A COMPLETENESS THEOREM FOR MULTIPLE CHAINING

by

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§0. Introduction. We consider a language whose sentences are finite disjunctions of inequalities among a finite set of constant symbols; there are no variables or quantifiers. We prove a completeness theorem for a system of deduction whose only non-trivial rule is chaining (reflecting transitivity of $<$). The system incorporates a number of restrictions designed to limit the number of alternatives a computer must examine in searching for a proof.

§1. Syntax. Let A be a finite alphabet. A literal of A is an expression of the form $a \leq b$ or $a < b$, where $a, b \in A$. A clause of A is a (finite) set of literals of A. We think of a clause as the disjunction of the literals in it, and we often exhibit clauses using a disjunction sign. Thus, $a \leq b \vee b < c$ and $b < c \vee a \leq b$ denote the same clause, $\{a \leq b, b < c\}$. We use $\square$ to denote the empty clause.

§2. Semantics. A model for A is a map, $\Gamma: A \to \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. We abuse notation by using $<$ also to denote the usual ordering of $\mathbb{R}$. It will become clear that we could allow maps into arbitrary totally ordered sets, but it will be notationally convenient to have a fixed target.

We say $\Gamma \models a < b$, or $a < b$ is true in Γ, iff $\Gamma(a) < \Gamma(b)$; likewise $\Gamma \models a \leq b$ iff $\Gamma(a) \leq \Gamma(b)$. If E is a clause, we say $\Gamma \models E$ iff at least one of the elements (disjuncts) of E is true in Γ. If S is a set of clauses, we say $\Gamma \models S$ iff $\Gamma \models E$ for every $E \in S$. In particular, for any Γ, $\Gamma \not\models \square$ and $\Gamma \models \emptyset$, where $\emptyset$ is the empty set of clauses; there is a minor abuse of notation here, since $\square$ and $\emptyset$ are actually the same object.

If S is a set of clauses and E a clause, $S \models E$, or E is a semantic consequence of S, iff for all Γ, $\Gamma \models S$ implies $\Gamma \models E$. S is semantically consistent iff $\Gamma \models S$ for some Γ iff $S \not\models \Box$. We call E a tautology iff $\emptyset \models E$ iff $\Gamma \models E$ for all Γ. If F is another clause, $F \models E$ means $\{F\} \models E$; we say E follows tautologically from F. Thus, $\Box \models E$ for all E; if E is a tautology then $F \models E$ for all F.

§3. Proof theory. There are two basic proof rules. One is deletion. If E is a clause, we define $\mathrm{del}(E)$ to be the result of deleting from E all literals of the form $a < a$ as well as all those literals of the form $a < b$ where $a \leq b$ occurs in E. The other is chaining. If $a \in A$ and E and F are clauses, $\mathrm{ch}(E,a,F)$, the result of chaining E and F on a, is defined as follows. Say

$$E = \bigvee\{b_i s_i a \colon i < n\} \vee E' ,$$

where each s_i is either the symbol $<$ or $\leq$, and all literals of E' do not have a on the right. Likewise, say

$$F = \bigvee\{a t_j c_j \colon j < m\} \vee F' ,$$

where each t_j is either $<$ or $\leq$ and the literals of F' do not have a on the left. Then we define

$$\mathrm{ch}(E,a,F) = \bigvee\{b_i u_{ij} c_j \colon i < n,\ j < m\} \vee E' \vee F' ,$$

where u_{ij} is $\leq$ if s_i and t_j are both $\leq$; otherwise u_{ij} is $<$. If $n = 0$, these definitions imply that $ch(E,a,F)$ is $E \vee F'$, while $m = 0$ implies that $ch(E,a,F)$ is $E' \vee F$.

Our proof theory has no logical axioms; a deduction from S must proceed by quoting clauses in S or applying the two proof rules. Because of this, the usual completeness theorem, $S \models E \Rightarrow S \vdash E$, is false. For a trivial example, $\emptyset \models a \leq a$, but there is no way of deriving anything from $\emptyset$. For a less trivial example,

Example 1. Let S be

$$\{a < b,\ c < d,\ a < b \vee c < d\}$$

and let E be

$$a < d \vee c < b \ .$$

Then S is closed under deletions and chainings, $S \models E$, and E does not follow tautologically from any single element of S.

Nevertheless, it is true that if $S \models \square$, then one can derive $\square$ from S. We shall prove this to be the case even when we put the following two restrictions on the allowable chainings. First, we forbid $ch(E,a,F)$ to be inferred from E and F if it follows tautologically from E or from F separately. Second, we list A is some order, say $A = \{a_1, a_2, \ldots\}$, and demand that the deduction first chain only on a_1, then chain only on a_2 between clauses which do not use a_1, and so forth.

