6.5 EXAMPLE: XOR PROBLEM (REVISITED)

To illustrate the procedure for the design of a support vector machine, we revisit the XOR (Exclusive OR) problem discussed in Chapters 4 and 5. Table 6.2 presents a summary of the input vectors and desired responses for the four possible states.

To proceed, let (Cherkassky and Mulier, 1998)

$$K(\mathbf{x}, \mathbf{x}_i) = (1 + \mathbf{x}^T\mathbf{x}_i)^2 \tag{6.43}$$

With $\mathbf{x} = [x_1, x_2]^T$ and $\mathbf{x}_i = [x_{i1}, x_{i2}]^T$, we may thus express the inner-product kernel $K(\mathbf{x}, \mathbf{x}_i)$ in terms of *monomials* of various orders as follows:

$$K(\mathbf{x}, \mathbf{x}_i) = 1 + x_1^2x_{i1}^2 + 2x_1x_2x_{i1}x_{i2} + x_2^2x_{i2}^2 + 2x_1x_{i1} + 2x_2x_{i2}$$

The image of the input vector $\mathbf{x}$ induced in the feature space is therefore deduced to be

$$\boldsymbol{\varphi}(\mathbf{x}) = \left[1, x_1^2, \sqrt{2}x_1x_2, x_2^2, \sqrt{2}x_1, \sqrt{2}x_2\right]^T$$

Similarly,

$$\boldsymbol{\varphi}(\mathbf{x}_i) = \left[1, x_{i1}^2, \sqrt{2}x_{i1}x_{i2}, x_{i2}^2, \sqrt{2}x_{i1}, \sqrt{2}x_{i2}\right]^T, \qquad i = 1, 2, 3, 4$$

From Eq. (6.41), we also find that

$$\mathbf{K} = \begin{bmatrix} 9 & 1 & 1 & 1 \\ 1 & 9 & 1 & 1 \\ 1 & 1 & 9 & 1 \\ 1 & 1 & 1 & 9 \end{bmatrix}$$

The objective function for the dual form is therefore (see Eq. (6.40))

$$Q(\alpha) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - \frac{1}{2}(9\alpha_1^2 - 2\alpha_1\alpha_2 - 2\alpha_1\alpha_3 + 2\alpha_1\alpha_4$$
$$+ 9\alpha_2^2 + 2\alpha_2\alpha_3 - 2\alpha_2\alpha_4 + 9\alpha_3^2 - 2\alpha_3\alpha_4 + 9\alpha_4^2)$$

Optimizing $Q(\alpha)$ with respect to the Lagrange multipliers yields the following set of simultaneous equations:

$$9\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4 = 1$$
$$-\alpha_1 + 9\alpha_2 + \alpha_3 - \alpha_4 = 1$$
$$-\alpha_1 + \alpha_2 + 9\alpha_3 - \alpha_4 = 1$$
$$\alpha_1 - \alpha_2 - \alpha_3 + 9\alpha_4 = 1$$

TABLE 6.2 XOR Problem

Input vector, $\mathbf{x}$	Desired response, d
$(-1, -1)$	-1
$(-1, +1)$	$+1$
$(+1, -1)$	$+1$
$(+1, +1)$	-1

Hence, the optimum values of the Lagrange multipliers are

$$\alpha_{o,1} = \alpha_{o,2} = \alpha_{o,3} = \alpha_{o,4} = \frac{1}{8}$$

This result indicates that in this example all four input vectors $\{\mathbf{x}_i\}_{i=1}^{4}$ are support vectors. The optimum value of $Q(\alpha)$ is

$$Q_o(\alpha) = \frac{1}{4}$$

Correspondingly, we may write

$$\frac{1}{2}\|\mathbf{w}_o\|^2 = \frac{1}{4}$$

or

$$\|\mathbf{w}_o\| = \frac{1}{\sqrt{2}}$$

From Eq. (6.42), we find that the optimum weight vector is

$$\mathbf{w}_o = \frac{1}{8}[-\varphi(\mathbf{x}_1) + \varphi(\mathbf{x}_2) + \varphi(\mathbf{x}_3) - \varphi(\mathbf{x}_4)]$$

$$= \frac{1}{8}\left[-\begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \\ 1 \\ -\sqrt{2} \\ -\sqrt{2} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -\sqrt{2} \\ 1 \\ -\sqrt{2} \\ \sqrt{2} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -\sqrt{2} \\ 1 \\ \sqrt{2} \\ -\sqrt{2} \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \\ 1 \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix}\right]$$

$$= \begin{bmatrix} 0 \\ 0 \\ -1/\sqrt{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The first element of $\mathbf{w}_o$ indicates that the bias b is zero.

