Controller Synthesis using Qualitative Models and Constraints

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Abstract

Many engineering systems require the synthesis of global behaviors in nonlinear dynamical systems. Multiple model approaches to control design make it possible to synthesize robust and optimal versions of such global behaviors. We propose a methodology called *Qualitative Heterogeneous Control* that enables this type of control design. This methodology is based on a separation of concerns between qualitative correctness and quantitative optimization. Qualitative sufficient conditions are derived, that define a space of quantitative control strategies. These sufficient conditions are used in conjunction with a numerical optimization procedure to synthesize nonlinear optimal controllers that are robust in practical implementations. We demonstrate this process of controller synthesis for the global control of an inverted pendulum system.

1. Introduction

One of the key requirements in the design of many complex controlled systems, such as robots, is the achievement of dynamical properties in the face of large, unstructured and *a priori* unknown, disturbances. The dynamical properties of interest may be as simple as stabilization of a single state variable at a fixed point or as complex as synchronized orbits in some subset of the state variables. Our approach to this design task factors the problem into two components. First, we develop a qualitative solution to the problem including weak conditions sufficient to guarantee the desired qualitative properties of the controlled system. Second, we use the remaining degrees of freedom of the design (within the space defined by the weak conditions) to optimize the design quantitatively according to any task-specific objective function. This factoring of qualitative correctness from quantitative optimization can also be applied in a machine learning context, to improve performance in an unknown environment while guaranteeing safe operation (Perkins & Barto, 2002).

In practical problems involving large disturbances and uncertainties, a single control strategy seldom provides the required guarantees across the entire state space of interest. This is especially so in robotics where different regions of the state space may require incompatible forms of dynamics from the same physical system, in which case the control strategy must admit multiple forms. For these types of problems, it is possible to design multiple local control strategies and to design a global strategy as the composition of these local strategies. The problem of characterizing the sufficient conditions becomes a two-part problem of first characterizing the local strategies and then characterizing the interaction of these local models according to the rules of the composition. If the conditions that characterize these local models and composition are sufficiently weak, it leaves the designer with many degrees of freedom to optimize the quantitative aspects of the global strategy.

The literature on qualitative reasoning contains a rich collection of techniques for making these weak characterizations. For our purposes, the appropriate representation is based on the phase space of dynamical system theory. This representation has been explored by many researchers in the context of analysis (Sacks, 1990; Sacks, 1991; Nishida, 1997), simulation (Kuipers, 1994; Sacks, 1990; Yip, 1990) and control (Bradley & Zhao, 1993; Zhao, Loh & May, 1999). We base our work on Qualitative Differential Equations (QDE) (Kuipers, 1994) that represent sets of ordinary differential equations (ODE) and have well defined behaviors as orbits in phase space. This representation allows us to take advantage of many results in control theory and the qualitative theory of differential equations (Guckenheimer & Holmes, 1983; Jordan & Smith, 1999) while also being able to make connections with symbolic reasoning techniques, such as using QSIM (Shults & Kuipers, 1997; Kuipers & Astrom, 1994).

Independent of this connection with qualitative reasoning, multiple model control design has been studied actively in the control systems community. The book by Murray-Smith & Johansen (1997) is a good introduction to this work. A related effort is the study of hybrid systems, i.e., systems involving nontrivial interactions between continuous and logical dynamics. This community is especially interested in powerful methods for abstracting continuous dynamics to symbolic finite state dynamics, to enable formal analysis.

In the context of robotics, there has been recent work involving the synthesis of multiple model controllers. In (Pratt, et. al., 2001), a controller for a bipedal robot is designed as a finite state machine based composition of simple and 'intuitive' local controllers. In (Burridge, et. al., 1999), a multiple model controller is designed as a sequential composition of multiple stabilizing controllers. In (Klavins & Koditschek, 2002), this work is extended to address several types of periodic orbits. In (Frazzoli, et. al., 2003), a hybrid controller for an aerial vehicle is designed as a maneuver automaton based on transitions between multiple stable trajectories.

In prior work (Kuipers & Ramamoorthy, 2002; Ramamoorthy & Kuipers, 2003) we proposed the Qualitative Heterogeneous Control framework that makes it possible to design control systems as a composition of qualitative local models. A key feature of this framework that makes it complementary to prior work such as (Burridge, et. al., 1999; Frazzoli, et. al., 2003) is the fact that we make a connection between classical techniques in control theory and powerful techniques for the analysis and representation of imprecise qualitative knowledge about dynamics. As noted in (McClamroch & Kolmanovsky, 2000), an outstanding research issue in the theory of multiple model control systems is that "*there is little guidance in the published literature on how the family of feedback functions should be selected*" for the local control strategies. This problem is exacerbated by the fact that this choice is intimately related to the tractability of the verification and design of the global strategy. By using an incompletely specified functional form as the basic unit of our design process, we are able to overcome this problem in a principled and intuitive manner. Our framework allows us to define a variety of types of local dynamics such as fixed points, limit cycles and even chaotic orbits using properties of QDEs. Many of our behavior guarantees are based on functionally specified constraints in the abstract QDE that translate to differential-algebraic constraints on concrete functions in the ODE. This enables us to optimize our quantitative controllers by systematically searching a well defined functional space. Also, use of the QDE representation makes it possible for a designer to utilize symbolic techniques for analysis and simulation as a part of the design process. In this sense, we hope to be able to design controllers that share the simplicity and intuitive appeal of Virtual Model Control (Pratt, et. al., 2001) but also entail rigorous robustness and optimality guarantees required in practical applications of autonomous systems.

In this paper, we use a familiar example from the control theory literature, the inverted pendulum, to demonstrate this approach to control design. We begin by using the QHC framework to derive sufficient conditions for the existence of a global swing-up behavior in an inverted pendulum system. Using these sufficient conditions, we demonstrate that we are able to synthesize linear controllers that achieve the same optimal performance as conventional control design techniques, e.g., Linear Quadratic Regulators (LQR) (Stengel, 1994). Techniques such as LQR are applicable only to single (local) controllers and do not address the optimality of a hybrid solution involving multiple local controllers. We demonstrate the use of QHC in the design of an optimal controller that optimizes performance across multiple regions with linear local controllers. This type of optimal control design is a nontrivial problem in control theory, see, e.g., (Barton & Lee, 2002). We are presenting a unique approach to the solution of this problem. We then extend this result by using nonlinear functions, i.e., sigmoids, as local controllers. This allows us to design nonlinear optimal controllers that provide better performance with less control effort. These results should make it evident that the use of qualitative models has resulted in a clear and useful separation between correctness and optimization concerns, thus enabling clear and better performing controller designs.

2. Synthesis of Global Behaviors using Qualitative Models

Qualitative Heterogeneous Control works by defining a hybrid system consisting of a set of qualitatively described control laws, each with its own operating region. Each control law is selected so as to enforce a well defined qualitative behavior in that local region of phase space. These behaviors are well defined in the sense that the controlled dynamical system executes a specific form of an attractor or orbit in phase space. The global behavior of the controller is then understood in terms of attractors and transitions between them.

The QHC design methodology can be summarized as follows.

1. Define a transition graph of local regions (based on topological definition of orbits) that guarantees the desired global behavior. This step defines the topological structure of the controller. For instance, if the desired behavior in phase space is to go from one point to another, one might design a strategy based on making the first point a repellor and the second point an attracting fixed point.

2. For each local region, select a qualitative behavior model that has the desired qualitative behavior over a region including the intended local region. The behavior model is typically a specific type of attractor (e.g., fixed points, limit cycles, chaotic attractors) or type of phase space orbit that can be defined for a nonlinear qualitative differential equation.

3. Select a control policy for each local region that transforms the uncontrolled plant into an instance of the qualitative behavior model over some region containing the intended model.

4. Define region boundaries with simple reliable descriptions (e.g., energy level curves, conservation laws, etc.) such that the local behaviors are guaranteed to cross the region boundaries exactly in the desired ways. These last two steps result in a set of qualitative constraints on the functions appearing in the qualitative differential equations.

5. Using the imprecisely specified functions and qualitative constraints, define quantitative control strategies. In this paper, we approach the process of quantitative design as a parametric optimization problem following the selection of specific functional forms for the control functions. An alternative approach would be to synthesize control functions directly in a nonparametric fashion, as in (Gazi, et. al., 1997; Say, 2000).

