Let $\mathcal{T}$ be any set of terms. If $\mathcal{S}$ is a set of clauses, we let gcons($\mathcal{S}$,0, $\mathcal{T}$) be $\mathcal{S}$, and let gcons($\mathcal{S}$, n+1, $\mathcal{T}$) be gcons($\mathcal{S}$,n, $\mathcal{T}$) unioned with the set of all clauses that can be obtained by one application of ground chaining or ground self-chaining on a term in $\mathcal{T}$ to clauses in gcons($\mathcal{S}$,n, $\mathcal{T}$). Let gcons($\mathcal{S}$, $\mathcal{T}$) be the union of all the gcons($\mathcal{S}$,n, $\mathcal{T}$). We shall often use without comment the obvious fact that gcons(gcons($\mathcal{S}$, $\mathcal{T}$), $\mathcal{T}$) = gcons($\mathcal{S}$, $\mathcal{T}$).

Let gcons!($\mathcal{S}$, $\mathcal{T}$) be the set of clauses in gcons($\mathcal{S}$, $\mathcal{T}$) which do not use any terms in $\mathcal{T}$. Our ground completeness theorem is expressed by:

THEOREM: If $\mathcal{S}$ is ground inconsistent and $\mathcal{T}$ is any set of terms, then gcons!($\mathcal{S}$, $\mathcal{T}$) is ground inconsistent.

As a special case, we may take $\mathcal{T}$ to be the set of all terms appearing in $\mathcal{S}$. The theorem then says that if $\mathcal{S}$ is ground inconsistent, we may derive $\square$ from it by ground chaining and self-chaining.

To prove the theorem, note first that by compactness, we may assume that $\mathcal{S}$ is finite. The proof will be by induction on the EXCESS LITERAL PARAMETER (introduced by Anderson and Bledsoe in [AB]), elp($\mathcal{S}$). If C is a clause, we let elp(C) be the number of literals in C minus one. If $\mathcal{S}$ is a finite set of clauses, we let elp($\mathcal{S}$) be the sum of all elp(C) for C in $\mathcal{S}$. If $\mathcal{S}$ does not already contain $\square$, elp($\mathcal{S}$) = 0 means that each clause in $\mathcal{S}$ consists of exactly one literal; this case will be the basis of the induction, which we state as a lemma.

LEMMA 1: If $\mathcal{S}$ is finite, elp($\mathcal{S}$) = 0, and $\mathcal{S}$ is ground inconsistent, then gcons!($\mathcal{S}$, $\mathcal{T}$) is ground inconsistent.

PROOF: We assume that $\square$ is not already in $\mathcal{S}$, so that each clause of $\mathcal{S}$ consists of a single literal. We say that $\mathcal{S}$ contains a BAD CYCLE iff there is a sequence of terms and symbols

$$\alpha_1 \; @_1 \; \alpha_2 \; @_2 \; \ldots \; \alpha_n \; @_n \; \alpha_1$$

where $\mathcal{A}$ contains the clause $\alpha_n \; @_n \; \alpha_1$, as well as the clauses $a_i \; \alpha_1 \; a_{i+1}$ for each $i < n$, and at least one of the symbols $@_i$ is $<$. If $\mathcal{A}$ contains such a cycle, say that m of the a_i are in $\mathcal{F}$. If $m < n$, then m applications of ground chaining on terms in $\mathcal{F}$ produce a bad cycle of length n-m containing only terms not in $\mathcal{F}$. If $m = n$, then n-1 applications of ground chaining plus one application of ground self-chaining produce $\square$. In either case, gcons!($\mathcal{A}$, $\mathcal{F}$) is ground inconsistent.

Now, assume that $\mathcal{A}$ does not contain a bad cycle; we construct a ground model for $\mathcal{A}$ by taking a transitive closure. Let $\mathcal{Q}$ be the set of all terms used in $\mathcal{A}$, and define the relation R on $\mathcal{Q}$ by: $\alpha R \beta$ iff either a and b are the same or at least one of the literals, $\alpha < \beta$, $\alpha \leq \beta$ is in $\mathcal{A}$. Let R^* be the transitive closure of R. R^* defines an equivalence relation E (as does any transitive reflexive relation) defined by aEb iff $\alpha R^* \beta$ and $\beta R^* \alpha$. Let $[\alpha]$ be the equivalence class of α, let $\mathcal{B}$ be the set of equivalence classes, and let $<$ be the partial order on $\mathcal{B}$ defined by $[\alpha] < [\beta]$ iff $\alpha R^* \beta$. Consider $(\mathcal{B}, <, F)$, where $F(\alpha)$ is $[\alpha]$; we show that this is a ground model for $\mathcal{A}$; recall that by Lemma 1 of Section 3.1, it is sufficient to produce a PARTIALLY ordered ground model. Clearly, if $\mathcal{A}$ contains either $\alpha < \beta$ or $\alpha \leq \beta$, then $F(\alpha) \leq F(\beta)$. Now suppose $\mathcal{A}$ contains $\alpha < \beta$. We cannot have $F(\alpha) = F(\beta)$, since this would imply $\beta R^* \alpha$, which would yield a bad cycle of the form

$$\alpha < \beta \; \ldots \; \alpha$$

Hence, $F(\alpha) < F(\beta)$.

The induction step in the proof of the ground completeness theorem uses a lemma that says that if we have a deduction from $\mathcal{A}$, we may tack on a subset of any clause D to each line of the deduction. An easy induction shows:

LEMMA 2: Suppose that D is any clause and that for each clause C in $\mathcal{J}$ there is a subset D'(C) of D such that C $\vee$ D'(C) is in $\mathcal{A}$. Let E be gcons($\mathcal{J}$, $\mathcal{F}$). Then there is a subset D" of D with E $\vee$ D" in gcons($\mathcal{A}$, $\mathcal{F}$).

In our applications of Lemma 2, D'(C) will actually be empty for all but one C.

PROOF OF THEOREM: By compactness, we may assume that $\mathcal{A}$ is finite, and we proceed by induction of elp($\mathcal{A}$). By Lemma 1, we may assume that $\square$ is not in $\mathcal{A}$ already, elp($\mathcal{A}$) > 0, $\mathcal{A}$ is ground inconsistent, and that whenever $\mathcal{J}$ is ground inconsistent and elp($\mathcal{J}$) < elp($\mathcal{A}$), we have gcons!($\mathcal{J}$, $\mathcal{F}$) ground inconsistent. Since elp($\mathcal{J}$) > 0, there is a clause in $\mathcal{A}$ of the form C $\vee$ D, where C and D are nonempty sets of literals. Let $\mathcal{J}$ be $\mathcal{A}$ with C $\vee$ D replaced by C. The inductive hypothesis applies to $\mathcal{J}$, so gcons!($\mathcal{J}$, $\mathcal{F}$) is ground inconsistent. By Lemma 2, for each E in gcons!($\mathcal{J}$, $\mathcal{F}$) there is a subset, D"(E) of D such that E $\vee$ D"(E) is in gcons($\mathcal{A}$, $\mathcal{F}$). Let $\mathcal{U}$(E) be obtained from $\mathcal{A}$ by replacing C $\vee$ D by D"(E). Then the inductive hypothesis applies to each $\mathcal{U}$(E), so gcons!($\mathcal{U}$(E), $\mathcal{F}$) is ground inconsistent. Let $\mathcal{V}$(E) be obtained from $\mathcal{A}$ by replacing C $\vee$ D by E $\vee$ D"(E). By Lemma 2, for each E in gcons!($\mathcal{J}$, $\mathcal{F}$) and each H in gcons!($\mathcal{U}$(E), $\mathcal{F}$) there is a subset E"(E,H) of E such that H $\vee$ E"(E,H) is in gcons($\mathcal{V}$(E), $\mathcal{F}$). However, $\mathcal{V}$(E) is a subset of gcons($\mathcal{A}$, $\mathcal{F}$), so each H $\vee$ E"(E,H) is in gcons($\mathcal{A}$, $\mathcal{F}$) and hence in gcons!($\mathcal{A}$, $\mathcal{F}$). But then gcons!($\mathcal{I}$, $\mathcal{F}$) is ground inconsistent. To see this, suppose it had a ground model. For each E in gcons!($\mathcal{I}$, $\mathcal{F}$) there must be an H in gcons!($\mathcal{U}$(E), $\mathcal{F}$) such that H is false in this model, since gcons!($\mathcal{U}$(E), $\mathcal{F}$) is ground inconsistent; so, for this E and H, E"(E,H) is true in this model. But E"(E,H) is a subset of E, so E is true in this model for each E in gcons!($\mathcal{I}$, $\mathcal{F}$) contradicting the fact that gcons!($\mathcal{I}$, $\mathcal{F}$) is ground inconsistent.

