DECISION PROCEDURES FOR SIMPLE EQUATIONAL THEORIES WITH COMMUTATIVE-ASSOCIATIVE AXIOMS:

COMPLETE SETS OF COMMUTATIVE-ASSOCIATIVE REDUCTIONS

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Lankford and Ballantyne (2) have established a semidecision procedure for the incompleteness of finitely terminating sets of permutative reductions. When the permutation equations are all commutative equations, Lankford and Ballantyne (1) have given an algorithm which decides unique termination for finitely terminating commutative rewrite rules. In this note we develop an algorithm which decides unique termination for finitely terminating commutative-associative rewrite rules.

It has been shown by Stickel (3) that commutativity and associativity can be built into the unification algorithm. In particular, he shows there is an algorithm CA which, given any two terms t and u of a commutative associative theory T, returns a finite set $CA(t,u)$ of substitutions such that if θ is any substitution for which $\vdash_T \; t\theta = u\theta$, then there exists $\mu \in CA(t,u)$ and a substitution λ such that $\theta = \mu\lambda$.

We first consider terms which contain only the commutative-associative function f and variables. Thus we are given permuters

$\mathcal{P}$: $f(f(x,y),z) = f(x,f(y,z))$, and

$f(x,y) = f(y,x)$

and a finite set of commutative-associative rewrite rules

$\mathcal{R}$: $\approx(L_1) \longrightarrow \approx(R_1)$,

...

$\approx(L_n) \longrightarrow \approx(R_n)$,

where the L_i and R_i are terms containing only variable symbols and the function symbol f .

<u>Theorem 1</u> If $\mathcal{R}$ has the finite termination property relative to $\mathcal{P}$, then $\mathcal{R}$ has the unique termination property relative to $\mathcal{P}$ iff (1) $\approx(R_i\mu)^* = \approx(R_j\mu)^*$ for all $i , j \leq n$ and all $\mu \in CA(L_i,L_j)$, (2) $\approx(f(x,R_i)\mu)^* = \approx(R_j\mu)^*$, for all $i , j \leq n$ and all $\mu \in CA(f(x,L_i),L_j)$, where x is a variable symbol that does not occur in L_i or L_j , and (3) $\approx(f(x,R_i)\mu)^* = \approx(f(y,R_j)\mu)^*$ for all $i , j \leq n$ and all $\mu \in CA(f(x,L_i),f(y,L_j))$ where x and y are variable symbols that do not occur in L_i or L_j .

$\underline{\text{Proof}}$ ($\Longrightarrow$) Consider each of the three cases. For example, in case 2, $\approx(f(x,R_i)\mu)$ and $\approx(R_j\mu)$ are immediate $\mathcal{P}$-reductions of $\approx(f(x,L_i)\mu)$, so that the hypothesis of unique $\mathcal{P}$-termination gives us $\approx(f(x,R_i)\mu)^* = \approx(R_j\mu)^*$, where the $*$ operator permutatively reduces each equivalence class $\approx(t)$ to an irreducible form $\approx(t)^*$ relative to $\mathcal{R}$. The other cases are done similarly.

($\Longleftarrow$) It suffices to establish the permutative diamond lemma, which we do in the following manner. Let $\vdash_{\mathcal{P}} t_1 = t_2$, u be the result of replacing one occurrence of $L_i\theta_1$ in t_1 by $R_i\theta$ and v be the result of replacing one occurrence of $L_j\theta_2$ in t_2 by $R_j\theta_2$. Because $\mathcal{P}$ is a commutative-associative theory, it follows that $\vdash_{\mathcal{P}} t_1 = L_i\theta_1$ or for some w_1, $\vdash_{\mathcal{P}} t_1 = f(w_1,L_i\theta_1)$ and $\vdash_{\mathcal{P}} t_2 = L_j\theta_2$ or for some w_2, $\vdash_{\mathcal{P}} t_2 = f(w_2,L_j\theta_2)$, which gives us three cases to consider. For example, case 2 is the hypothesis which allows us to establish the permutative diamond lemma when $\vdash_{\mathcal{P}} t_1 = L_i\theta_1$ and $\vdash_{\mathcal{P}} t_2 = f(w_2,L_j\theta_2)$. In that case we have $\vdash_{\mathcal{P}} L_i\theta_1 = f(w_2,L_j\theta_2)$, so it follows that $\vdash_{\mathcal{P}} L_i\theta = f(x,L_j)\theta$, where we assume that L_i and L_j have been standardized so as to have no variables in common and that x is a variable which does not occur in L_i or L_j. Thus there is some $\mu \in CA(L_i,f(x,L_j))$ and a substitution λ such that $\theta = \mu\lambda$. It is clear that $\vdash_{\mathcal{P}} u = R_i\mu\lambda$ and $\vdash_{\mathcal{P}} v = f(x,R_j)\mu\lambda$, so it follows that $\approx(u)$ and $\approx(v)$

permutatively reduce to a common term, in this case $\approx (R_i \mu)^* \lambda$. The other cases are done similarly. Q.E.D.

