Midterm 2 for CS301K Fall 2013

Answers

A. Provide counterexamples disproving the following propositions. (Explain why they are counterexamples). (5 points each).

1. The $\leq$ relation on $\mathbb{Z}$ is an equivalence relation

   The pair $(1, 2)$ is in the $\leq$ relation because $1 \leq 2$. However, $(2, 1)$ is not in the $\leq$ relation because $2 \not\leq 1$. This means that $\leq$ is not symmetric, and if it is not symmetric, it is also not an equivalence relation.

2. If $A$, $B$, $C$, and $D$ are sets, then $(A \cup C) \times (B \cup D) \subseteq (A \times B) \cup (C \times D)$

   Let $A = \{a\}$, $B = \emptyset$, $C = \emptyset$, $D = \{d\}$. Then $(A \cup C) \times (B \cup D) = (\{a\} \cup \emptyset) \times (\emptyset \cup \{d\}) = \{a\} \times \{d\} = \{(a, d)\}$. But $(A \times B) \cup (C \times D) = (\{a\} \times \emptyset) \cup (\emptyset \times \{d\}) = \emptyset \cup \emptyset = \emptyset$. Because $\{(a, d)\} \subseteq \emptyset$ is false, we have found a counterexample.

B. Enumerate every element of the following sets. (5 points each).

1. $\mathcal{P}(\{a\}) \times \mathcal{P}(\{b\}) = \\
   \{ (\emptyset, \emptyset), (\emptyset, \{b\}), (\{a\}, \emptyset), (\{a\}, \{b\}) \}$

2. Define relation $R$ on $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ as $aRb \iff [a \equiv b \pmod{3}]$.
   Then $[1]_R = \\
   \{ 1, 4, 7 \}$
C. Prove the following theorems. (10 points each).

1. Define the relation $\preceq$ on $\mathbb{R} \times \mathbb{R}$ as: $(a, b) \preceq (x, y) \leftrightarrow a \leq x \land b \leq y$.

   Theorem: $\preceq$ is transitive. (Hint: take the fact that $\leq$ is transitive as given)

   1. Assume $(a, b) \preceq (c, d) \land (c, d) \preceq (e, f)$ for $a, b, c, d, e, f \in \mathbb{R}$
   2. $a \leq c \land b \leq d \{\text{simplification, def. } \preceq: 1\}$
   3. $c \leq e \land d \leq f \{\text{simplification, def. } \preceq: 1\}$
   4. $a \leq e \land b \leq d \leq f \{\text{conjunction: 2,3}\}$
   5. $a \leq e \land b \leq f \{\leq \text{ is transitive}\}$
   6. $(a, b) \preceq (e, f) \{\text{Def. } \preceq\}$
   7. Therefore $(a, b) \preceq (c, d) \land (c, d) \preceq (e, f) \rightarrow (a, b) \preceq (e, f)$
   8. $\preceq$ is transitive $\{\forall \text{ generalization, def. transitive}\}$
2. Theorem: If $A$ and $B$ are sets, then $A \cup (A^C \cap B) = A \cup B$

Proof:
1. Let $x \in A \cup (A^C \cap B)$
2. $\equiv x \in A \lor x \in A^C \cap B$ {Def. $\cup$}
3. $\equiv x \in A \lor (x \in A^C \land x \in B)$ {Def. $\cap$}
4. $\equiv x \in A \lor (x \notin A \land x \in B)$ {Def. complement}
5. $\equiv (x \in A \lor x \notin A) \land (x \in A \lor x \in B)$ {distribute $\lor$ over $\land$}
6. $\equiv T \land (x \in A \lor x \in B)$ {Def. $\lor$ negation}
7. $\equiv x \in A \lor x \in B$ {Def. $\lor$ identity}
8. $\equiv x \in A \cup B$ {Def. $\cup$}
D. Decide whether or not the following proposition is true, and then either prove or disprove it. (10 points).

This problem depends on the definition of relation composition provided in the book (not in class). As a reminder, if \( R \) is a relation from \( A \) to \( B \), and \( S \) is a relation from \( B \) to \( C \), then \( S \circ R \) is a relation from \( A \) to \( C \) where

\[
S \circ R = \{(a, c) \in A \times C | \exists b \in B [(a, b) \in R \land (b, c) \in S]\}.
\]

Claim: Assume \( R \) is an equivalence relation on \( A \), and \( S \) is a partial order on \( A \). Then \( S \circ R \) is symmetric.

False, Counterexample:

Let \( A = \{1, 2\} \). Then \( R = i_A = \{(1, 1), (2, 2)\} \) is an equivalence relation on \( A \). Then define \( S = \leq_A = \{(1, 1), (1, 2), (2, 2)\} \), which is a partial order on \( A \). Then \( S \circ R = \leq_A \circ i_A = \leq_A = \{(1, 1), (1, 2), (2, 2)\} \), which is clearly not symmetric because it contains \((1, 2)\) but lacks \((2, 1)\).