This algorithm is non-deterministic in the sense that earlier choices must be made correctly for later choices to be possible, so backtracking may be necessary. However, if the algorithm terminates, the resulting design is guaranteed to have the desired properties. Another aspect of this algorithm is that it represents a sequence of choices made by the control designer, not all of which are explicitly available in traditional techniques. In step 1, the designer makes choices at a very high level by selecting the topology of the orbits. At this level, the problem may be modeled in many different ways. For instance, the task of swinging up a pendulum may be achieved via the use of either a repellor or a chaotic attractor, depending on the goals of the problem. This is an area where the QHC approach differs from many traditional control design techniques, whose emphasis is almost exclusively on stabilization and tracking problems. QHC allows the designer to use any orbit that can be generated via the evolution of a nonlinear differential equation. In steps 2 - 4, since the models are qualitative, the resulting design describes an entire family of control laws, all of which are guaranteed to have the desired properties. In step 5, after the qualitative design is complete, one would need to optimize the model with respect to specific cost criteria so as to obtain an optimal and provably correct controller for the nonlinear, multivariable, hybrid control problem.

3. Some Results Concerning the Qualitative Behavior of Second Order Systems

In this paper, we demonstrate our control design methodology by using the example of a second order system. Before we discuss the detailed design, we state some results that enable it. We consider a second order QDE of the form,

$$\ddot{x} + f(\dot{x}) + g(x) = 0 \tag{1}$$

To make qualitative modeling, simulation and design possible, we restrict our attention to "reasonable" functions, which are continuously differentiable functions with a few additional constraints that make qualitative reasoning possible (Kuipers, 1994; Kuipers & Ramamoorthy, 2002). M_0^+ is the set $f \in M^+$ of monotonic functions such that $f(0) = 0$. The sign function is denoted $[x]_0 = sign(x) \in \{+, 0, -\}$. P_0^+ is defined as the set of reasonable functions $f : [a, b] \rightarrow \Re^*$ such that $[f(x)]_0 = [x]_0$ over (a, b).

The results presented in this section may be proved by hand as in (Kuipers & Ramamoorthy, 2002). Alternatively, these results may be proved in an automated manner by the use of QSIM and a temporal logic model checker, as in (Shults & Kuipers, 1997). While both approaches produce the same result, we anticipate that advances in qualitative reasoning algorithms would enable the derivation of similar results for larger, higher dimensional QDEs, enabling the QHC methodology to be used in a scalable way.

Lemma 1. Qualitative damped spring model. Let S be a dynamical system described by the QDE

$$\ddot{x} + f(\dot{x}) + g(x) = 0 \tag{2}$$

where $f, g \in P_0^+$. Then S has a Lyapunov function

$$V(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + \int_0^x g(x)dx$$

such that $V(0, 0) = 0$, $V > 0$ elsewhere, and $dV/dt < 0$ everywhere except that $dV/dt = 0$ where $\dot{x} = 0$. Therefore, S is asymptotically stable at $(0, 0)$.

Lemma 2. Qualitative anti-damped spring model. Let S be a dynamical system described by the QDE

$$\ddot{x} - f(\dot{x}) + g(x) = 0 \tag{3}$$

where $f, g \in P_0^+$. Then $(0, 0)$ is the only fixed point of S, it is unstable, and there are no limit cycles.

4. Qualitative Control Design for a Pivot-Torque Actuated Pendulum

By appealing to the qualitative properties of solutions to the general models in section 3, we can give a simple and natural derivation for a controller for the pendulum, able to pump it up and stabilize it in the inverted position. The goal is to design a global controller that can respond to arbitrarily large disturbances and still recover to the stable fixed point.

The first step in the design process is the selection of the various orbits that will be used to define the local behaviors. We wish to stabilize the pendulum at the point $\phi = 0$ with no velocity (We use the variable ϕ to refer to the counter-clockwise angular position of the pole, see figure 1). So, the region around the point $(\phi, \dot{\phi}) = (0, 0)$ needs to be a basin of attraction of a fixed point. We define a mode, **Balance**, that has this behavior. We wish to move the pendulum away from the point $\theta = 0$ (θ also refers to angular position, but it takes the value $\theta = 0$ when the pole is hanging straight down, see figure 1). Therefore, the point $(\theta, \dot{\theta}) = (0, 0)$ needs to be made into an unstable (repelling) fixed point. We define a mode, **Pump**, with this behavior. Both these behaviors are defined in the vicinity of the stable and unstable fixed points of the uncontrolled pendulum. We define a third mode, **Spin**, whose qualitative behavior is identical to **Balance** but whose region of applicability covers the remaining phase space not covered by **Balance** and **Pump**. The utility of **Spin** is to accelerate convergence towards the desired fixed point by adding to the natural damping of the system. The three modes are depicted in the discrete transition graph in figure 2.

This transition graph encodes the desired global behavior by requiring the following properties to be true,

1. Any trajectory that enters the **Balance** region stays within that region and terminates at a desired fixed point.

2. Any trajectory beginning inside the region of applicability of **Pump** moves towards and terminates inside the region of applicability of **Balance**.

3. Any trajectory beginning inside the region of applicability of **Spin** moves towards and terminates inside the region of applicability of **Balance**.

4. No trajectory begins inside the region of applicability of **Spin** to enter and remain inside region of applicability of **Pump**. Correspondingly, no trajectory begins inside the region of applicability of **Pump** to enter and remain inside the region of applicability of **Spin**. What this implies is that trajectories beginning inside these regions must either terminate inside **Balance** or move along a one dimensional manifold that defines the boundaries of these two regions. We will explain this aspect of the behavior in more detail in section 4.4.

In the following sections, we demonstrate by hand that these requirements are satisfied. It is also possible to express these statements in a logic such as CTL*, enabling automatic verification using a temporal logic model checker (Shults & Kuipers, 1997).

4.1 Stabilization in the Balance Region

The dynamics of the pendulum are derived from three main terms. The angular acceleration due to gravity is $k \sin\phi$ (k is used here to collect multiple terms that appear in the detailed physical model). There is a small amount of damping friction $f(\dot\phi)$, where $f \in M_0^+$. A control action $u(\phi, \dot\phi)$ exerts angular acceleration at the pivot. The resulting model of the pendulum is:

$$\ddot\phi + f(\dot\phi) - k\sin\phi + u(\phi, \dot\phi) = 0 \tag{4}$$

Our goal is to design $u(\phi, \dot\phi)$ so that the system is asymptotically stable at $(\phi, \dot\phi) = (0, 0)$.

In this section, we consider only $\phi \in (-\pi/2, \pi/2)$ [1].

Lemma 1 provides a simple sufficient condition: make the pendulum behave like a monotonic damped spring. We define the controller for the **Balance** region to be:

$$u(\phi, \dot\phi) = g_b(\phi) \tag{5}$$

such that $[g_b(\phi) - k\sin\phi]_0 = [\phi]_0$.

Since $k\sin\phi$ increases monotonically with ϕ over $(-\pi/2, \pi/2)$, $g_b(\phi)$ must increase at least as fast in order to ensure that $[g_b(\phi) - k\sin\phi]_0 = [\phi]_0$.

We can get faster convergence by augmenting the natural damping $f(\dot\phi)$ with a damping term $h_b(\dot\phi)$ included in the control law, giving us

$$u(\phi, \dot\phi) = g_b(\phi) + h_b(\dot\phi) \tag{6}$$

where $[g_b(\phi) - k\sin\phi]_0 = [\phi]_0$; $[h_b(\dot\phi)]_0 = [\phi]_0$.

If there is a bound $u_{\max}$ on the control action u, then the limiting angle $\phi_{\max}$ beyond which the controller cannot restore the pendulum to $\phi = 0$ is given by the constraint,

$$u_{\max} = k\sin\phi_{\max} \tag{7}$$

The maximum velocity $\dot\phi_{\max}$ that the **Balance** controller can tolerate at $\phi = 0$ is then determined by the constraint

$$\frac{1}{2}\dot\phi_{\max}^2 = \int_0^{\phi_{\max}} g_b(\varphi) - k\sin\varphi d\varphi \tag{8}$$

which represents the conversion of the kinetic energy of the system (4) at $(0, \dot\phi_{\max})$ into potential energy at $(\phi_{\max}, 0)$.

