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April, 1976

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April 12, 1976

ABSTRACT

We investigate certain recently reported results about the modification method and point out a fallacious step in the argument by which these results were reached. In particular, it has not been established conclusively that if S is equality unsatisfiable then the modification S' and $x = x$ is unsatisfiable. We prove that if there is a refutation of the empty clause from S by resolution and uniform paramodulation which does not use the functional reflexive axioms and which does not allow substitution onto variable positions then there is a refutation of the empty clause from the uniform modification S' and $x = x$ by resolution alone. We also prove that resolution on uniformly modified equality unsatisfiable sets is refutation complete in the presence of $x = x$ when no clause contains more than one equation.

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It has been common knowledge in the theorem proving community for several years that a computationally powerful and efficient method of proving theorems in the first order predicate calculus with equality is to translate axioms like $x \cdot 1 = x$ into $P_{\cdot}(x,1,x)$ so that paramodulation can be simulated by resolution. Only recently has this approach been studied in any depth, and it is because of certain errors in these recent studies (Brand 1973,1975) that this note is being written.

Let us take a formal glance at the translation method. The basic rule of translation is: replace the literal $\pm X(t_1,\ldots,t_n)$ by $\pm X(x_1,\ldots,x_n) \vee t_1 \neq x_1 \vee \ldots \vee t_n \neq x_n$, where X is any predicate and the x_i are variable symbols which do not occur in $\pm X(t_1,\ldots,t_n)$. This rule is used recursively until all arguments of predicates are variables, with the exception that left hand arguments of $\neq$ may be of the form $f(y_1,\ldots,y_n)$ where the y_i are variables. For example, replace $f(a) = b$ by $x_1 = x_2 \vee f(a) \neq x_1 \vee b \neq x_2$ which in turn is replaced by $x_1 = x_2 \vee f(x_3) \neq x_1 \vee b \neq x_2 \vee a \neq x_3$. This final clause is called the modification of $f(a) = b$.

The basic idea behind the modification method is to begin with an equality unsatisfiable set S, transform it into the corresponding modification S', and using resolution, but not paramodulation, derive a refutation of the empty clause from S'. From the theoretical standpoint this would seem to be a good idea because it eliminates all explicit need for paramodulation. Of course, the paramodulation steps in a refutation are still there, simulated by resolution steps. One could get the erroneous impression that the translation method provides a constructive proof of the existential result (Anderson 1970) that all paramodulants can be done before any resolvents are formed ("Since each step of the modification process corresponds to a resolution with an equality axiom, the whole process can be thought of as applying the equality axioms a fixed number of times, and then throwing them away." Brand 1975, pp. 419); however, a study of the proof of Theorem 5.1 (Brand 1975) reveals that some of the resolution steps in the modified refutation correspond to paramodulation steps in the unmodified refutation.

The complete modification method actually includes one more transformation which detracts somewhat from its appeal. For each clause C we must add new clauses $C_1, \dots, C_n$, where the C_i are the clauses obtained from C by permuting each equation of C. For example the clause $a = x \vee b = x$ is translated into $y_1 = x \vee y_2 = x \vee a \neq y_1 \vee b \neq y_2$ and then the additional clauses $x = y_1 \vee y_2 = x \vee a \neq y_1 \vee b \neq y_2$, $x = y_1 \vee x = y_2 \vee a \neq y_1 \vee b \neq y_2$, and $y_1 = x \vee x = y_2 \vee a \neq y_1 \vee b \neq y_2$ must be added. Thus for a translated clause with n equations we must add $2^n - 1$ clauses. Fortunately, for most interesting theories equations occur primarily as unit clauses, and so the additional permutations only tend to double the number of clauses with equations.