4. Real Completeness Results

4.1 Intent

In this section, we discuss some completeness results for actual systems based on the rules of inference discussed above. As the rules become more restrictive, the completeness theorems become harder to prove.

4.2 A Simple Result

On the most trivial level, if we allowed ARBITRARY substitutions as legitimate inferences, we could prove a completeness theorem by using Herbrand's theorem plus the ground completeness theorem. Such a deductive system however, would certainly not be practical. Somewhat less trivially, we can admit only chaining, self-chaining, and factoring, and obtain an exact analog of Robinson's completeness theorem for resolution. More precisely, let $\mathcal{S}$ be a set of clauses. Let cons($\mathcal{S}$,0) be the set of all renamings of clauses in S, and cons($\mathcal{S}$,n+1) be cons($\mathcal{S}$,n) unioned with the set of all renamings of clauses which can be obtained by one application of chaining, self-chaining, or factoring to clauses in cons($\mathcal{S}$,n). Let cons($\mathcal{S}$) be the union, for all n, of cons($\mathcal{S}$,n). Clearly, cons($\mathcal{S}$) and each cons($\mathcal{S}$,n) are closed under arbitrary renamings.

For the remainder of Section 4.2 we shall discuss further two aspects of this "cons" deductive system: why its completeness theorem is easy, and why it is not a good system.

THEOREM: If $\mathcal{S}$ is any set of clauses such that $\mathcal{S}$ has no totally ordered model and $\mathcal{E}\mathcal{E}$ is a subset of $\mathcal{S}$, then $\Box$ is in cons($\mathcal{S}$).

(See Section 2.7 for the definition of $\mathcal{E}\mathcal{E}$). We omit the proof because it is exactly like the proof of completeness of ordinary resolution and because "cons" is not really the system we are interested in anyway. In brief, by Herbrand's theorem (Section 3.2), subinst($\mathcal{A}$) is ground inconsistent, so by the ground completeness theorem (Section 3.3), $\square$ is in gcons(subinst($\mathcal{A}$)). We may now take a ground inference of $\square$ from subinst($\mathcal{A}$) and lift it to an inference of $\square$ from $\mathcal{A}$. Factoring may be necessary along the way because a ground literal may lift to a disjunction of several literals. The following simple example shows that factoring is an essential rule. Let $\mathcal{A}$ have as its only member, C:

$$f(x) < f(y) \vee f(z) < f(t).$$

Then subinst($\mathcal{A}$) contains the clause $f(x) < f(x)$ which yields $\square$ by ground self-chaining, but $\square$ cannot be obtained from $\mathcal{A}$ by chaining and self-chaining alone; in fact, it is easily proved by induction that any clause so obtained has a renaming of C as a subset (remember: before we can chain two clauses, we must rename the variables of one of them to be disjoint from those of the other.)

There are two defects of this "cons" deductive system. First, it is only complete for total orders, not dense total orders. Second, it would be very inefficient to implement. We now take up these defects in order.

The rules of chaining, self-chaining, and factoring are sound for all total orders. Thus, for example, if C is the clause $x \leq d$, every consequence of C by these rules is valid in all total orders in which C is valid (i.e., in which d is interpreted as a largest element). In particular, we could never derive $\square$ from C. Of course, if we allow variable elimination, then $\square$ is concluded directly. In fact, if we add variable elimination, the system becomes complete for dense total orders without endpoints, but our proof of this will be rather indirect. First, we shall prove completeness of a weaker system under the additional assumption that $\mathcal{S}$ contains the axioms of dense total orders without endpoints, and we shall then show that our proof rules are set up so that these axioms can never be used.

We now state explicitly what these axioms are. When they are stated in ordinary predicate calculus, they involve existential quantifiers, which in our framework become Skolem functions. If p and q are one-place function symbols and r is a two-place function symbol, we let $\mathcal{DTO}(p,q,r)$ be the set of clauses,

1. $p(x) < x$
2. $x < q(x)$
3. $y \leq x \vee x < r(x,y)$
4. $y \leq x \vee r(x,y) < y$
5. EE(p)
6. EE(q)
7. EE(r).

(See Section 2.7 for the definition of EE). The Theorem above has as an immediate corollary:

COROLLARY: Suppose $\mathcal{S}$ is a set of clauses which is not valid in any dense total order without endpoints. Say p,q,r are not used in $\mathcal{S}$. Let $\mathcal{EE}$ be the set of all EE(f) for function symbols f occurring in S. Then $\square$ is in

$$\text{cons}(\mathcal{S} \cup \mathcal{EE} \cup \mathcal{DTO}(p,q,r)).$$

So for this we do not need variable elimination at all, but we have the inelegant inclusion of extra function symbols q,q,r, and new axioms about them.

We now take up the second defect of this system; namely, that it is not very efficient. We envision our system operating by starting with a finite set $\mathcal{S}$ of clauses and generating cons($\mathcal{S}$,n) successively for n = 0,1,..., until we get $\square$. We have succeeded in making each cons($\mathcal{S}$,n) finite (up to renamings of variables), so that this procedure can be implemented in principle, but unfortunately cons($\mathcal{S}$,n) is very large, making the procedure impossible in practice. The main problem is that a variable can unify with every term. Thus, for each occurrence of a variable, x, on, say, the left side of a literal, we must consider, for EACH other literal, the unification of x with the right side of that literal. The fact that we must add in $\mathcal{DIO}$ exacerbates the problem. Even if $\mathcal{S}$ is empty, $\mathcal{DIO}$ alone contains 17 literals and 24 occurrences of variables, resulting in 17*24 = 408 possible chainings to be investigated at the first step.

We now proceed to show that if we allow variable elimination, it will be possible to put some restrictions on chaining and still have a complete system. These restrictions will greatly increase efficiency for two reasons. First, we shall see by a syntactic argument that with this restricted chaining, the axioms in $\mathcal{DIO}$(p,q,r) can never be used, and therefore never need be considered by the prover. Second, our restrictions will, among other things, not allow chaining on a variable.

4.3 Restricted Chaining

Let us call a SHIELDING TERM any term which is neither a variable nor ground. Then x is eligible in a clause C iff C does not contain any shielding term which contains x; such a term would "shield" x from variable elimination.

There are two motivations for our restrictions on chaining. First, we want to forbid chaining on variables, since those will be eliminated by variable elimination. Second, one might expect, in view of our ground completeness result, that if $\mathcal{A}$ is inconsistent and one chained only on shielding terms, one would obtain a set $\mathcal{J}$ of clauses which is inconsistent and contains no shielding terms. If we also allow variable elimination, then variables can be removed as the shielding terms disappear, so $\mathcal{J}$ will contain only ground terms.

