A generalization of the Sheffer Stroke for n-valued logic.

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For binary variables --i.e. ranging over $\{0, 1\}$-- x and y the "Sheffer Stroke" --alias "alternative denial" or "nand"-- is the operator --or binary function-- denoted by

$$x|y$$

and (usually) defined as

$$1 - x.y \quad .$$

As $\quad 1 - x.y = (x.y + 1) \underline{\text{mod}}\ 2$

and $\quad x.y = \min(x, y)$

an alternative definition is:

$$x|y = (\min(x, y) + 1) \underline{\text{mod}}\ 2 \quad .$$

It is well-known [1] that any binary function of k binary arguments can be expressed using the Sheffer Stroke as the only primitive operator.

In the following we regard n-ary variables --i.e. ranging over $\{0, 1, \ldots, n-1\}$-- and shall demonstrate that any n-ary function of k n-ary arguments can be expressed, using as the only operator the generalization

$$x|y = (\min(x, y) + 1) \underline{\text{mod}}\ n \quad .$$

In the following we shall use the notation

$$\text{suc}(x) = (x + 1) \underline{\text{mod}}\ n \quad ,$$

in terms of which we can rewrite:

$$x|y = \text{suc}(\min(x, y)) \quad .$$

Theorem 1. The function suc --as defined above-- is expressible.

Proof. $\quad \text{suc}(x) = x|x$

Theorem 2. The function pred --defined by $\text{pred}(x) = (x - 1) \underline{\text{mod}}\ n$-- is expressible.

Proof. $\quad \text{pred}(x) = \text{suc}^{n-1}(x)$

Theorem 3. The function min is expressible.

Proof. $\min(x, y) = \mathrm{pred}(x | y)$

Note. Because $\min(x, y, z) = \min(\min(x, y), z)$ etc.

also the minimum of more than two arguments is expressible. (End of note.)

Theorem 4. For any n-ary constant c the discriminator function $d_c(x)$, given by

$$\begin{aligned} d_c(x) &= n - 1 && \text{for} \quad x = c \\ &= 0 && \text{for} \quad x \neq c \end{aligned}$$

is expressible.

Proof. Because

$$\begin{aligned} \mathrm{suc}^{n-c}(x) &= 0 && \text{for} \quad x = c \\ &> 0 && \text{for} \quad x \neq c \end{aligned}$$

we have

$$\begin{aligned} \min(1, \mathrm{suc}^{n-c}(x)) &= 0 && \text{for } x = c \\ &= 1 && \text{for } x \neq c \quad , \end{aligned}$$

so that

$$d_c(x) = \mathrm{pred}(\min(1, \mathrm{suc}^{n-c}(x))) \quad .$$

Theorem 5. For any n-ary constant c the weighten discriminator function $w_c(x, y)$, given by

$$\begin{aligned} w_c(x, y) &= y && \text{for } x = c \\ &= n - 1 && \text{for } x \neq c \end{aligned}$$

is expressible.

Proof.

$$\begin{aligned} \min(d_c(x), \mathrm{suc}(y)) &= \mathrm{suc}(y) && \text{for } x = c \\ &= 0 && \text{for } x \neq c \end{aligned}$$

and because $\mathrm{pred}(\mathrm{suc}(y)) = y$, we find

$$w_c(x, y) = \mathrm{pred}(\min(d_c(x), \mathrm{suc}(y))) \quad .$$

Theorem 6. The selector function, defined by

$$\mathrm{sel}(x, y_0, y_1, \ldots , y_{n-1}) = y_i \quad \text{for } x = i$$

is expressible.

Proof.

$$\mathrm{sel}(x, y_0, y_1, \ldots , y_{n-1}) = \min(w_0(x, y_0), w_1(x, y_1), \ldots , w_{n-1}(x, y_{n-1})) \quad .$$

The final induction from 2 to k arguments is straightforward.

[1] Sheffer, H.M., A set of five independent postulates for Boolean algebras, with applications to logical constants. Trans.Amer.Math.Soc. vol 14, pp.481-488