We would like to define the boundary of the **Balance** region as a level curve of the Lyapunov function for the controlled system (4), from Lemma 1.

$$V(\phi, \dot\phi) = \frac{1}{2}\dot\phi^2 + \int_0^{\phi} g_b(\varphi) - k\sin\varphi d\varphi \tag{9}$$

It is easy to check that $V(\phi_{\max}, 0) = V(0, \dot\phi_{\max})$, so these intercepts lie on the same level curve of V. When a trajectory intersects this level curve, $V(\phi, \dot\phi) = \frac{1}{2}\dot\phi_{\max}^2$. Expanding and dividing by $\frac{1}{2}\dot\phi_{\max}^2$, we get

$$\frac{\dot\phi^2}{\dot\phi_{\max}^2} + \frac{\int_0^{\phi} g_b(\varphi) - k\sin\varphi d\varphi}{\frac{1}{2}\dot\phi_{\max}^2} = 1 \tag{10}$$

Substituting the definition of $\dot\phi_{\max}^2$ (8) into the second term gives,

$$\frac{\dot\phi^2}{\dot\phi_{\max}^2} + \frac{\int_0^{\phi} g_b(\varphi) - k\sin\varphi d\varphi}{\int_0^{\phi_{\max}} g_b(\varphi) - k\sin\varphi d\varphi} = 1 \tag{11}$$

Because $[g_b(\phi) - k\sin\phi]_0 = [\phi]_0$, we know that both the integrals are non-negative, so equation (11) defines an "ellipse-like" curve that intersects the axes at $\pm\phi_{\max}$& $\pm\dot\phi_{\max}$. Furthermore, the curve changes monotonically between the intercepts.

In the special case where $g_b(\phi) - k\sin\phi \cong \phi$, we can evaluate the integrals and show that the level curve of V is an ellipse:

$$\frac{\dot\phi^2}{\dot\phi_{\max}^2} + \frac{\phi^2}{\phi_{\max}^2} = 1 \tag{12}$$

Note that the shapes of the non-linear functions g_b and h_b are only very weakly constrained. The qualitative constraints in (6) provide weak sufficient conditions guaranteeing the stability of the inverted pendulum controller. However, there is plenty of freedom available to the designer to select the properties of g_b and h_b to optimize any desired criterion.

[1] The derivation here applies over the larger interval $(-\pi, \pi)$, but the maximum control force is required at $\phi = \pm\pi/2$. The controller design problem is less interesting if the controller is powerful enough to lift the pendulum directly to $\phi = 0$ from any value of ϕ.

4.2 Pumping up the Hanging Pendulum

With no input, the stable state of the pendulum is hanging straight down. We use the variable θ to measure the angular position counter-clockwise from straight down (figure 1). The goal is to pump energy into the pendulum, swinging it progressively higher, until it reaches the region where the inverted pendulum controller can balance it in the upright position.

Angular acceleration due to gravity is $-k\sin\theta$. As before, damping friction is $-f(\dot\theta)$, where $f \in M_0^+$, and the control action exerts an angular acceleration $u(\theta,\dot\theta)$ at the pivot. The resulting model of our system is:

$$\ddot\theta + f(\dot\theta) + k\sin\theta + u(\theta,\dot\theta) = 0 \tag{13}$$

Without control action, since $[\sin\theta]_0 = [\theta]_0$ over $-\pi < \theta < \pi$, the model exactly matches the monotonic damped spring model of Lemma 1, so we know that it is asymptotically stable at $(\theta,\dot\theta) = (0,0)$. Unfortunately, this is not where we want it.

Fortunately, Lemma 2 gives us a sufficient condition to transform the stable attractor at (0,0) into an unstable repellor. We define the controller for the **Pump** region so that the system is modeled by a spring with negative damping, pumping energy into the system. That is, define

$$u(\theta,\dot\theta) = -h_p(\dot\theta) \tag{14}$$

such that $h_p - f \in P_0^+$

Starting with any perturbation from $(0,0)$, this controller will pump the pendulum to higher and higher swings. Lemma 2 is sufficient to assure us that there are no limit cycles in the region $-\pi < \theta < \pi$ to prevent the trajectory from approaching $\theta = \pi$ so the **Balance** control law can stabilize it in the inverted position.

4.3 The Spinning Pendulum

The **Spin** region represents the behavioral mode of the pendulum when it is spinning freely at high speed. In the **Spin** region, a simple qualitative controller augments the natural friction of the system with additional damping, to slow the system down toward the two other regions.

$$u(\theta,\dot\theta) = h_s(\dot\theta) \tag{15}$$

such that $h_s \in P_0^+$.

4.4 Bounding the Pump and Spin Regions

One of our specifications is to ensure that no trajectories oscillate between **Pump** and **Spin** or terminate inside these regions without reaching **Balance**. We ensure this by defining a suitable boundary between the **Pump** and **Spin** regions, and showing that the **Pump** and **Spin** controllers together define a sliding mode controller (Slotine & Li, 1991), forcing nearby trajectories to converge to the boundary.

A boundary with the desired properties is the separatrix of the same pendulum,

$$\ddot\theta + k\sin\theta = 0 \tag{16}$$

without damping friction or control action. It turns out that this boundary will lead straight into the heart of the **Balance** region.

A *separatrix* is a trajectory that starts at an unstable fixed-point of the system and ends at another fixed-point. In the case of the pendulum, the separatrices are the trajectories where the pendulum starts upright and at rest, then swings around once and returns to the upright position, at rest. It is the locus of points $(\theta,\dot\theta)$ such that the total energy of the system is exactly equal to the potential energy of the motionless pendulum in the upright position.

$$KE + PE = \frac{1}{2}\dot\theta^2 + \int_0^\theta k\sin\varphi\, d\varphi = 2k \tag{17}$$

Evaluating the integral and simplifying, we get an equation $s(\theta,\dot\theta) = 0$ that defines the separatrix, i.e., the boundary between **Spin** ($s > 0$) and **Pump** ($s < 0$).

$$s(\theta,\dot\theta) = \frac{1}{2}\dot\theta^2 - k(1+\cos\theta) = 0 \tag{18}$$

We use the method for defining a sliding mode controller from (Slotine & Li, 1991) to ensure that trajectories always approach $s = 0$.

Differentiating (18) and substituting for $\ddot\theta$, we get:

$$\dot s = \dot\theta\ddot\theta + k\sin\theta\dot\theta$$

$$\dot s = \dot\theta(-f(\dot\theta) - k\sin\theta - u(\theta,\dot\theta)) + k\sin\theta\dot\theta$$

$$\dot s = -\dot\theta f(\dot\theta) - \dot\theta u(\theta,\dot\theta) \tag{19}$$

Now, examine the **Pump** region, inside the separatrix where $s < 0$, and substitute the **Pump** control law (14) for $u(\theta,\dot\theta)$. $\dot s_{pump} = -\dot\theta f(\dot\theta) + \dot\theta h_p(\dot\theta)$ where $h_p - f \in P_0^+$

$$\dot s_{pump} = \dot\theta(h_p - f)(\dot\theta) \geq 0 \tag{20}$$

Similarly, for the **Spin** region where $s > 0$, substituting its control law (15). $\dot s_{spin} = -\dot\theta f(\dot\theta) - \dot\theta h_s(\dot\theta)$ where $h_s \in P_0^+$

$$\dot s_{spin} = -\dot\theta(f + h_s)(\dot\theta) \leq 0 \tag{21}$$

This shows that the **Pump** control law moves the system toward the separatrix from the inside, and the **Spin** control law approaches the separatrix from the outside: the existing control laws define a sliding mode controller with the separatrix $s = 0$ as the attractor (a one-dimensional manifold). Once the system gets sufficiently close to the boundary, it will follow the separatrix, directly into the **Balance** region. In particular, it is impossible for an aggressive **Pump** controller to overshoot the **Balance** region.

4.5 The Heterogeneous Control Strategy

In the foregoing discussion, we have derived qualitative constraints on local control laws along with their regions of applicability. The global controller is defined in its entirety by these constraints and the region of applicability of each mode, i.e., any quantitative controller that satisfies all these requirements is guaranteed to possess the desired qualitative global behavior. These qualitative constraints and the switching rules are summarized in figure 3.