In light of these motivations, we define restricted chaining and self-chaining as follows. Suppose C is $\beta < \gamma \vee D$ and σ is mgu(β,γ). We call $D\sigma$ a result of RESTRICTED SELF-CHAINING of C on β and γ if neither β nor γ is a variable and one of them is a shielding term. Likewise, if C is

$$\alpha \mathbin{@_1} \beta \quad \vee \quad D,$$

E is

$$\gamma \mathbin{@_2} \delta \quad \vee \quad F,$$

and $@$ is tr$(@_1,@_2)$ (see Section 2.4), then we call

$$(\alpha \mathbin{@} \delta \vee D \quad \vee \quad F)\sigma$$

a result of RESTRICTED CHAINING of C and E on β and γ if neither β nor γ is a variable and one of them is a shielding term, and C and E have no variables in common. Let rcons$(\mathcal{A},0)$ be the set of all renamings of clauses in $\mathcal{A}$, and let rcons$(\mathcal{A},n+1)$ be rcons$(\mathcal{A},n)$ unioned with the set of all renamings of clauses obtained from rcons$(\mathcal{A},n)$ by factoring, variable elimination, restricted chaining, and restricted self-chaining. Let rcons$(\mathcal{A})$ be the union of all the rcons$(\mathcal{A},n)$, and let rcons!$(\mathcal{A})$ be the set of all ground clauses in rcons$(\mathcal{A})$.

In the next section, we shall show that if $\mathcal{S}$ is inconsistent, then so is rcons!($\mathcal{S}$). In particular, if the language contains no constant symbols, rcons!($\mathcal{S}$) will be $\{\square\}$.

This result suggests two possibilities in implementations. First, one may simply expand the proof rules to allow also chainings and self-chainings when the clauses involved are ground; this procedure will be complete by the above and the ground completeness theorem. However, it is fairly quick in practice to check whether a set of clauses is ground inconsistent (although in theory this problem is NP complete). As with checking tautologies in propositional logic, there are better algorithms then merely searching for formal deductions. Thus, a second possibility is to start with $\mathcal{S}$ and inductively construct rcons($\mathcal{S}$), periodically sending the list of ground clauses to the ground inconsistency checker.

Of course, just because the restricted rules are complete does not require us to use them only. There are certainly cases where a judicious use of additional chainings or self-chainings will lead to a shorter proof; for example, one should probably always conclude D from $\alpha < \alpha \vee D$ immediately, even if α is not a shielding term; see Section 5.1 for more on this.

Although we have used the ground result to motivate our restrictions on chaining, we have not yet proved anything, and one in fact must be careful to state the restrictions correctly, or the system will not be complete. Naively, one might expect from the ground result that we could state our restrictions to require BOTH β and γ to be shielding terms, but this is wrong; the problem is that a shielding term and a non-shielding term could unify to the same term.

Specifically, consider how to derive □ from the clause $f(x) < f(c)$. More generally, the fact that variables, shielding terms and ground terms may all have common substitution instances makes the proof of the completeness result rather more difficult than the ground theorem.

We can now make precise our statement in the previous section that axioms such as $\mathcal{BTB}(p,q,r)$ can never be used in a deduction from $\mathcal{S}$ unioned with $\mathcal{BTB}(p,q,r)$ (assuming that p,q,r do not occur in $\mathcal{S}$). The general result is the following, which we shall apply with $\mathcal{BTB}(p,q,r)$ as the $\mathcal{T}$.

THEOREM: Suppose that $\mathcal{S}$ and $\mathcal{T}$ are sets of clauses such that no function symbol occurs in both $\mathcal{S}$ and $\mathcal{T}$. Then

$$\text{rcons}(\mathcal{S} \cup \mathcal{T}) = \text{rcons}(\mathcal{S}) \cup \text{rcons}(\mathcal{T}), \text{ and}$$

$$\text{rcons!}(\mathcal{S} \cup \mathcal{T}) = \text{rcons!}(\mathcal{S}) \cup \text{rcons!}(\mathcal{T}).$$

PROOF: There is no restricted chaining possible between a clause in rcons($\mathcal{S}$) and one in rcons($\mathcal{T}$).\

COROLLARY: Suppose that $\mathcal{S}$ and $\mathcal{T}$ are sets of clauses such that no function or constant symbol occurs in both $\mathcal{S}$ and $\mathcal{T}$, and suppose that rcons($\mathcal{S} \cup \mathcal{T}$) is ground inconsistent. Then at least one of rcons!($\mathcal{S}$) and rcons!($\mathcal{T}$) is ground inconsistent.

PROOF: rcons!($\mathcal{S}$) and rcons!($\mathcal{T}$) have no terms in common. If they both had ground models, these models could easily be combined to form a model for their union, which is inconsistent by the theorem.\

The logician will recognize this corollary as a version of A. Robinson's Consistency Theorem (see [E]), since $\mathcal{A}$ and $\mathcal{B}$, when reducted to their common language ($<$ and $=$), both yield the complete theory of dense total order without endpoints.

The theorem would be false if chaining were not restricted (i.e., if we tried to replace rcons by cons). For example, let $\mathcal{A}$ be $\{x < y \vee c < d\}$, and let $\mathcal{B}$ be $\{z < w \vee c_1 < d_1\}$; by chaining and then self-chaining, cons($\mathcal{A} \cup \mathcal{B}$) contains the clause $c < d \vee c_1 < d_1$, which is neither in cons($\mathcal{A}$) nor in cons($\mathcal{B}$).

As an example of the material of this section, let $\mathcal{A}$ be the set of clauses:

1. $f(x) \leq x$
2. $y \leq z \vee f(y) \leq f(z)$
3. $g(w) < w$.

These clauses cannot be valid in any total order, since if we fix y and set $z = g(f(y))$, we have by (1) and (3), $f(z) \leq z < f(y) \leq y$, contradicting (2). Thus, there is a deduction of $\square$ from these that does not use variable elimination at all. However, an algorithm that searches for a deduction using unrestricted chaining (with or without variable elimination) by methodically listing cons($\mathcal{A}$,1) cons($\mathcal{A}$,2), ..., will waste an enormous amount of time enumerating all the possible chainings on variables. For example, there are a total of 4 literals in clauses (1)...(3). Any of the four variables x,y,z,w, can be chained with a term in any of these four literals, yielding $4*4 = 16$ distinct clauses that (among others) appear in cons($\mathcal{S}$,1). Using restricted chaining. we obtain □ very quickly. rcons($\mathcal{S}$,1) contains only:

4. $y \leq x \vee f(y) \leq x$. (1,2 x/z)

rcons($\mathcal{S}$,2) contains □ (from 4 and variable elimination) plus just one other clause obtained from (4) and (2). Note that (3) was not used at all; in fact, by the theorem above, any restricted-chaining consequence of (1)...(3) must be either a consequence of (1), (2), or a consequence of (3) alone, so that (3) can never be used.

One might attempt to construct a prover along these lines that works for arbitrary total orders, rather than dense total orders without endpoints. To make the rules sound, some forms of variable elimination would have to be dropped. One could still conclude $c \leq d$ from $c < x \vee x \leq d$ but not from $c \leq x \vee x \leq d$ since it would not be excluded that $d < c$ with nothing between them. However, this prover would not be nearly so efficient as our prover for dense total orders without endpoints, since, as the above example shows, some chaining on variables MUST be allowed. If not, then clause (3) could never be used, but one cannot derive □ from (1) and (2) alone in a system sound for all total orders since they have a totally ordered model. Such a model must have a first element, and our use of variable elimination in concluding □ from (4) corresponded to the assertion that there is no first element.

4.4 Completeness of Restricted Chaining

The main result of this section is the following Theorem.

THEOREM: If subinst($\mathcal{S}$) is ground inconsistent, then rcons!($\mathcal{S}$) is ground inconsistent.

Thus, one may derive $\square$ from $\mathcal{A}$ by restricted chaining followed by ground chaining applied to purely ground clauses. Before we proceed to the proof, we note the following corollary.

COROLLARY: If there is no model for $\mathcal{A}$ which is a dense total order without endpoints, and $\mathcal{E}\mathcal{E}$ is a subset of $\mathcal{A}$, then rcons!($\mathcal{A}$) is ground inconsistent.