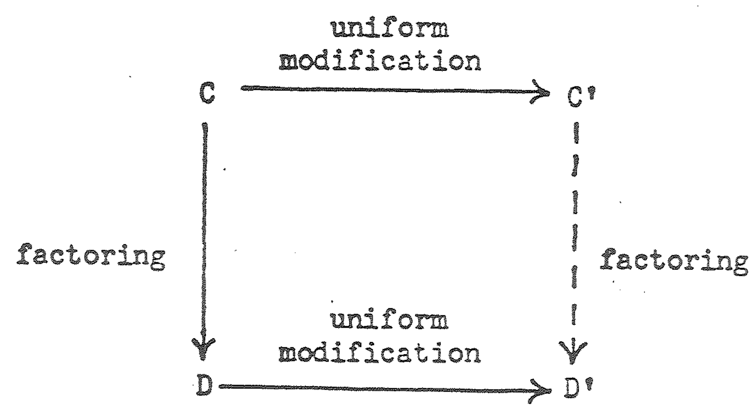
The details of our modification method above are not identical to the recent studies of modification (Brand 1973, 1975), but the spirit is the same. An important claim of Brand's reports is that given an equality unsatisfiable set S , a refutation of the empty clause can be obtained from the modified set S' and $x = x$ (all permutations of equations have been added) by resolution alone. While this conjecture may be true, we believe that Brand's proof is lacking. The crutial point in his proof is that given a modified set S' (without the permutations of equations added) and a model M of S' which is also a model of $x = x$ and <u>transitivity</u> $x \neq y \vee x \neq z \vee y = z$, then there is an equality model of S' (Theorem 2.1 Brand 1975). The difficulty is that he assumes it is enough to show that he can construct models of <u>symmetry</u> $x \neq y \vee y = x$ and S' with arbitrarily large initial segments that are E-consistent. One need not know Brand's exact definition of E-consistent to understand what is being said here. Intuitively, an E-consistent initial segment of a model is an initial segment (relative to some well-ordering of ground literals, such as the lexical order) from which one can derive no complementary pairs by substitution of equals restricted to that initial segment. Thus it is generally the case that a model with an E-consistent initial segment is not an equality model. Now it might be hoped that these models with arbitrarily large E-consistent initial segments could be arranged in some way that an equality model could be constructed inductively; but this does not seem possible, and more important, Brand does not discuss how the equality model is to be constructed. Thus Brand's results must be reinterpreted as saying if a modified set of clauses S' (including permutations of equations) is unsatisfiable when $x = x$ is included, then there is a refutation of the empty clause from the unmodified set S and $x = x$ using resolution and paramodulation, where substitution is never done onto a variable, and where the functional reflexive axioms are never used.

It is natural to conjecture the converse of Brand's result, that is, if there is a refutation of the empty clause from S and $x = x$ using resolution and paramodulation, where substitution onto variables is not permitted, and where the functional reflexive axioms are never used, then there is a refutation of the empty clause from the modification S' and $x = x$ (where S' contains all permutations of equations) using resolution alone. We have been unable to establish this conjecture for reasons that we will point out later, but we have been able to establish a related conjecture. While the modification method replaces each occurrence of a term t in a clause by a new variable x and adds the disjunction $\ldots \vee t \neq x$, by contrast we define uniform modification to be the rule that replaces all occurrences. For example, the uniform modification of $P(a) \vee Q(a)$ is $P(x) \vee Q(x) \neq x$, while the (ordinary) modification is $P(x) \vee Q(u) \vee a \neq x \vee a \neq y$. A uniform paramodulant of $t = u \vee C$ and D is defined to be $C\theta \vee D\theta'$ where θ is the most general unifier of t (u) and a subterm v of D that is not a variable, and where $D\theta'$ is obtained form $D\theta$ by replacing all occurrences of $v\theta$ in $D\theta$ by $u\theta$ ($t\theta$). For example, $P(f(a),a)$ is a uniform paramodulant of $P(f(f(a)),f(a))$ and $f(a) = a$. The connection between the modification method and uniform paramodulation is the following.

Theorem 1 If S is a set of clauses which has a refutation of the empty clause using resolution and uniform paramodulation, then there is a refutation of the empty clause from the uniform modification S' (to which all permutations of equations have been added) and $x = x$ by resolution alone. Throughout, we consider resolution to consist of two steps, factoring and binary resolution.

