+b}{2}\right)^n = 1 \quad \text{is not less than} \quad \left(\frac{a^n + b^n}{2}\right) = 1, \quad \text{this}
\]
\text{disproves the claim.}

4. \(|2x - 6| > x \rightarrow |x - 4| > 2. \quad \text{TRUE.}

1. Split into cases \(2x - 6 < 0\) and \(2x - 6 \geq 0\)
2. Case 1: \(2x - 6 < 0\)
3. Then \(12x - 6) = 6 - 2x\)
4. Since \(12x - 6) > x\), \(6 - 2x > x\)
5. Add \(2x = 6 \geq 3x\)
6. Divide by \(3 = 2 > x\)
7. Mult. by \(-1: -2 < -x\)
8. Add \(4: 2 < 4 - x\)
9. Also, from line 2, divide by \(2\)
\text{to get} \(x - 3 < 0\)
10. Add \(-1 < 0\) \text{to get} \(x - 4 < 0\)
11. Then \(1x - 4) = 4 - x\)
12. Combine lines 8 and 11 to \text{get} \(|x - 4| > 2\)
13. Case 1 proven.

14. Case 2: \(2x - 6 \geq 0\)
15. Then \(|2x - 6| = 2x - 6\)
16. Therefore \(2x - 6 > x \quad \text{given}\)
17. Subtract \(x: x - 6 > 0\)
18. Add \(2 > 0\): \(x - 4 > 0\)
19. Then \(|x - 4| = x - 4\)
20. Add 2 to line 17 to \text{get} \(x - 4 > 2\)
21. Combine lines 21 and 19 \text{to get} \(|x - 4| > 2\)
22. Case 2 proven.
23. Combining both cases proves the theorem.
I. Extra Credit: Prove the following theorem. (6 points).

\[ P(A - B) - (P(A) - P(B)) = \emptyset. \] (Hint: If a set \( X \) is a subset of \( A - B \), then \( X = \emptyset \) or \( (X \subseteq A \land X \subseteq B) \))

1. Let \( x \in P(A - B) - (P(A) - P(B)) \)
   \[ \equiv \exists \text{def.} - 3 \]
2. \( x \in P(A - B) \land \neg(x \in P(A) - P(B)) \)
   \[ \equiv \exists \text{def.} - 3 \]
3. \( x \in P(A - B) \land \neg(x \in P(A) \land x \notin P(B)) \)
   \[ \equiv \exists \text{De Morgan} \]
4. \( x \in P(A - B) \land (x \notin P(A) \lor x \in P(B)) \)
   \[ \equiv \exists \text{def. powerset} \]
5. \( x \subseteq A - B \land (x \notin A \lor x \subseteq B) \)

6. Since \( X \subseteq A - B \), either \( X = \emptyset \) or \( (X \subseteq A \land x \notin B) \)

7. If \( x = \emptyset \), the theorem is proven, so consider the other case

8. \( x \subseteq A - B \land (x \notin A \lor x \subseteq B) \)
   \[ \equiv \exists \text{Assumption that } x \subseteq A - B = x \subseteq A \land x \notin B \]
9. \( x \subseteq A \land x \notin B \land (x \notin A \lor x \subseteq B) \)
   \[ \equiv \exists \text{Distribute} \lor \text{over} \land \]
10. \( x \subseteq A \land [(x \notin B \land x \notin A) \lor (x \notin B \land x \subseteq B)] \)
    \[ \equiv \exists \land \text{negation and} \lor \text{identity} \]
11. \( x \subseteq A \land (x \notin B \land x \notin A) \)
    \[ \equiv \exists \land \text{commutativity,} \land \text{associativity} \]
12. \( (x \subseteq A \land x \notin A) \land x \notin B \)
    \[ \equiv \exists \land \text{negation and} \land \text{dominance} \]
13. \( F \)
14. Since this case results in false, it must be that \( x = \emptyset \).
15. Since any \( x \) in \( P(A - B) - (P(A) - P(B)) \) ends up being \( \emptyset \), the set equals \( \exists \emptyset \).