PROOF OF COROLLARY (assuming Theorem): Let $\mathcal{L}$ be the set of function and constant symbols used in $\mathcal{A}$. Then $\mathcal{E}\mathcal{E}$ means the set of all EE(f) for function symbols f in $\mathcal{L}$. Let p,q,r be the function symbols which do not occur in $\mathcal{L}$, with p and q one-place, and r 2-place. Let $\mathcal{L}'$ be $\mathcal{L} \cup \{p,q,r\}$ and let $\mathcal{A}'$ be $\mathcal{A} \cup \mathcal{DJO}(p,q,r)$. Then $\mathcal{A}$ has no totally ordered model, and by definition of $\mathcal{DJO}$, $\mathcal{A}'$ contains EE(f) for all function symbols f of $\mathcal{L}'$. By Herbrand's theorem (Section 3.2) applied to $\mathcal{A}'$, subinst($\mathcal{A}'$) is ground inconsistent. Thus, by the Theorem, rcons!($\mathcal{A}'$) is ground inconsistent. By the corollary in Section 4.3, rcons!($\mathcal{A}$) is ground inconsistent, since rcons!($\mathcal{DJO}$(p,q,r)) is not ground inconsistent (and is in fact the empty set). \

We now begin a sequence of lemmas heading towards a proof of the theorem. Call a substitution function μ REPLETE iff for each term α of $\mathcal{L}$, there are infinitely many variables x such that $x\mu$ is α. We shall concentrate our efforts on the special case where $\mathcal{A}$ is finite and there is a replete μ such that $\mathcal{A}\mu$ is ground inconsistent; later, in the proof of the theorem, we shall see that this special case was sufficient. This μ will be called the MAIN substitution, and will not change throughout our argument. Another substitution, σ will be called CONSISTENT with μ iff for each variable x, $x\mu = x\sigma\mu$ (and hence, $\alpha\mu = \alpha\sigma\mu$ for all terms α). An application of a proof rule (chaining, self-chaining, or renaming) will be called CONSISTENT with μ iff the substitution function used is consistent with μ. We shall in fact show that one can derive $\square$ using only steps consistent with μ ; of course, we also allow variable elimination, which, since it does not involve a substitution, will always be considered to be consistent with μ . It is easily checked that if α and β are terms and $\alpha\mu = \beta\mu$, then mgu(α,β) is consistent with μ . Thus, for example, if we are arguing from $\alpha < \beta$ and $\beta' < \gamma$ and $\beta\mu = \beta'\mu$, then there is a possibility of chaining (assuming that our other restrictions on chaining are met); if the variables in these two clauses are not disjoint, we may first apply renaming; there is a consistent renaming because μ is replete.

Let reconsl($\mathcal{A}$,0) be the set of all consistent (with μ) renamings of clauses in $\mathcal{A}$ (μ will always be clear from context). Let reconsl($\mathcal{A}$,n+1) be rconsl($\mathcal{A}$,n) unioned with all consistent renamings of clauses obtained from rconsl($\mathcal{A}$,n) by one application of variable elimination or of a factoring, restricted chaining, or restricted self-chaining consistent with μ . Let rconsl($\mathcal{A}$) be the union of all reconsl($\mathcal{A}$,n). Let reconsl!($\mathcal{A}$) be the set of all ground clauses in rconsl($\mathcal{A}$).

We wish to show under suitable hypotheses (see Lemma 7) that reconsl!($\mathcal{A}$) is ground inconsistent. As a first step, we see that we can remove all eligible variables.

LEMMA 1: Assume that $\mathcal{A}\mu$ is ground inconsistent. Then there is a set $\mathcal{J}$ of clauses obtained from $\mathcal{A}$ by variable eliminations alone such that $\mathcal{J}\mu$ is ground inconsistent and no clause of $\mathcal{J}$ has any eligible variables.

PROOF: For each C in $\mathcal{S}$, let C' be obtained from C by variable eliminations so that C' either has no eligible variable or C' contains a literal of the form $x \leq x$ (see Lemma 2 of Section 2.5). Let $\mathcal{J}$ be the set of those C' which do not contain any literals of the form $x \leq x$. Then any ground model for $\mathcal{J}\mu$ can be extended to a ground model for $\mathcal{S}\mu$. \

The main idea behind the proof of the theorem will be to try to reduce the number of shielding terms of $\mathcal{S}$ by chaining; observe that if $\mathcal{S}$ had no shielding terms, the $\mathcal{J}$ of Lemma 1 would be purely ground, so reconsl!($\mathcal{S}$), which contains $\mathcal{J}$ (= $\mathcal{J}\mu$ if $\mathcal{J}$ is ground) would trivially be ground inconsistent. Let ST($\mathcal{S}$) be

$$\{\alpha\mu \ : \ \alpha \text{ is a shielding term in } \mathcal{S}\}.$$

This will be more useful than the set of shielding terms of $\mathcal{S}$, since it will not change under renamings consistent with μ. However, there may be a problem chaining away members of ST($\mathcal{S}$) since under μ a shielding term could match a variable, making some chainings illegal. For example, let $\mathcal{S}$ be $\{x < c,\ c < f(y)\}$, with $x\mu = f(y)$ and $y\mu = y$. Then our restrictions forbid us from chaining on x and f(x) to obtain $c < c$. Of course, in this case we conclude $\Box$ directly by variable elimination. In general, the proof of the theorem will be by first applying variable elimination to $\mathcal{S}$ as per Lemma 1 to remove eligible variables, and then show (Lemma 3) that then no member of ST($\mathcal{S}$) of maximal height (or complexity) can match a variable, so that these can be chained away (Lemma 6). Then the proof of the theorem will involve (in Lemma 7) an induction on the maximal height of the members of ST($\mathcal{S}$). Formally, we define the HEIGHT, ht(α), of any term α by:

The height of a variable or constant symbol is 0.

$$ht(f(\alpha_1 \ldots \alpha_n)) = 1 + max(ht(\alpha_1) \ldots ht(\alpha_n)).$$

If $\mathcal{S}$ is finite, let MH($\mathcal{S}$) be the maximal height of all elements of ST($\mathcal{S}$). If ST($\mathcal{S}$) is empty, we set MH($\mathcal{S}$) to be 0.

LEMMA 2: Assume that $\mathcal{S}$ is finite, the variable x and the shielding term α both occur in $\mathcal{S}$, and x is used in α. Then $ht(x\mu) < MH(\mathcal{S})$.

PROOF: $ht(x\mu) < ht(\alpha\mu) \leq MH(\mathcal{S})$.\

In particular, if x is not eligible in any clause of $\mathcal{S}$, x is used within some shielding term, so

LEMMA 3: Assume that $\mathcal{S}$ is finite and has no eligible variables and x is a variable occurring in $\mathcal{S}$. Then $ht(x\mu) < MH(\mathcal{S})$.\

We now shall attempt to show (Lemma 6) that we may start with $\mathcal{S}$ having no eligible variables and eliminate all members of ST($\mathcal{S}$) of maximal height by chaining. This will be an induction on the excess literal parameter resembling the ground completeness theorem. It will be important for the induction that variable elimination is not used at this point. Let rcons2($\mathcal{S}$,0,k) be the set of all consistent (with μ) renamings of clauses in $\mathcal{S}$. Let rcons2($\mathcal{S}$,n+1,k) be recons2($\mathcal{S}$,n,k) unioned with all consistent renamings of clauses obtained from rcons2($\mathcal{A}$,n,k) by one application of a factoring, restricted chaining, or restricted self-chaining consistent with μ, where, in the chaining and self-chaining, we have the additional restriction that the terms β, γ chained upon satisfy that ht($\beta\mu$) (= ht($\gamma\mu$)) $\geq$ k. Let rcons2($\mathcal{A}$,k) be the union of all rcons2($\mathcal{A}$,n,k). Let rcons2!($\mathcal{A}$,k) be the set of those clauses C in rcons2($\mathcal{A}$,k) such that C contains no shielding terms α with ht($\alpha\mu$) $\geq$ k. We shall show (Lemma 6) that if $\mathcal{A}\mu$ is ground inconsistent, μ is replete, and ht(xμ) < k for all variables x occurring in $\mathcal{A}$, then (rcons2!($\mathcal{A}$,k))μ is ground inconsistent. The proof will be an induction on the excess literal parameter resembling the proof of the ground completeness theorem in Section 3.3; the $\mathcal{F}$ there is analogous to the set of shielding terms α with ht($\alpha\mu$) $\geq$ k here. Analogously to the ground case, we state a result saying that if we have a deduction from $\mathcal{A}$, a clause D may be stuck onto each line as a disjunct. The actual statement of the lemma is slightly more complicated, since the various chainings in the deduction from $\mathcal{A}$ will introduce substitution instances of D. Call D' a STRENGTHENING of D iff there are substitutions, $\sigma_1, \ldots, \sigma_p$ consistent with μ such that D' is a subset of

$$D\sigma_1 \vee \ldots \vee D\sigma_p .$$

An easy induction shows:

LEMMA 4: Suppose that D is any clause and that for each clause C in $\mathcal{J}$, there is a strengthening D'(C) of D such that C $\vee$ D'(C) is in $\mathcal{A}$. Let E be in rcons2($\mathcal{J}$,k). Then there is a strengthening D" of D with E $\vee$ D" in rcons2($\mathcal{A}$,k).\

It is important here that we are not allowing variable elimination. For example, if C is the clause $x \leq c$, and D is the clause $f(x) < g(x)$, then variable elimination would allow us to conclude $\Box$ from C, whereas there are no inferences possible from $C \vee D$.

In our applications of Lemma 4, D'(C) will actually be empty for all but one C. It may still be the case that the D'' obtained may very well be longer than D, as it may be a disjunction of several different substitution instances of D. For that reason, our proof of Lemma 6 will be by induction on the excess literal parameter of $\mathcal{A}\mu$, not of $\mathcal{A}$.

We first list some elementary properties of strengthenings which will be used without comment in the proof of Lemma 6.

LEMMA 5: Let D' be a strengthening of D. Then

a. $D'\mu$ is a subset of $D\mu$.

b. If $ht(x\mu) < k$ for all variables x occurring in D, then the same is true for D'.

c. If $ht(\alpha\mu) < k$ for all shielding term α occurring in D, then the same is true for D'.

LEMMA 6: If μ is replete, $ht(x\mu) < k$ for all variables x occurring in $\mathcal{A}$, and $\mathcal{A}\mu$ is ground inconsistent, then $(rcons2!(\mathcal{A},k))\mu$ is ground inconsistent.

PROOF: By compactness, we may assume that $\mathcal{A}$ is finite, and we proceed by induction on the excess literal parameter, elp($\mathcal{A}\mu$). If this is zero, we eliminate a bad cycle as in the ground case, but we must be somewhat careful because of our restrictions on chaining. Let bc($\mathcal{A}$) be the shortest length of a bad cycle in $\mathcal{A}\mu$ (defined only when elp($\mathcal{A}\mu$) is 0). Observe that if C' is a unit clause (i.e., of length 1) in $\mathcal{A}\mu$, then by factoring there is a unit clause C in $\mathcal{A}$ such that C' is $C\mu$. Thus, if bc($\mathcal{A}$) is n, rcons2($\mathcal{A}$,k) contains clauses of the form:

$$\alpha_1 @_1 \beta_1 \;, \ldots, \; \alpha_n @_n \beta_n \;,$$

where $i < n$, $(\beta_i)\mu = (\alpha_{i+1}))\mu$, $(\beta_n)\mu = (\alpha_1)\mu$, and at least one of the $@_i$ is $<$. The conclusion is trivial unless for some i, either α_i is a shielding term and $ht((\alpha_i)\mu) \geq k$ or β_i is a shielding term and $ht((\beta_i)\mu) \geq k$. For definiteness, say β_1 is a shielding term and $ht((\beta_1)\mu) \geq k$. Since $(\alpha_2)\mu = (\beta_1)\mu$ has height $\geq k$, α_2 is not a variable. It follows that restricted chaining can be applied to $\alpha_1 @_1 \beta_1$ and $\alpha_2 @_2 \beta_2$ to shorten the length of the bad cycle. In the special case n = 1, we apply self-chaining instead.

For the induction step, we assume elp($\mathcal{A}\mu$) > 0, and that the lemma holds for smaller values of elp. By definition of elp, $\mathcal{A}\mu$ must contain a non-unit clause, so it must contain a clause of the form $C' \vee D'$, where C' and D' are non-empty. Then $\mathcal{A}$ contains a clause of the form $C \vee D$, where $C\mu$ is C' and $D\mu$ is D'. Let $\mathcal{J}$ be obtained from $\mathcal{A}$ by replacing $C \vee D$ by C. The inductive hypothesis applies to $\mathcal{J}$, so there is a finite subset, $\mathcal{E}$, of rcons2($\mathcal{J}$,k), such that $\mathcal{E}\mu$ is ground inconsistent and for each shielding term α occurring in $\mathcal{E}$, $ht(\alpha\mu) < k$. By Lemma 4, for each E in $\mathcal{E}$ there is a strengthening, D"(E) of D such that E $\vee$ D"(E) is in rcons2($\mathcal{S}$). Let $\mathcal{U}$(E) be obtained from $\mathcal{S}$ by replacing C $\vee$ D by D"(E). Then the inductive hypothesis applies to each $\mathcal{U}$(E), so there is a finite subset $\mathcal{F}$(E) of rcons2($\mathcal{U}$(E)) such that ($\mathcal{F}$(E))μ is ground inconsistent and for each shielding term α occurring in $\mathcal{F}$(E), ht(αμ) < k. Let $\mathcal{V}$(E) be obtained from $\mathcal{S}$ by replacing C $\vee$ D by E $\vee$ D"(E). By Lemma 4, for each E in $\mathcal{E}$ and each F in $\mathcal{F}$(E), there is a strengthening E"(E,F) of E such that F $\vee$ E"(E,F) is in rcons2($\mathcal{V}$(E)). However, every clause in $\mathcal{V}$(E) is in rcons2($\mathcal{S}$), so each F $\vee$ E"(E,F) is in rcons2($\mathcal{S}$) and hence in rcons2!($\mathcal{S}$). But then (rcons2!($\mathcal{S}$,k))μ is ground inconsistent. To see this, suppose it had a ground model. For each E in $\mathcal{E}$, there must be an F in $\mathcal{F}$(E) such that Fμ is false in this model, since ($\mathcal{F}$(E))μ is ground inconsistent; for this E and F, (E"(E,F))μ is true in this model. But (E"(E,F))μ is a subset of Eμ, so Eμ is true in this model for each E in $\mathcal{E}$, contradicting the fact that $\mathcal{E}$μ is ground inconsistent.

Observe the asymmetric nature of the proof of Lemma 6. For each E in D'(E) may be a different variant of D, with differing shielding terms and ground terms. Thus, as E ranges of $\mathcal{E}$, the $\mathcal{F}$(E) and the way we obtain them may bear no relationship to each other.

LEMMA 7: If $\mathcal{S}$ is finite, μ is replete, and $\mathcal{S}$μ is ground inconsistent, then rcons1!($\mathcal{S}$) is ground inconsistent.