5. Quantitative Optimization of Controllers

In section 4, we derived a set of sign equality and monotonicity constraints that guarantee the correctness of a qualitatively defined control strategy for the global control of the inverted pendulum system. There are numerous concrete functions that could satisfy these qualitative constraints. Correspondingly, given a concrete instance of a pendulum, each of these functions would result in varying levels of performance. In this section we demonstrate this process of quantitative design by considering specific examples of concrete functional forms. We select parameterized functions that satisfy the qualitative constraints and then optimize the parameters according to a cost function. This procedure is fairly general and many other nonlinear functions, including architectures such as neural networks, could be designed in this way.

5.1 Optimization of a Single Linear Controller

When we talk about optimality and performance of a control system, there are a wide variety of concerns that could be addressed. Performance specifications could range from simple parameterized measurements such as rise time and settling time to more complex definitions such as norms of a frequency domain description of a linearized system. In optimal control theory, the emphasis is on minimizing some cost function by appropriate selection of a control strategy. For the purpose of this paper, this is the measure of performance we will use. Specifically, we will define performance as a time integral of the control effort and state variables, with the goal of taking the state variables to the desired values as quickly as possible using as little control effort as possible.

In order to understand the basic optimal control problem in a concrete setting, we consider how a controller is designed within the optimal control theory framework. We design a Linear Quadratic Regulator (LQR) which is an optimal linear controller for a linear time invariant plant (Stengel, 1994).

LQR design begins with a linear time invariant plant in the state space form,

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{22}$$

where x is a vector of states, u is a vector of control inputs and y is the vector of outputs. The goal of LQR design is to find a control function $u(t)$ that minimize a cost function of the form,

$$J = x^T(T_f)Q_{T_f}x(T_f) \int_0^{T_f} (x^T(t)Qx(t) + u^T(t)Ru(t))dt \tag{23}$$

where Q_{T_f} and Q are symmetric positive semidefinite matrices and R is a symmetric positive definite matrix. It can be shown that the optimal control action that minimizes this cost is,

$$u = -R^{-1}B^T K(t)x \tag{24}$$

where $K(t)$ must satisfy a continuous time matrix differential equation (the Riccati equation),

$$\dot{K}(t) = -K(t)A - A^T K(t) + K(t)BR^{-1}B^T K(t) - Q$$

$$K(T_f) = Q_{T_f} \tag{25}$$

In practice, one designs a steady state controller with $T_f = \infty$ in which case one calculates a gain K by taking the steady state solution of the Riccati equation. When the necessary assumptions on the matrices are satisfied, this equation can be solved numerically. Many computer aided control design packages provide canned routines for performing this computation.

Consider a concrete version of the pendulum equation,

$$\ddot{\phi} + c\dot{\phi} - k\sin\phi + u(\phi, \dot{\phi}) = 0$$

This equation can be linearized near the origin to yield,

$$\begin{bmatrix} \ddot{\phi} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} -c & k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \phi \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \tag{26}$$

Further, we define the cost matrices,

$$Q = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}, [R] = [1] \tag{27}$$

This allows us to compute the gains for an optimal controller that will stabilize the pendulum at the origin, minimizing control effort and state deviation. For this numerical computation, we utilized the LQR function from the LabVIEW Control Design Toolkit (National Instruments, 2004). The resulting control function is,

$$u = \begin{bmatrix} 14.07 & 100.02 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \phi \end{bmatrix}$$

We note that, in essence, we have designed a linear control function that can optimally stabilize a linearized version of the plant. We then ask the question: Is it possible to arrive at the same result starting from the qualitative constraints derived in section 3?

The qualitative controller, in the Balance region, is defined by the equation,

$$u(\phi, \dot{\phi}) = g_b(\phi) + h_b(\dot{\phi})$$

$$[g_b(\phi) - k\sin\phi]_0 = [\phi]_0$$

$$[h_b(\dot{\phi})]_0 = [\dot{\phi}]_0$$

Consider a linear control function,

$$u(\phi, \dot{\phi}) = (c_{11} + k)\phi + c_{12}\dot{\phi} \tag{28}$$

These qualitative constraints imply that this function can be a **Balance** controller if $c_{11}, c_{12} > 0$. What is required is an optimization procedure that can determine the optimal values of these parameters.

We perform this optimization numerically. The basic operation in the optimization procedure is the evaluation of the cost function. We evaluate this cost by performing a dynamic simulation of the controlled system over a finite time horizon, for a given set of control parameters, and using the resultant trajectory to compute cost. The dynamic simulation is implemented as a 4^{th} order Runge-Kutta numerical integration scheme that is coded using the LabVIEW Simulation Module (National Instruments, 2004). The time step for integration is set at 1ms and the horizon is set at 10 seconds. In order to enable comparisons with LQR, we use the cost function J from (23, 27).

The parameters are selected by a nonlinear programming procedure that is implemented using the *Optimize* function in the XMath Optimization Module (National Instruments, 2004). This function solves the optimization problem,

$$\begin{array}{l} \min F(p) \\ G(p) = 0 \\ h_l \leq H(p) \leq h_u \\ p_l \leq p \leq p_u \end{array} \tag{29}$$

where $F(p)$ is the objective function, $G(p)$ is an equality constraint, $H(p)$ defines an inequality constraint and p is bounded within a parameter range. F, G, H represent nonlinear functions that are representable using the LabVIEW Simulation Module.

The procedure for computation of the optimum is summarized below:

1. Construct a linearly constrained optimization problem with an augmented Lagrangian objective function.
2. Use an iterative algorithm to compute the linearized constraints.
3. Solve the resulting problem using a sequential quadratic programming algorithm based on Broyden - Fletcher - Goldfarb - Shanno weight updates.
4. Use multiple restarts to escape local optima.

For the purposes of our current problem, $F(p) = J$; $p = [\ c_{11} \quad c_{12}\ ]$. Using this procedure, we obtain the control function,

$$u = \left[\ 13.1427 \quad 95.9079\ \right] \left[\begin{array}{c} \dot{\phi} \\ \phi \end{array} \right]$$

We simulated the action of both the LQR and the QHC versions of the **Balance** controller, using the same initial conditions and constraints. The results are shown in figure 4. The LQR controller achieves a cost of J = 39.5602 with a maximum control effort of 23.5215 while the QHC based controller achieves the cost J = 39.5983 with a maximum control effort of 22.4673. The difference between the two sets of results is sufficiently small for us to conclude that we have been able to reproduce the LQR optimum using our qualitative control methodology.

5.2 Optimization of Global Behavior: Multiple Linear Controllers

Our original goal was to design a controller that is capable of achieving a global behavior specification, that the pendulum should be stabilized from all points in the state space. The LQR controller is only valid in a small region of state space defined by the local linearization. The QHC based **Balance** controller is also applicable only in a limited region, constrained by the strength of the actuator. We seek a controller that can deliver the global behavior, including optimal performance, over the entire state space. This is a nontrivial task when solved by many traditional methods, see e.g., (Barton & Lee, 2002).

We now demonstrate our approach to the synthesis of such an optimal global controller. We begin with the linear **Balance** controller,

$$u(\phi, \dot{\phi}) = (c_{11} + k)\phi + c_{12}\dot{\phi} \tag{30}$$

As derived in section 4, the extent of **Balance** is defined as $\phi_{\max} = u_{\max}/k$. This relationship is derived from energy considerations as the maximum angle from which a finite strength actuator could counteract gravitational force and stabilize the pendulum. In practice, in order to avoid saturation of the actuator, one might also want to determine a point where the maximum commanded control action is within this limit. For the linear **Balance** controller, this point is $\phi_{\max} = u_{\max}/(c_{11} + k)$. Equation 8 is then used to determine $\dot{\phi}_{\max}$.

In addition, we need **Pump** and **Spin** controllers. The qualitative constraints for these functions are,

$$[(h_p - f)(\dot{\phi})]_0 = [\dot{\phi}]_0$$

$$[h_s(\dot{\phi})]_0 = [\dot{\phi}]_0$$

We could implement these with the concrete functions,

$$u(\dot{\phi}) = (c + c_2)(\dot{\phi}) \tag{31}$$

$$u(\dot{\phi}) = (c_3)(\dot{\phi}) \tag{32}$$

where $c_2 > 0, c_3 > 0$.