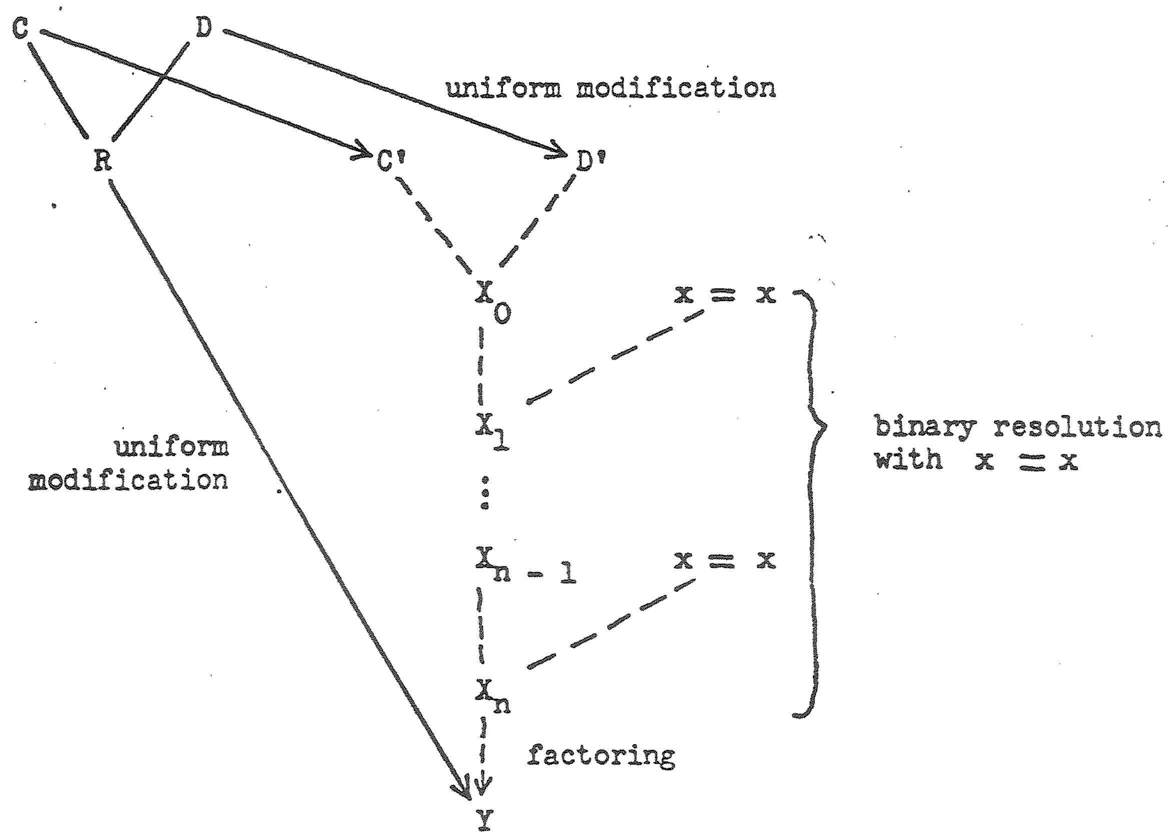
The full details of a proof of Theorem 1 will be the subject of an expanded version of this paper. Here we content ourselves with an outline of the proof. Basically, our proof of Theorem 1 consists of establishing three lemmas, one for each of the basic refutation steps, factoring, binary resolution, and uniform paramodulation. These three lemmas show that a factoring step in the given refutation corresponds to a factoring step in the uniform modification refutation, that a binary resolution step in the given refutation corresponds to a binary resolution in the uniform modification refutation followed by a finite number of binary resolution steps with $x = x$ that is in turn followed by a factoring step, and that uniform paramodulation transforms like binary resolution.

<u>Lemma 1</u> If D is a factor of C, C' is the uniform modification of C, and D' is the uniform modification of D, then D' is a factor of C'. This lemma can be captured by the diagram below.



The corresponding conjecture for (ordinary) modification does not hold, as can be seen by examining the factor of $P(f(a),x,h(c)) \vee P(y,g(b),h(c)) \vee Q(x) \vee R(y)$. It is because of this failure that we have been unable to establish the converse of Brand's result.

Lemma 2 If R is a resolvent of C and D then the following diagram can be completed as indicated.



Lemma 3 If P is a uniform paramodulant of C and D then the previous daigram, with R replaced by P , can be completed as previously indicated.

As we have said, proofs of these three lemmas will be the subject of an expanded article. Given the three lemmas, a proof of Theorem 1 is easily established. Each step of the given refutation is transformed by the appropriate lemma into a corresponding step or sequence of steps, and the result is a refutation from the uniformly modified input set and $x = x$.

Corollary 1 If S is an equality unsatisfiable set of clauses such that each equation occurs only in a unit clause, then there is a refutation of the empty clause from the uniform modification S' and $x = x$ (all permutations of equations are included) by resolution.

Proof It has been shown (Richter 1975) that when all equations occur only as units then there is a refutation of the empty clause from S by resolution and uniform paramodulation that prohibits substituting onto variable positions, and that does not use the functional reflexive axioms. Now Corollary 1 follows immediately from Theorem 1.

In conclusion, let us summarize our findings on the modification method. First of all, we have established the refutation completeness of the uniform modification method when all equations occur as units. Since we can simulate uniform modification by (ordinary) modification and factoring, it follows that modification in general is refutation complete when all equations occur as units. The case when some clause contains more than one equation is presently an open question. The functional reflexive conjecture is still unsolved. We conjecture that the modification method can be shown equivalent to the uniform paramodulation conjecture. Half of this equivalence has been established by Theorem 1. Finally, we believe that additional insight into the workings of equality may be a byproduct of deeper investigation of the modification method, for example, translation of various restrictions of resolution back into corresponding facts about paramodulation.

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