PROOF: This will be an induction on MH($\mathcal{S}$), so assume that k = MH($\mathcal{S}$) and that Lemma 7 holds for all sets with MH less than k. By Lemma 1, there is an $\mathcal{S}'$ obtained from $\mathcal{S}$ by variable elimination alone such that $\mathcal{S}'$μ is ground inconsistent and $\mathcal{S}'$ has no eligible variables. Then MH($\mathcal{S}'$) $\leq$ k. If x is any variable used in S', then by Lemma 3, ht($x\mu$) < k. In particular, if k is 0, then $\mathcal{S}'$ contains no variables whatever, so $\mathcal{S}'\mu$ is $\mathcal{S}'$, and the lemma follows trivially. If k > 0, observe that MH(rcons2!($\mathcal{S}'$,k)) is less than k by definition of rcons2!. By Lemma 6, rcons2!($\mathcal{S}'$,k) is ground inconsistent, so by the inductive hypothesis, reconsl!(rcons2!($\mathcal{S}'$,k)) is ground inconsistent; but this is contained in rconsl!($\mathcal{S}$).\

PROOF OF THEOREM: Assume subinst($\mathcal{S}$) is ground inconsistent. By compactness, some finite subset of subinst($\mathcal{S}$), $\mathcal{G}$, is also ground inconsistent, and $\mathcal{G}$ involves substitution instances of some finite set $\mathcal{J}$ of $\mathcal{S}$. Since renaming is a proof rule, we may assume that any two clauses in $\mathcal{J}$ have disjoint sets of variables. Furthermore, we may assume that $\mathcal{G}$ contains exactly one substitution instance of each clause of $\mathcal{J}$; if $\mathcal{G}$ contains more than one instance of a given clause, we may instead replace that clause by a finite number of renamings of itself. Since variables occurring in distinct clauses are distinct, there is ONE substitution, μ, such that $\mathcal{G}$ is the set of all $C\mu$ for C in $\mathcal{J}$. Since what μ does to variable not occurring in $\mathcal{J}$ is irrelevant, we may modify it on these variables to make it replete. Then by Lemma 7, rconsl!($\mathcal{J}$), and hence rcons!($\mathcal{S}$) is ground inconsistent.\

4.5 Equality Axioms and Skolem Functions

In practice, our set $\mathcal{S}$ of clauses usually is obtained, via Skolemization and other syntactical transformations (see Section 1.5), from a set of sentences in ordinary predicate logic. Up to now, this has not been relevant, since we dealt directly with $\mathcal{A}$ without caring about how it was obtained. In particular, our theory has made no distinction between the function symbols used in the original sentences and the new Skolem functions introcuced in obtaining $\mathcal{A}$. However, there is an important distinction which we examine in this section, namely, it is never necessary to use equality axioms for the Skolem functions.

We begin with some remarks on the role of equality axioms in ordinary predicate logic. It is possible to treat the symbol $=$ as just another binary relation symbol, provided that in our structures, we allow $=$ to be interpreted as an arbitrary binary relation; call such a structure a WEAK STRUCTURE. With this in mind, let EQ be the sentence that states that $=$ is an equivalence relation. Let $\mathcal{A}$ be a set of sentences in a language $\mathcal{L}$. If f is an n-place function symbol, let EE'(f) be the sentence

$$\forall x_1 \ldots \forall x_n \forall y_1 \ldots \forall y_n ((x_1 = y_1 \ \& \ \ldots \ \& \ x_n = y_n) \longrightarrow$$
$$f(x_1 \ldots x_n) = f(y_1 \ldots y_n)).$$

If P is an n-place predicate symbol, let EE'(P) be the sentence

$$\forall x_1 \ldots \forall x_n \forall y_1 \ldots \forall y_n ((x_1 = y_1 \ \& \ \ldots \ x_n = y_n) \longrightarrow$$
$$P(x_1 \ldots x_n) \longleftrightarrow P(y_1 \ldots y_n)).$$

Call a WEAK MODEL for $\mathcal{A}$ any weak structure satisfying all the sentences in $\mathcal{A}$, whereas a STRONG MODEL is one in which $=$ is interpreted as real equality. Let $\mathcal{EE}'$ be the set containing EQ plus all the EE'(f) and EE'(P) for f and P in $\mathcal{L}$. If $\mathcal{EE}'$ is a subset of $\mathcal{A}$, then $\mathcal{A}$ has a weak model iff $\mathcal{A}$ has a strong model, since in any weak model, the equivalence relation defined by the interpretation of $=$ may be factored out to produce a strong model. Also $\mathcal{A}$ may be skolemized to produce a set $\mathcal{J}$ of universal sentences in a larger language, $\mathcal{L}^+$; so $\mathcal{L}^+$ contains some function symbols not present in $\mathcal{L}$. Then $\mathcal{A}$ has a weak model iff $\mathcal{J}$ has a weak model, and $\mathcal{A}$ has a strong model iff $\mathcal{J}$ has a strong model; the proof here has nothing to do with the presence or absence of equality axioms; we simply note that any model (weak or strong) for $\mathcal{J}$ is a model for $\mathcal{A}$, whereas any model for $\mathcal{A}$ can be made into a model for $\mathcal{J}$ by appropriate choice of interpretations for the Skolem functions. Thus, if $\mathcal{E}\mathcal{E}'$ is a subset of $\mathcal{A}$, we have that $\mathcal{A}$ has a weak model iff $\mathcal{J}$ has a weak model iff $\mathcal{A}$ has a strong model iff $\mathcal{J}$ has a strong model. Observe that $\mathcal{E}\mathcal{E}'$ contains only equality axioms for the symbols of $\mathcal{L}$, and not for the new Skolem functions of $\mathcal{L}'$.

It thus follows, in the general framework of resolution, that one never needs to use an equality axiom, EE'(f), when f is a Skolem function. This is also true in the system discussed here, but requires some additional discussion, since we do not have a symbol for $=$, and we expressed the equality axioms $\mathcal{E}\mathcal{E}$ in terms of $<$ and $\leq$; the axiom EQ is never used, as it is subsumed in the transitivity of $\leq$.

A PREORDER on a set A is any relation, $\leq$, on A, which is transitive and reflexive. Such a preorder defines an associated equivalence relation on A, $\sim$, defined by $x \sim y$ iff $x \leq y$ and $y \leq x$, and it defines an associated partial order, $<$, defined by $x < y$ iff $x \leq y$ and not $y \leq x$. In general, $x \leq y$ does NOT imply that $x < y$ or $x = y$, since we may have distinct x and y with $x \sim y$.

Let $\mathcal{L}$ be a language in ordinary predicate logic such that $\mathcal{L}$ contains the binary relation symbols $=$, $<$, and $\leq$, plus some (possibly empty) set of function and constant symbols; we now view $=$ as a non-logical symbol. Let $\mathcal{A}$ be a set of sentences of $\mathcal{L}$. A WEAK PREORDERED MODEL for $\mathcal{A}$ is a structure $\mathfrak{A}$ which satisfies $\mathcal{A}$ and in which $\leq$ is interpreted as a preorder and $=$ and $<$ are interpreted as the associated equivalence relation and partial order respectively. A WEAK PRE-TOTALLY-ORDERED MODEL for $\mathcal{A}$ is a weak preordered model that satisfies $\forall x \forall y(x \leq y \vee y \leq x)$; i.e., $\leq$ defines a total ordering in the usual sense on the equivalence classes. Likewise, a WEAK PRE-DENSE-TOTALLY-ORDERED MODEL is a weak pre-totally-ordered model in which $\leq$ defines a dense total ordering without endpoints on the equivalence classes. If we drop the "weak", we get the corresponding notions of an ordered model, totally ordered model, and dense-totally-ordered model, in which the equivalence relation corresponding to $\leq$ (i.e., the interpretation of $=$) is required to be identity. Such a model may be obtained from a weak model by passing to the set of equivalence classes, PROVIDED that the functions of $\mathcal{L}$ factor through the equivalence relation. We thus have the following definition and lemma.

If f is an n-place function symbol, let EE'(f) be the universal closure of the clause EE(f) (see Section 2.7); so the only distinction between EE and EE' is that EE is a clause whereas EE' is the corresponding sentence in predicate logic. Let EE'($\mathcal{L}$) be the set of all EE'(f) for f in $\mathcal{L}$. Let EE($\mathcal{L}$) be the set of all clauses EE(f) for f in $\mathcal{L}$.