If we implement these three functions in the multiple model strategy outlined in section 4, we are again able to use the procedure to numerically solve a differential equation, compute the cost and to minimize this cost. Our experiments yielded the optimal parameters,

$$\begin{array}{l} c_{11} = 5.66897, c_{12} = 7.01115 \\ c_2 = 1.87099, c_3 = 0.886154 \end{array}$$

This results in a cost of J=513.743 with a maximum control effort of 23.3069. The results of a simulation using this parameter set are shown in figure 5. What we have been able to demonstrate here is that we can begin with a set of qualitative constraints that guarantee correctness of the global controller and use it to define a parameter optimization problem to design a quantitative controller to minimize a specified cost function.

5.3 Optimization of Global Behavior: Multiple Nonlinear Controllers

Having designed a linear multiple model controller to optimize a cost function, we ask whether this is the most optimal performance we might be able to obtain. The answer is no. This optimality result assumes a linear structure for the control functions. Our original qualitative constraints did not require this. So, there may be room for better functional forms that deliver better performance.

To explore this idea, we investigate the use of sigmoids. The sigmoid function is defined as,

$$\sigma(a, m, \phi) = a\left(\frac{1 - e^{-m\phi}}{1 + e^{-m\phi}}\right) \tag{33}$$

The sigmoids based controller is defined as,

$$u(\phi, \dot{\phi}) = \begin{cases} k\phi + \sigma(a_{bp}, m_{bp}, \phi) + \sigma(a_{bv}, m_{bv}, \dot{\phi}), \textbf{Balance} \\ -c\dot{\phi} - \sigma(a_p, m_p, \dot{\phi}), \textbf{Pump} \\ \sigma(a_s, m_s, \dot{\phi}), \textbf{Spin} \end{cases} \tag{34}$$

As we did in section 5.2, we define the boundary of **Balance** by solving the following equation for $\phi_{\max}$,

$$u_{\max} = k\phi_{\max} + \sigma(a_{bp}, m_{bp}, \phi_{\max}) \tag{35}$$

Once we have this value of $\phi_{\max}$, we determine the maximum allowable velocity in the **Balance** region as,

$$\frac{1}{2}\dot{\phi}^2_{\max} = \int_0^{\phi_{\max}} \sigma(a_{bp}, m_{bp}, \phi) + k\phi - k\sin\phi d\phi \tag{36}$$

This gives the desired values to implement the QHC control strategy. The result of the parameter optimization for this structure is,

$$\begin{aligned} a_{bp} &= 11.6213, m_{bp} = 11.5874 \\ a_{bv} &= 11.5401, m_{bv} = 6.17449 \\ a_p &= 12.1986, m_p = 0.75895 \\ a_s &= 2.81266, m_s = 1.64864 \end{aligned}$$

This yields a cost of J = 480.572 and a maximum control effort of 12.424. This shows that we are able to achieve a lower cost than the linear control strategy (J = 513.743) with a maximum control effort that is half that for the linear strategy (23.3069). The results of the dynamic simulation with this parameter set are shown in figure 6.

5.4 Robustness of global behaviors obtained using QHC

One of the primary objectives of the QHC methodology is to enable the design of robust control strategies for real world problems in domains such as robotics. We have demonstrated how this methodology can be used to ensure coverage of the entire operational state space of the system so as to assign suitable strategies to each point in this space. This makes the global behavior robust and makes it possible to derive guarantees regarding this behavior. What happens to these guarantees when the measurements are noisy and the implementation is based on discrete time control updates? It turns out that due to the inherently physical nature of the models and constraints, the controller designed using the QHC methodology is tolerant towards such disturbances. We present results from an experiment to demonstrate this aspect of the controller.

In order to understand the effect of discrete implementation, we implemented a simulation at a step size of 0.05 seconds, which is coarser than the time step used in the parameter optimization experiments. In addition, we simulated stochasticity in the period of operation of the control loop by defining the effective control action as follows,

$$u = \begin{cases} u_{QHC}; r > 0.5 \\ 0; r \leq 0.5 \end{cases} \tag{37}$$

where r is a uniform random variable $r \in [0, 1]$ and u_{QHC} is the control action as defined in figure 3 and section 5.3.

The result of this simulation is shown in figure 7. Note that the topology of the desired global behavior is preserved although the actual trajectory looks slightly different. We also found that the addition of sensor noise (implemented in our simulation as Gaussian white noise with amplitude ±12.5% of signal amplitude) did not have any appreciable detrimental effect on the controller operation.

6. Conclusions

QHC is a methodology for hybrid nonlinear control design. This methodology enables a control designer to systematically derive constraints based on increasingly more concrete specifications. At the highest level, the specifications take the form of a transition graph between orbits that are defined by their topological properties in phase space. The topological properties are used to derive functional constraints using a QDE representation, based on insights from the qualitative theory of differential equations. This imprecise representation of the controller is sufficient to ensure correctness of the desired behaviors. Once we have proved correctness, the designer has many degrees of freedom to design quantitative controllers that optimize various performance metrics. In this way, we have demonstrated the use of the step by step methodology outlined in section 2. Using this methodology, we have demonstrated that we are able to match the performance of traditional methods like linear quadratic regulators. We extend this result by optimizing a hybrid controller consisting of multiple linear controllers. Further, we demonstrate that the same set of qualitative constraints allow us to synthesize a nonlinear optimal controller with better global performance than the linear optimal controller. Lastly, we demonstrate that the resulting optimal controllers are robust in the face of noise and stochasticity in sampling time. We have implemented a controller designed using this methodology on physical apparatus in our laboratory. This has also been achieved at the Technische Universitat Graz in Austria, by Florian & Hofbaur (2004).

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References

Barton, P.I., & Lee, C.K. (2002). Modeling, simulation, sensitivity analysis, and optimization of hybrid systems. *ACM Transactions on Modeling and Computer Simulation*, *12*(4), (pp. 256–289).

Bradley, E., & Zhao, F. (1993). Phase-space control system design. *IEEE Control Systems Magazine*, *13*(2), (pp. 39–47).

Burridge, R.R., Rizzi, A.A., & Koditschek, D.E. (1999). Sequential composition of dynamically dexterous robot behaviors. *International Journal of Robotics Research*, *18*, (pp. 534–555).

Florian, K., & Hofbaur, M. (2004). Personal Communication.

Frazzoli, E., Dahleh, M.A., & Feron, E. (2003). A maneuver based hybrid control architecture for autonomous vehicle motion planning. In G. Balas & T. Samad (Eds.), *Software Enabled Control: Information Technology for Dynamical Systems*. Wiley – IEEE Press.

Gazi, E., Seider, W.D., & Ungar, L. H. (1997). A nonparametric Monte-Carlo technique for controller verification. *Automatica*. *33*(5), (pp. 901–906).

Guckenheimer, J., & Holmes, P. (1983). *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Berlin: Springer-Verlag.

Jordan, D.W., & Smith, P. (1999). *Nonlinear Ordinary Differential Equations. An introduction to dynamical systems*. Third edition. Oxford: Oxford University Press.

Klavins, E., & Koditschek, D.E. (2002). Phase regulation of decentralized cyclic robotic systems. *International Journal of Robotics Research*. *21*(3), (pp. 257–275).

Kuipers, B.J. (1994). *Qualitative Reasoning: modeling and simulation with incomplete knowledge*. Cambridge, Massachusetts: MIT Press.

Kuipers, B.J., & Astrom, K. (1994). The composition and validation of heterogeneous control laws. *Automatica*, *30*, (pp, 233–249).

Kuipers, B.J., & Ramamoorthy, S. (2002). Qualitative modeling and heterogeneous control of global system behavior. In C. J. Tomlin and M. R. Greenstreet (Eds.), *Hybrid Systems: Computation and Control*, Lecture Notes in Computer Science 2289. Springer Verlag.

McClamroch, N.H., & Kolmanovsky, I. (2000). Performance benefits of hybrid controller design for linear and nonlinear systems. *Proceedings of the IEEE*. *88*, (pp. 1083–1096).