When additional functions are present, we show below that by adding certain additional superpositions, unique termination can again be decided. Let $\mathcal{P}$ be a commutative-associative theory and let $\mathcal{R}$ be permutative rewrite rules containing arbitrary function symbols.

<u>Theorem 2</u> If $\mathcal{R}$ has the finite termination property relative to $\mathcal{P}$, then $\mathcal{R}$ has the unique termination property relative to $\mathcal{P}$ iff the Knuth and Bendix superposition test is satisfied for $\{L_1 \longrightarrow R_1, \ldots, L_n \longrightarrow R_n\} \cup \{f(x,L_i) \longrightarrow f(x,R_i) \mid$ the leading function symbol of L_i is $f\}$ where unification is replaced by commutative-associative unification and reduction is replaced by commutative-associative reduction. The variable symbol x is assumed not to occur in L_i or R_i.

<u>Proof</u> This proof is like the proof of Theorem 1, but also involves recursion on terms of lesser complexity. Also, some commutative-associative matches need not be formed, namely, those matches against subterms with leading function symbol f which are immediate arguments of some f. Q.E.D.

Theorem 1 and Theorem 2 can also be extended to the case when there are a finite number of commutative-associative functions by using

Stickel's method of generalization and recursive calls on the unification algorithm for terms of lesser complexity (3). We state the extension of Theorem 2 below.

Let us be given

$$\mathcal{P}: \quad f_1(f_1(x,y),z) = f_1(x,f_1(y,z)) ,$$
$$f_1(x,y) = f_1(y,x) ,$$
$$\ldots$$
$$f_m(f_m(x,y),z) = f_m(x,f_m(y,z)) ,$$
$$f_m(x,y) = f_m(y,x) ,$$
$$\mathcal{R}: \quad \approx(L_1) \longrightarrow \approx(R_1) ,$$
$$\ldots$$
$$\approx(L_n) \longrightarrow \approx(R_n) ,$$

and a commutative-associative unification algorithm CA for the commutative-associative functions $f_1 , \ldots , f_m$.

<u>Theorem 3</u> If $\mathcal{R}$ has the finite termination property relative to $\mathcal{P}$, then $\mathcal{R}$ has the unique termination property relative to $\mathcal{P}$ iff the Knuth and Bendix superposition test is satisfied for $\{ L_1 \longrightarrow R_1 , \ldots , L_n \longrightarrow R_n \} \cup \{ f_j(x,L_i) \longrightarrow f_j(x,R_i) \mid$ the leading function symbol of L_i is $f_j \}$ where unification is replaced by CA-unification and reduction is replaced by permutative reduction.

<u>Proof</u> Imitate the proof of Theorem 2 with CA-unification. Q.E.D.

The semi-decision procedure for non-unique termination (2) was used as a completion-attempting procedure, and produced the sets of commutative-associative reductions below. In each case, the non-unique termination semi-decision procedure did not halt, but attempted to generate additional permuters and reducers (even though none were generated). When the examples were tested by a LISP implementation of Theorem 3, all were found to be complete sets of commutative-associative reductions. Of course, Theorem 3 could also have been used as the basis of a completion-attempting procedure, since all the permuters in the example are commutative-associative. However, in practice, arbitrary equational theories can seldom be transformed into complete sets of permutative reductions such that all the permuters are commutative-associative. We have recently learned that Stickel and Peterson (4) have independently discovered and implemented a commutative-associative completion-attempting procedure based on a approach equivalent to Theorem 3, and that their program derived decision procedures for Abelian groups, commutative rings with unit, and distributive lattices.

EXAMPLES OF COMPLETE SETS OF COMMUTATIVE ASSOCIATIVE REDUCTIONS

$X^2 = 1$, IDENTITY, COMMUTATIVITY, AND ASSOCIATIVITY

$\mathcal{P}$: P1. $(X \cdot Y) \cdot Z = X \cdot (Y \cdot Z)$

P2. $X \cdot Y = Y \cdot X$

$\mathcal{R}$: R1. $\{X \cdot X\} \longrightarrow \{1\}$

R2. $\{X \cdot 1 \;,\; 1 \cdot X\} \longrightarrow \{X\}$

ABELIAN GROUP THEORY

$\mathcal{P}$: P1. $(X \cdot Y) \cdot Z = X \cdot (Y \cdot Z)$

P2. $X \cdot Y = Y \cdot X$

$\mathcal{R}$: R1. $\{X \cdot 1 \;,\; 1 \cdot X\} \longrightarrow \{X\}$

R2. $\{X \cdot X^{-1} \;,\; X^{-1} \cdot X\} \longrightarrow \{1\}$

R3. $\{1^{-1}\} \longrightarrow \{1\}$

R4. $\{(X^{-1})^{-1}\} \longrightarrow \{X\}$

R5. $\{(X \cdot Y)^{-1} \;,\; (Y \cdot X)^{-1}\} \longrightarrow \{X^{-1} \cdot Y^{-1} \;,\; Y^{-1} \cdot X^{-1}\}$

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