The optimal hyperplane is defined by (see Eq. (6.33))

$$\mathbf{w}_o^T\varphi(\mathbf{x}) = 0$$

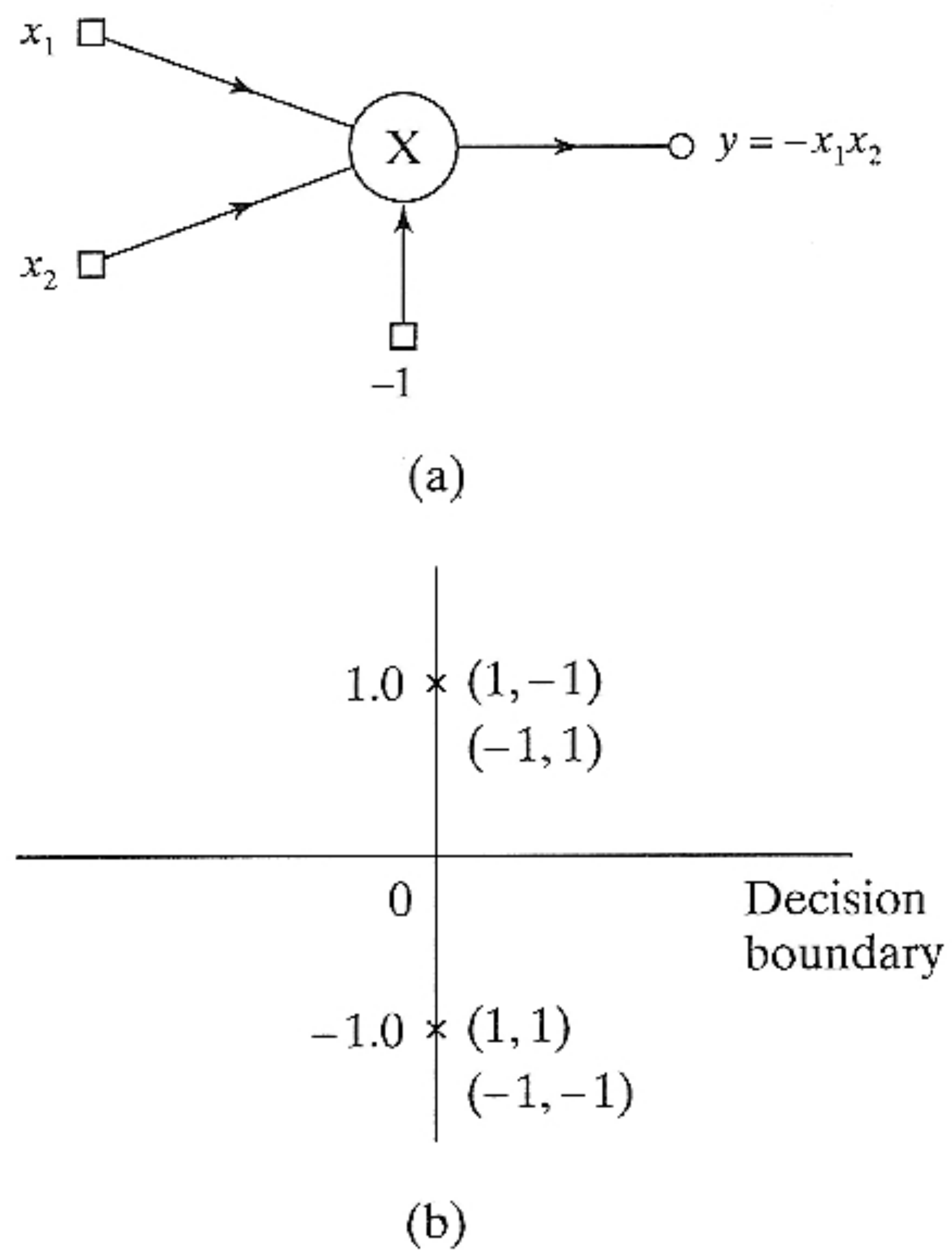

FIGURE 6.6 (a) Polynomial machine for solving the XOR problem. (b) Induced images in the feature space due to the four data points of the XOR problem.

That is,

$$\left[0, 0, \frac{-1}{\sqrt{2}}, 0, 0, 0\right] \begin{bmatrix} 1 \\ x_1^2 \\ \sqrt{2}x_1x_2 \\ x_2^2 \\ \sqrt{2}x_1 \\ \sqrt{2}x_2 \end{bmatrix} = 0$$

which reduces to

$$-x_1x_2 = 0$$

The polynomial form of support vector machine for the XOR problem is as shown in Fig. 6.6a. For both $x_1 = x_2 = -1$ and $x_1 = x_2 = +1$, the output $y = -1$; and for both $x_1 = -1, x_2 = +1$ and $x_1 = +1$ and $x_2 = -1$, we have $y = +1$. Thus the XOR problem is solved as indicated in Fig. 6.6b.

6.6 COMPUTER EXPERIMENT

In this computer experiment we revisit the pattern-classification problem that we studied in Chapters 4 and 5. The experiment involved the classification of two overlapping two-dimensional Gaussian distributions labeled 1 (class $\mathscr{C}_1$) and 2 (Class $\mathscr{C}_2$). The scatter plots for these two sets of data are shown in Fig. 4.14. The probability of correct classification produced by the Bayesian (optimum) classifier is calculated to be

$$p_c = 81.51 \text{ percent}$$