Murray-Smith, R., & Johansen, T.A. (Eds.). (1997). *Multiple Model Approaches to Nonlinear Modeling and Control*. London, U.K.: Taylor & Francis.

National Instruments Corp. (2004). LabVIEW Control Design Toolkit User Manual.

National Instruments Corp. (2004). LabVIEW Simulation Module User Manual.

National Instruments Corp. (2004). XMath Optimization Module User Manual.

Nishida, T. (1997). Grammatical description of behaviors of ordinary differential equations in two-dimensional phase space. *Artificial Intelligence*, *91*(1), (pp. 3–32).

Perkins, T.J., & Barto, A.G. (2002). Lyapunov design for safe reinforcement learning. *Journal of Machine Learning Research*, *3*, (pp. 803–832).

Pratt, J., Chew, C-M., Torres, A., Dilworth, P., & Pratt, G. (2001). Virtual Model Control: An intuitive approach for bipedal locomotion. *International Journal of Robotics Research*, *20*(2).

Ramamoorthy, S., & Kuipers, B.J. (2003). Qualitative heterogeneous control of higher order systems. In O. Maler and A. Pnueli (Eds.), *Hybrid Systems: Computation and Control*, Lecture Notes in Computer Science 2623.Springer Verlag.

Sacks, E.P. (1990). Automatic qualitative analysis of ordinary differential equations using piecewise linear approximations. *Artificial Intelligence*, *41*(3), (pp. 313–364).

Sacks, E.P. (1990). A dynamic systems perspective on qualitative simulation. *Artificial Intelligence*, *42*(2-3), (pp. 159–188).

Sacks, E.P. (1991). Automatic analysis of one-parameter planar ordinary differential equations by intelligent numeric simulation. *Artificial Intelligence*, *48*(1), (pp. 27–56).

Say, A.C. (2000). A new technique for Monte Carlo generation of monotonic functions. *Proc. Fourteenth Intl. Workshop on Qualitative Reasoning*. Morelia, Mexico.

Shults, B., & Kuipers, B.J. (1997). Proving properties of continuous systems: qualitative simulation and temporal logic. *Artificial Intelligence*, *92*, (pp. 91–129).

Slotine, J.J.E., & Li, W. (1991). *Applied Nonlinear Control*. Englewood Cliffs, NJ: Prentice Hall.

Stengel, R.F. (1994). *Optimal Control and Estimation*. Dover Publications.

Yip, K.M. (1990). Generating global behaviors using deep knowledge of local dynamics. In D. S. Weld and J. deKleer (Eds.), *Qualitative Reasoning about Physical Systems*. Palo Alto, CA: Morgan Kaufmann.

Zhao, F., Loh, S.C. , & May, J.A. (1999). Phase-space nonlinear control toolbox: The maglev experience. In P.J. Antsaklis, W. Kohn, M. Lemmon, A. Nerode and S. Sastry (Eds.), *Hybrid Systems V*, Lecture Notes in Computer Science 1567. Springer Verlag.

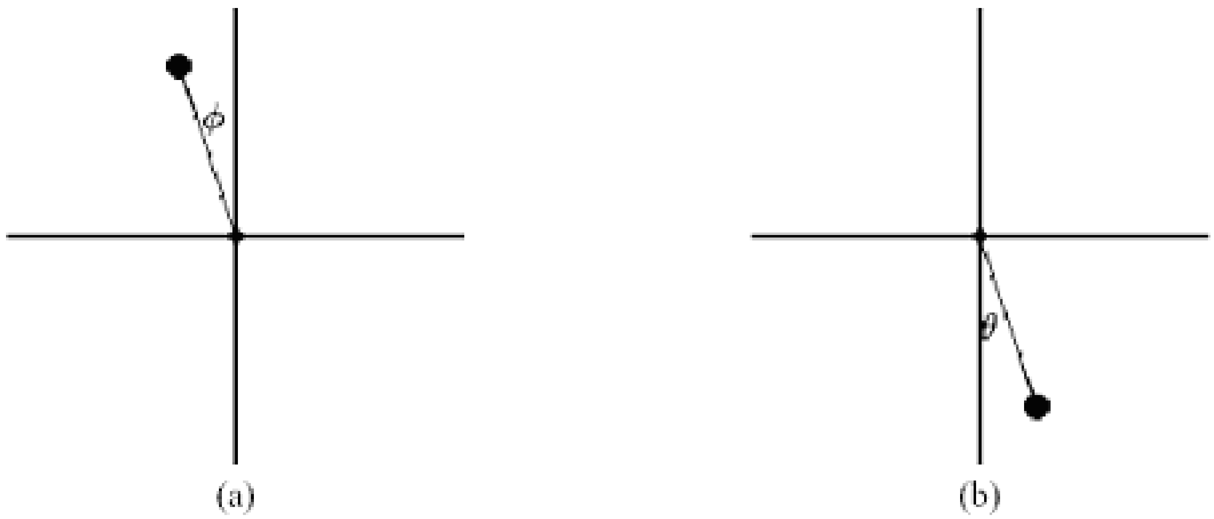

Figure 1: Local models of the pendulum: (a) $\phi = 0$ at the unstable and (b) $\theta = 0$ at the stable fixed-point.

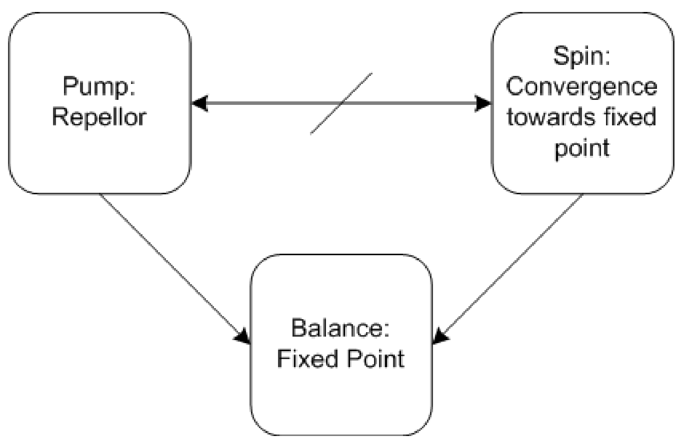

Figure 2: A discrete transition graph that encodes the structure of the desired global behavior.

Given a model,

$$\ddot{\phi} + f(\dot{\phi}) - k\sin\phi + u(\phi,\dot{\phi}) = 0,$$

we apply one of the following control laws,

Balance: $u(\phi,\dot{\phi}) = g_b(\phi) + h_b(\dot{\phi})$, such that $\left[g_b(\phi) - k\sin\phi\right]_0 = \left[\phi\right]_0, [h_b(\dot{\phi})]_0 = [\dot{\phi}]_0$

Pump: $u(\phi,\phi) = -h_p(\dot{\phi})$, such that $[(h_p - f)(\dot{\phi})]_0 = [\dot{\phi}]_0$

Spin: $u(\phi,\dot{\phi}) = h_s(\dot{\phi})$, such that $[h_s(\dot{\phi})]_0 = [\dot{\phi}]_0$

The selection of the control region depends on the values of two parameters:

$$\alpha = \frac{\dot{\phi}^2}{\dot{\phi}^2_{\max}} + \frac{\int_0^{\phi} g_b(\phi) - k\sin\phi d\phi}{\int_0^{\phi_{\max}} g_b(\phi) - k\sin\phi d\phi} \text{ and } s(\phi,\dot{\phi}) = \tfrac{1}{2}\dot{\phi}^2 - k(1-\cos\phi).$$

$\alpha \leq 1$ describes the region of applicability of the **Balance** controller based on physical limitations ($\phi_{\max}$ and $\dot{\phi}_{\max}$) and the requirement that the system should not exit the **Balance** region due to control actions, once it has entered it. $s(\phi,\dot{\phi})$ represents the energy of the system, with the *separatrix* of the pendulum (the locus of points where $s(\phi,\dot{\phi}) = 0$) serving as the boundary between the **Pump** and **Spin** regions. The rule for selecting control mode is:

If $\alpha \leq 1$ then **Balance**
else if $s < 0$ then **Pump**
else **Spin**

Figure 3: A summary of the qualitative control laws, qualitative constraints and regions of applicability. Any quantitative controller that has this structure and that satisfies these constraints is guaranteed to possess the desired global behavior.