LEMMA 1: Let $\mathcal{A}$ be any set of sentences in $\mathcal{L}$. Then $\mathcal{A}$ has an ordered model iff $\mathcal{A} \cup$ EE'($\mathcal{L}$) has a weak pre-ordered model. Likewise for totally ordered models and dense totally ordered models.\

Now, the herbrandization procedure described in Section 1.5 obtained from $\mathcal{A}$ a set herb($\mathcal{A}$) of clauses, expressed in a language herb($\mathcal{L}$) equal to $\mathcal{L}$ plus some Skolem functions. Besides purely logical manipulations, this procedure involved the equivalence of a = b with (a $\leq$ b) & (b $\leq$ a), which is valid in all weak pre-ordered models, and the equivalence of a < b with $\neg$ (b $\leq$ a), which is valid in all weak pre-totally-ordered models. Thus,

LEMMA 2: Let $\mathcal{A}$ be any set of sentences in $\mathcal{L}$. Then $\mathcal{A}$ has a weak pre-totally-ordered model iff herb($\mathcal{A}$) does. Likewise for weak pre-dense-totally-ordered models.\

Since herb(EE'($\mathcal{L}$)) is just EE($\mathcal{L}$), we may apply Lemmas 1 and 2 to get

LEMMA 3: Let $\mathcal{A}$ be any set of sentences in $\mathcal{L}$. Then $\mathcal{A}$ has a totally ordered model iff herb($\mathcal{A}$) $\cup$ EE($\mathcal{L}$) has a weak pre-totally ordered model. Likewise for dense totally ordered models.\

Observe that there is no need here for the equality axioms applied to the Skolem functions. We now proceed to see what the results of this paper say in the context of weak models.

In the discussion of ground models (see Section 3.1), little is changed, since we are regarding terms as constant symbols. Thus, call a WEAK GROUND MODEL for $\mathcal{A}$ a quadruple ($\mathcal{Q}$, <, $\leq$,F), where $\leq$ is a preorder and < is the associated partial order, and F is as in the definition of Section 3.1. If $\sim$ is the associated equivalence relation, we may form ($\mathcal{Q}$, <, $\leq$,F)/$\sim$.

LEMMA 4: If ($\mathcal{Q}$, <, $\leq$, F) is a weak ground model for $\mathcal{A}$, then ($\mathcal{Q}$, <, $\leq$, F)/$\sim$ is a ground model for $\mathcal{A}$.\

This passage to the quotient structure was used also in the proof of the ground completeness theorem (see Lemma 1 of Section 3.3).

LEMMA 5: $\mathcal{A}$ has a ground model iff $\mathcal{A}$ has a weak ground model iff $\mathcal{A}$ has a weak ground model in which F is 1-1.

PROOF: Any weak ground model can be factored into a ground model by Lemma 4. Conversely, given any ground model, $(\mathcal{B}, <, G)$, we may define a weak ground model $(\mathcal{Q}, <, \leq, F)$ in which $\mathcal{Q}$ is the set of terms occurring in $\mathcal{A}$ and F is the identity.\

We may also state the analogue of Herbrand's theorem. It is actually somewhat easier here, since when F is 1-1, we do not have to worry about whether the interpretations of the function symbols are well-defined. As in our previous Herbrand theorem, we do need to observe that any preorder may be extended to a pre-total order.

LEMMA 6: If subinst($\mathcal{A}$) is ground consistent, then $\mathcal{A}$ has a weak pre-totally-ordered model.\

We now examine the completeness results in Section 4.4 in our new context. The main theorem remains unchanged, since the notions of rcons! and ground inconsistency have not changed. The corollary, modified for weak models, becomes,

LEMMA 7: If $\mathcal{A}$ has no weak pre-dense-totally-ordered model, then rcons!($\mathcal{A}$) is ground inconsistent.

PROOF: Exactly as in Section 4.4, but without the mention of EE. In particular, $\mathcal{DTO}$ (see Section 4.2) now contains only 4 clauses.\

The following theorem now says that one never needs equality axioms for the Skolem functions.

THEOREM: Let $\mathcal{S}$ be any set of sentences in $\mathcal{L}$ such that $\mathcal{S}$ has no dense totally ordered model. Then rcons!(herb($\mathcal{S}$) $\cup$ EE($\mathcal{L}$)) is ground inconsistent.

PROOF: By Lemma 3, herb($\mathcal{S}$) $\cup$ EE($\mathcal{L}$) has no weak pre-dense-totally-ordered model, so Lemma 7 applies.\

5. Conclusion

5.1 Comparison of Theory and Practice

The actual system of Bledsoe and Hines [BH] is somewhat more complicated than what we have described here. It employs a number of heuristics designed to obtain deductions more quickly. In some cases, these heuristics lead to incompleteness.

One additional rule applied is that of reduction. If C is any clause, let REDUCE(C) be the result of deleting from C all literals of the form $\alpha < \alpha$. In the system of [BH], each C, as it is obtained, is immediately replaced by REDUCE(C). It is not hard to see that this rule leaves the system complete.

Another rule involves the elimination of tautologies. C is called a TAUTOLOGY iff C is valid in all totally ordered structures. Since a tautology "says nothing", it is tempting to throw out tautologies as they are produced; equivalently, to allow an application of a proof rule only when the resultant clause is a tautology. Observe that it is decidable, by a transitive closure algorithm, whether C is a tautology; however, the actual system of [BH] only looks for tautologies of a very special kind, for which the decision procedure is quicker. Say C is a type 1 tautology iff C contains a literal of the form $\alpha \leq \alpha$. Say C is type 2 iff C contains literals of the form $\alpha @_1 \beta$ and $\beta @_2 \alpha$, where at most one of $@_1$ and $@_2$ is $<$. It is easily seen that the system remains complete if it is modified to throw out type 1 tautologies as they are produced. The system of [BH] throws out both type 1 and type 2 tautologies, however, and this leads to incompleteness, as the following example shows.

Let $\mathcal{S}$ be the set of

$$c \leq f(x) \lor d \leq f(x) \lor e \leq f(x)$$

$$f(x) < c \lor f(x) < d \lor f(x) < e$$

$$d \leq c$$

$$c \leq d$$

$$e \leq c$$

$$c \leq e.$$

Then $\mathcal{S}$ is ground inconsistent. However, with our restrictions, every possible chaining on $\mathcal{S}$ results in a type 2 tautology.

Another feature of the [BH] system is that variable elimination is mandatory. Thus, as soon as C is produced, C is replaced by a clause, D, obtained by applying variable elimination to C until all eligible variables are removed. This seems reasonable since C and D are equivalent in all dense total orders without endpoints, and D is simpler. However, we do not know whether the system remains complete with this rule.

5.2 An Example

We present here an example which illustrates some of the points in this paper. The example expresses the fact that if a and c are real numbers, there is no continuous f with $f(a) < c$ and $\forall x > a(f(x) \geq c)$. Clauses (1) and (2) below express this property of f; here, a and c are constant symbols. Continuity is expressed by

$$\forall x\ \forall y_1\ \forall y_2\ \exists x_1\ \exists x_2\ \forall v(y_1 < f(x) < y_2 \longrightarrow$$

$$x_1 < x < x_2\ \&\ (x_1 < v < x_2 \rightarrow y_1 < f(v) < y_2)),$$

which becomes clauses (3) to (6) below; (7) is the equality axiom for f. As we pointed out in Section 4.5, it is not necessary to postulate equality axioms for the Skolem functions L or R. So, we shall derive $\square$ from:

1. $f(a) < c$
2. $x \leq a \ \vee\ c \leq f(x)$
3. $f(x) \leq y_1 \ \vee\ y_2 \leq f(x) \ \vee\ L(x,y_1,y_2) < x$
4. $f(x) \leq y_1 \ \vee\ y_2 \leq f(x) \ \vee\ x < R(x,y_1,y_2)$
5. $f(x) \leq y_1 \ \vee\ y_2 \leq f(x) \ \vee\ v \leq L(x,y_1,y_2) \ \vee$

   $R(x,y_1,y_2) \leq v \ \vee\ y_1 < f(v)$
6. $f(x) \leq y_1 \ \vee\ y_2 \leq f(x) \ \vee\ v \leq L(x,y_1 . y_2) \ \vee$

   $R(x,y_1,y_2) \leq v \ \vee\ f(v) < y_2$
7. $x < y \ \vee\ y < x \ \vee\ f(x) \leq f(y)$

This example serves to illustrate a number of points. First, it requires the order to be dense; that is, (1) - (7) can be satisfied in non-dense total orders; thus, variable elimination (VE) would be required in the derivation of $\square$ even if we dropped all restrictions on chaining. Second, it will serve to illustrate how the inductive proof of completeness can be used (by humans) to construct a derivation of $\square$; if one has in mind an informal proof of contradiction, an examination of that proof will yield an inconsistent set of ground instances of the given clauses, and then the procedure of eliminating shielding terms, starting from the ones whose ground instances have greatest height will produce a derivation of $\square$. Finally, this example illustrates the level of difficulty with which provers based on the present system can be expected to cope. It was tried on the prover set up at Austin by Bledsoe and Hines (see [BH]). Note that clauses (5) and (7) are not needed for the proof; when the prover was given the list with (5) and (7) deleted, it found a proof of $\square$, but when it was given the entire list, it failed.