More precisely, if $a \in A$ and S is a set of clauses of A, define

$$R_a(S) = S \cup \{del(E): E \in S\} \cup \{ch(E,a,F): E,F \in S \text{ and } E \not\models ch(E,a,F) \text{ and } F \not\models ch(E,a,F)\} .$$

Let $R_a^0(S) = S$, $R_a^{n+1}(S) = R_a(R_a^n(S))$, and $R_a^\infty(S) = \bigcup_n R_a^n(S)$. Let $R_a^*(S)$ be the set of clauses in $R_a^\infty(S)$ which do not mention the letter a.

3.1. <u>Completeness Theorem</u>. Suppose that $n \geq 1$, $A = \{a_1, \ldots, a_{n+1}\}$, S is a set of clauses of A, and $S \models \Box$. Then

$$\Box \in R_{a_n}^*(R_{a_{n-1}}^*(\cdots(R_{a_1}^*(S))\cdots)) .$$

<u>Example 2</u>. Let $S = \{E,F,G,H,I\}$, where

$$E \text{ is } b < a \vee a < b$$

$$F \text{ is } c < a \vee a < c$$

$$G \text{ is } c \leq b, \; H \text{ is } b \leq a, \; I \text{ is } a \leq c .$$

Then $S \models \Box$. $R_a^1(S)$ is S together with the 5 clauses

$$ch(E,a,I): b < c \vee a < b$$

$$ch(F,a,I): c < c \vee a < c$$

$$ch(H,a,E): b < a \vee b < b$$

$$ch(H,a,F): c < a \vee b < c$$

$$ch(H,a,I): b \leq c .$$

We do not obtain $b < c \vee a < b \vee c < a$ $(ch(E,a,F))$, since it follows tautologically from E and from F separately, although it is not a tautology. Further deletions and chainings yield, in $R_a^\infty(S)$, $b < a$, $a < c$, $b < c$, among others. Thus $R_a^*(S)$ contains $c \le b$ and $b < c$, so $\square \in R_b^*(R_a^*(S))$.

Before proving Theorem 3.1, we establish, as a lemma, a completeness result for chaining on one letter only.

3.2. Lemma. Suppose that $a \in A$, $A_0 = A \setminus \{a\}$, S is a set of clauses of A, and $\Gamma_0 \colon A_0 \to \mathbb{R}$. Suppose further that for every $\Gamma\colon A \to \mathbb{R}$ which extends Γ_0, $\Gamma \not\models S$. Then $\Gamma_0 \not\models R_a^*(S)$.

Proof. Say $A_0 = B_1 \cup \ldots \cup B_n$, $r_1, \ldots, r_n \in \mathbb{R}$, $r_1 < r_2 < \ldots < r_n$, and $\Gamma_0(b) = r_i$ whenever $b \in B_i$. For $1 \le i \le n$, let $C_i \in S$ be such that $\Gamma \not\models C_i$ when Γ is the extension of Γ_0 such that $\Gamma(a) = r_i$. For $1 \le i \le n-1$, let $D_i \in S$ be such that $\Gamma \not\models D_i$ when Γ is some (any) extension of Γ_0 with $r_i < \Gamma(a) < r_{i+1}$. Likewise, let $D_0 \in S$ contradict $\Gamma(a) < r_1$ and let $D_n \in S$ contradict $r_n < \Gamma(a)$.

We now produce E_i and F_i in $R_a^\infty(S)$ for $1 \le i \le n$ such that for each Γ extending Γ_0,

$$\Gamma(a) < r_i \Rightarrow \Gamma \not\models E_i \quad \text{and}$$
$$\Gamma(a) \le r_i \Rightarrow \Gamma \not\models F_i ,$$

We also produce $E_{n+1} \in R_a^\infty(S)$ such that $\Gamma \not\models E_{n+1}$ for any Γ extending Γ_0.

Let E_1 be D_0. Given E_i, let F_i be $ch(E_i, a, C_i)$ unless $ch(E_i, a, C_i)$ follows tautologically from E_i or C_i, in which case we let F_i be any one of E_i, C_i from which $ch(E_i, a, C_i)$ follows tautologically. Likewise, given F_i, let E_{i+1} be $ch(F_i, a, D_i)$ unless $ch(F_i, a, D_i)$ follows tautologically from F_i or D_i, in which case E_{i+1} is any one of F_i, D_i from which $ch(F_i, a, D_i)$ follows tautologically. It is easily verified inductively that the E_i and F_i have the properties claimed.

Finally, observe that E_{n+1} can involve the letter a only in the combination $a < a$, so that $del(E_{n+1})$ is an element of $R_a^*(S)$ which contradicts Γ_0, proving the lemma.

As a special case, if $S \models \Box$ we may apply Lemma 3.2 to every $\Gamma_0 : A_0 \to \mathbb{R}$ and obtain the following.

3.3. <u>Corollary</u>. Suppose that $a \in A$, S is a set of clauses of A, and $S \models \Box$. Then $R_a^*(S) \models \Box$.

Finally, we prove Theorem 3.1 by induction on n. For $n = 1$, it is easy by inspection, and the induction step is immediate from Corollary 3.3.

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