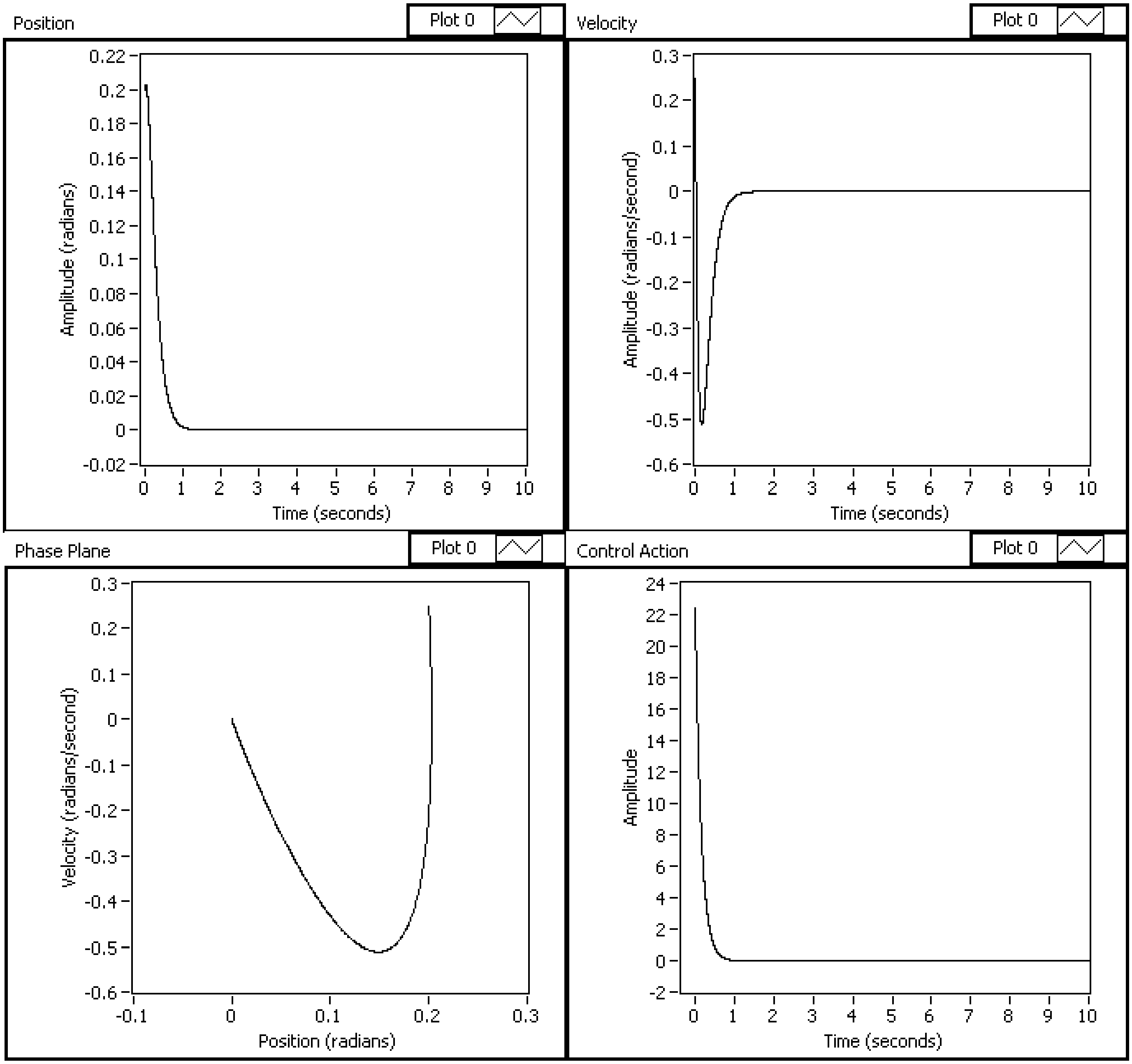

Figure 4: Results of dynamic simulation of system in the **Balance** region with a controller designed using QHC constraints and parameter optimization. The system reaches equilibrium within the **Balance** region within 1 second. The LQR controller yields results that are visually identical to this.

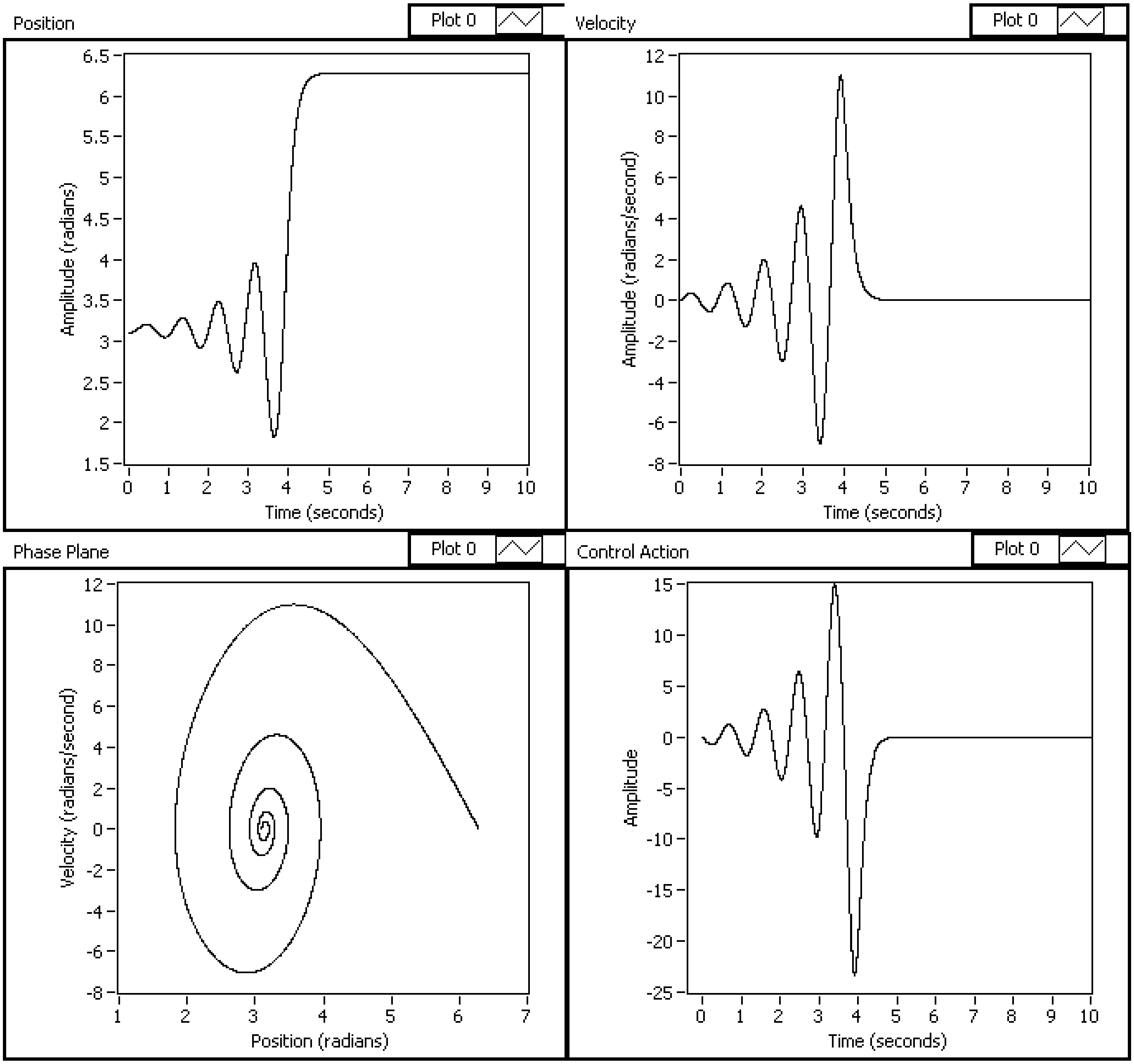

Figure 5: Results of dynamic simulation of system with the multiple model controller designed using QHC constraints and parameter optimization. The system enters the **Balance** region at approximately 4 seconds.