We now give an informal mathematical proof that (1) - (7) are inconsistent. This proof requires that the order is dense and without endpoints. For such an order, we may, as in the discussion of $\mathcal{SJO}$ in Section 4.2, introduce functions p and r so that for any x we have $p(x) < x$, and whenever $x < y$, we have $x < r(x,y) < y$. Now, by (1), $p(f(a)) < f(a) < c$. So by (3) and (4), $L(a,p(f(a)),c) < a < R(a,p(f(a)),c)$. Then (6) gives $f(r(a,R(a,p(f(a)),c)) < c$, contradicting (2), since $a < r(a,R(a,p(f(a)))$.

This informal proof tells us what the inconsistent ground instances of (1)-(7) are; of course, these are not inconsistent alone, but they are inconsistent if put together with some instances of $\mathcal{SJO}$. Then when we follow our procedure of eliminating first the shielding terms whose ground instances are of greatest height, this translates into some variable eliminations if those shielding terms begin in p or r. We display these ground instances after the :: following the clause.

1. $f(a) < c$

2. $x \leq a \vee c \leq f(x) :: r(a,R(a,p(f(a)),c))/x$

3. $f(x) \leq y_1 \vee y_2 \leq f(x) \vee L(x,y_1,y_2) < x ::$

   $a/x,\ p(f(a))/y_1, c/y_2$

4. $f(x) \leq y_1 \vee y_2 \leq f(x) \vee x < R(x,y_1,y_2) ::$

   $a/x,\ p(f(a))/y_1, c/y_2$

6. $f(x) \leq y_1 \vee y_2 \leq f(x) \vee v \leq L(x,y_1,y_2) \vee$

   $R(x,y_1,y_2) \leq v \vee f(v) < y_2$

   $::\ a/x,\ p(f(a))/y_1,\ c/y_2,\ r(a,R(a,p(f(a)),c))/v$

The indicated substitutions will lead to a set of ground clauses that is inconsistent with the following ground instances of $\mathcal{DIO}$:

a. $p(f(a)) < f(a)$

b. $a < r(a,R(a,p(f(a)),c))$

c. $r(a,R(a,p(f(a)),c)) < R(a,p(f(a)),c))$

The shielding term of greatest height in its ground instance is the f(x) from (2) and f(v) from (6), so we should chain (2) and (6), after which they will no longer be used:

8. $v \leq a \vee c < y_2 \vee f(x) \leq y_1 \vee y_2 \leq f(x) \vee$

$v \leq L(x,y_1,y_2) \vee R(x,y_1,y_2) \leq v$

(2 (v/x), 6).

:: z/x, $p(f(a))/y_1$, c/y_2, $r(a,R(a,p(f(a)),c))/v$

We now have (1), (3), (4), (8) as active clauses, and the shielding term of next greatest height is the v of (8), which becomes r(a,R(a,p(f(a)),c); since r comes from $\mathfrak{SJO}$, we eliminate it by VE. This corresponds to the ground step of chaining it against (b) and (c).

9. $r(x,y_1,y_2) \leq a \vee c < y_2 \vee f(x) \leq y_1 \vee$

$y_2 \leq f(x) \vee R(x,y_1,y_2) \leq L(x,y_1,y_2)$

(8, VE)

:: a/x, $p(f(a))/y_1$, c/y_2

We now have (1), (3), (4), (9) as active clauses, and the next shielding terms in ground instance are L(a,p(f(a)),c) and R(a,p(f(a)),c), so we eliminate them in order to get:

10. $R(x,y_1,y_2) \leq a \vee c < y_2 \vee f(x) \leq y_1 \vee$

$y_2 \leq f(x) \vee r(x,y_1,y_2) < x$

(3,9)

11. $f(x) \leq y_1 \vee y_2 \leq f(x) \vee x < a \vee c < y_2$

$r(x,y_1,y_2) < x$

(4,10)

12. $f(x) \leq y_1 \vee y_2 \leq f(x) \vee x < a \vee c < y_2$

(4,11) :: a/x, $p(f(a))/y_1$, c/y_2

In obtaining (12), we used REDUCE (see Section 5.1) to drop the $x \not< x$. Now (1) and (12) are active clauses, and the next shielding term in ground instance is $p(f(a))$, which is removed by VE. This corresponds to the ground step of chaining against (a).

13. $y_2 \leq f(x) \vee c < y_2 \vee x < a$ (12, VE) :: a/x, c/y_2

y_2 is likewise removed by VE to get to a situation where there are no eligible variables. This VE step does not correspond to an instance of $\mathcal{DLO}$, since it would be valid in any total order; we could have also obtained (14) from (13) by self-chaining on y_2, except that this is forbidden in our system because y_2 is not a shielding term.

14. $c \leq f(x) \vee x < a$ (13, VE)

Now we eliminate f(x) by chaining with (1)

15. $c < c \vee a < a$ (1, 14, a/x)

(15) is ground inconsistent. The actual prover would REDUCE it immediately to $\square$.

5.3 Possible Generalizations

The logician will recognize some features of the system described here which might apply in a more general context. In ordinary predicate logic, let $\mathcal{J}$ be the theory of dense total orders without endpoints, expressed in a language $\mathcal{L}$ consisting of symbols for $<$ and $=$. Then $\mathcal{J}$ is complete and decidable, and moreover admits quantifier elimination; that is, every formula of $\mathcal{L}$ is provably from $\mathcal{J}$ equivalent to a quantifier-free formula. Now, let $\mathcal{L}'$ be the language $\mathcal{L}$ with a finite set of function and constant symbols added, and let $\mathcal{J}'$ be $\mathcal{J}$ plus a finite set of new sentences of $\mathcal{L}'$. What we have presented, in the format of resolution is a prover for consequences of $\mathcal{J}'$ that uses the quantifier elimination for $\mathcal{J}$ to obtain a significant speedup over a pure predicate calculus prover applied to $\mathcal{J}'$.

We do not know the extent to which the results of this paper can be generalized to other theories that admit quantifier elimination. Some of our arguments certainly do, but other features, such as the specific definition of chaining and our restrictions on chaining shielding terms seem specific to total orders. The system of [BH] actually handles $+$ as well; that is, $\mathcal{J}$ is essentially the theory of densely ordered Abelian groups. One may neglect the fine points discussed in Section 5.1 and use this $\mathcal{J}$ to define rcons in direct analogy with Section 4. We do not know whether the analogous completeness result holds.

6. References

[AB] R. Anderson and W. W. Bledsoe, A linear format for resolution and a new technique for establishing completeness, J. ACM 17(1979) 525-534.

[BH] W. W. Bledsoe and Larry M. Hines, Variable elimination and chaining in a resolution-based prover for inequalities, Proc. 5th Conference on Automated Deduction, Les Arcs, France July 1980, Springer-Verlag.

[E] H. B. Enderton, A Mathematical Introduction to Logic, Academic Press, 1972.

[L] D. W. Loveland, Automated Theorem Proving, North-Holland, 1978.

[R] J. A. Robinson, Logic: From and Function, North-Holland, 1979.

[V] R. Vaught, Applications of the Löwenheim-Skolem-Tarski theorem to problems of completeness and decidability, Indag-Math 16 (1954) 467-472.