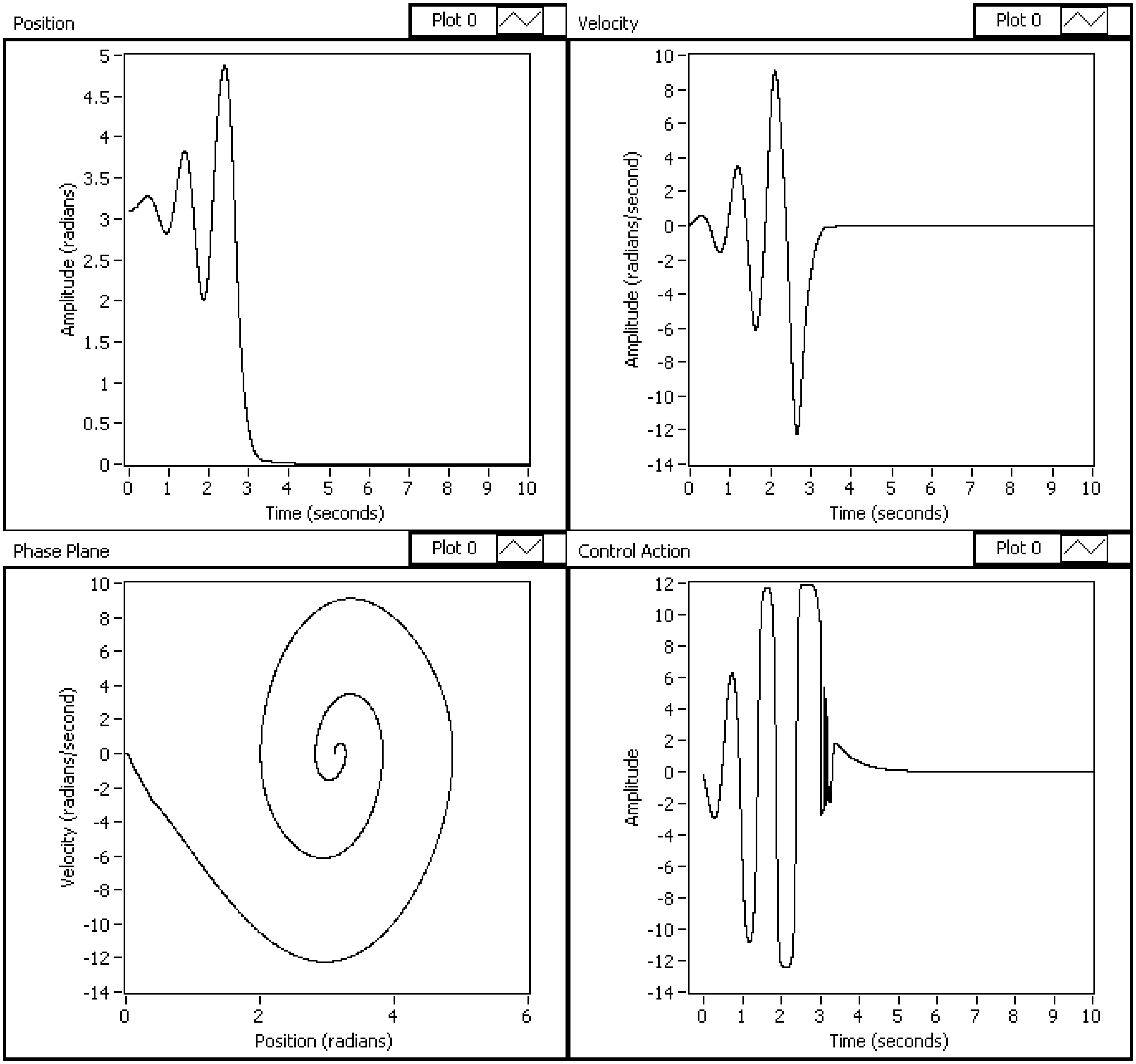

Figure 6: Results of dynamic simulation of system with the multiple model nonlinear controller designed using QHC constraints and parameter optimization. This system is briefly in the **Pump-Spin** sliding mode near 3 seconds before entering the **Balance** region shortly thereafter.

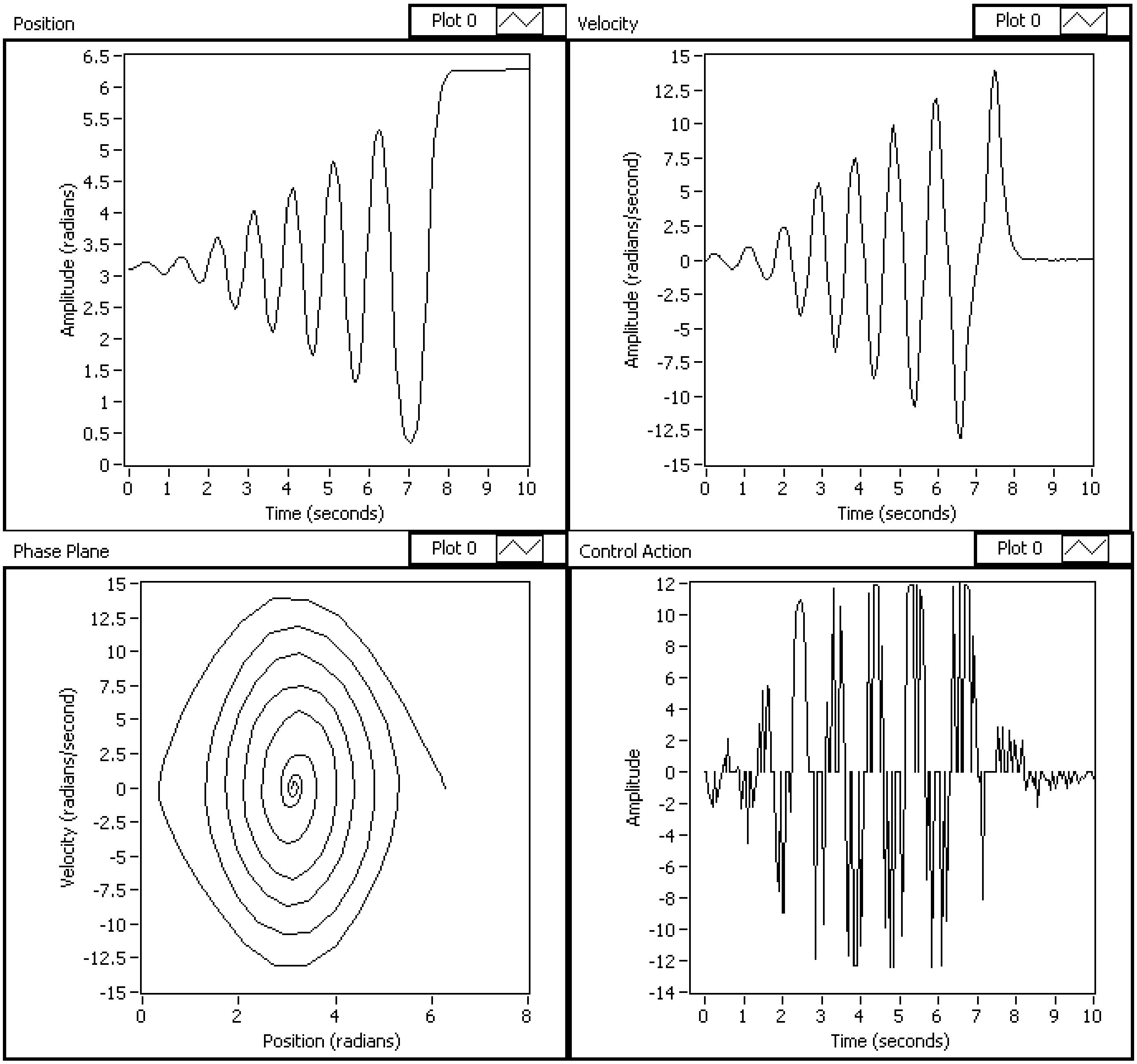

Figure 7: Results of dynamic simulation of system with the multiple model nonlinear controller designed using QHC constraints and parameter optimization. This simulation is based on a step size of 0.05 seconds and the stochastic control action of equation (37). Although the qualitative behavior is exactly as before, convergence of global behavior is slower due to the stochastic nature of the control action that pumps in less energy per unit time.