A Description for Calculations Related to the Poincare Conjecture

by

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POINCARE CONJECTURE: If M^3 is a connected compact, 3-manifold without boundary, and $\pi_1(M^3_3) = 1$ (i.e., M^3 is simply connected), then M^3 is homeomaophic to S^3.

This conjecture was originally given for k-manifolds, but all cases have been proved but k = 3. So we will consider only the 3-manifold case.

We will try to get ideas about proving (or disproving) the conjecture by looking at a number of examples. The natural way to proceed is to construct a number of examples of compact, simply connected, connected, 3-manifolds, without boundary, and

(1) Verify that each is indeed homeomorphic to S^3, or
    Verify that one is not (and thereby disprove the conjecture),

(2) Try to determine a pattern in these examples that will suggest a general proof,

(3) Try to generate conjectures, which if true, can act a lemmas to help in a general proof.

One problem with the above plan is that it is very difficult to construct examples of non-trivial, compact, simply connected, connected, 3-manifolds, without boundary. So we have tried to transform the problem into one in which such constructions are easier, and where computers can be employed to quickly generate many such examples.

FACT: Every compact, simply connected, connected, 3-manifold without boundary, M^3, can be represented as a handlebody H of some genus n (number of handles), together with a set $\{J_i\}_{i=1}^{n}$ of distinct simple closed curves on the boundary, Bd H, of H, for which Bd H $- \bigcup_{i=1}^{n} J_i$ is connected. This representation corresponds to writing $M^3 = H \cup \bigcup_{i=1}^{n} P_i \cup B^3$, where each P_i is a pillbox glued to H along an annular neighborhood of J_i and B is a 3-cell glued to $H \cup \bigcup_{i=1}^{n} P_i$ along the 2-sphere boundary of $H \cup \bigcup_{i=1}^{n} P_i$.

If n = 1, then H is a torus.

If n = 2, the H is a bonelike object, with holes in the ends.

<u>FACT</u>: Every compact, connected, 3-manifold, without boundary, M^3, can be <u>triangulated</u>, by images of tetrahedrons. (Since M^3 is a 3-manifold it is locally like three space, and since it is compact, it can be covered by a finite number of 3-space image "patches". Furthermore, we can assume that M^3 is covered by a finite number of compact patches whose interiors cover M^3.

Each of these compact patches is triangulated; the overlapping parts of the "tetrahedrons" (continuous images of tetrahedrons) are then adjusted (see) so that the result is a trangulatipon of all of M^3 by the tetrahedrons.

The one-dimensional skeleton of this triangulation (i.e., the edges of the tetrahedrons) is used to build the handlebody: the handlebody is just a neighborhood of this skeleton. (Such a neighborhood exists which preserves the basic structure of the skeleton, as can be seen by subdividing the tetrahedra in a special way and making the neighborhood of the J-skeleton a union of these smaller pieces (see).

The complement of the first handlebody is also a handlebody.

If we are given one of these handlebodies H, of genus n, and n curves on the boundary of H (satisfying certain conditions) we can discard the second handlebody (and recover it if needed). Those n curves correspond to the one-dimensional sides of some of the 2-dimensional faces of some tetrahedrons in the triangulation. (See).

Simply connected. Given the set $\{J_i\}_{i=1}^{n}$ of simple closed curves in the representation of M^3, one can determine whether M^3 is simply connected by checking whether a certain group is trivial. This will be explained below when we consider the case where H has genus 2.

Case $n = 2$

We want to consider first the special case where H has genus 2 (n=2). Our objective is to examine the curves J_1 and J_2 on H in the case when M^3 is simply connected, and hope to see some special features of J_1 and J_2 which will allow us to conclude that M is homeomorphic to S^3.

So we seek a method of representing curves on Bd H.

The first step is to represent a single simple closed curve J on Bd H. Any such curve J can be broken into pieces, namely the part of J on T_1 (the left hand torus), the part of J on Cy (the connecting cylinder), and the part of J on T.

has $(p,q) = (1, -1)$.

For T_i, $i = 1, 2$, let the collections be (p_{i1},q_{i1}), (p_{i2},q_{i2}), (p_{i3}, q_{i3}). (We will call these 'red arcs', 'white arcs', and 'blue arcs', respectively). (See p.).

We see that the boundary points for all these arcs, are interlaced around the curves γ_i at the ends of Cy. Thus boundary points (in γ_i) are divided into six subsets

$$A_i, B_i, C_i, A'_i, B'_i, C'_i ,$$

where all red arcs (those in the (p_{i1},q_{i1}) collection) go from A_i to A'_i, the white arcs go from B_i to B'_i, and the blue arcs go from C_i to C'_i. (See p.).

In the interesting cases, (Note: The only case where this cannot be done is when J can be pushed entirely onto one end.) J can be maneuvered into a neat position where $J \cap Cy$ consists of parallel straight arcs running from one end of Cy to the other and for each T_1, T_2, $J \cap T_i$ consists of arcs which fall into three collections wehre each collection of arcs is a collection of parallel arcs on T_i. Each collection is a (p,q) curve on T_i where (p,q) are relatively prime integers and p represents how many times longitudinally the curve goes around T_i and q is how many times meridionally.

For example, assuming a fixed orientation, the following arc

has (p,q) = (0,1), while the arc

has (p,q) = (3,2), and the arc

So any curve on Bd H in 'nice' position (i.e., straightened out this way) can be specified by:

* The pairs (p_{ij},q_{ij}), i = 1, 2; j = 1, 2, 3 (actually, I believe that once you have settled on (P_{13},q_{13}) is determined.)
* the number of points a_i, b_i, c_i, in A_i, B_i, C_i, (i.e., $a_i = |A_i|$, $b_i = |B_i|$, $c_i = |C_i|$,
* and how the points in A_i, B_i, C_i, A'_i, B'_i, C'_i, are connected to A_i, B_i, C_i, A'_i, B'_i, C'_i, by arcs in Cy.

We also let $N_i = a_i + b_i + c_i$, the number of arcs on torus T_i. Note: $N_1 = N_2$.

Similarly, if we have two or more simple closed curves J_1, J_2, ... J_i, on H, which do not intersect, then $(J_1 \cup J_2 \cup \ldots \cup J_j) \cap$ Cy consists of straight arcs in Cy from γ_1 to γ_2, and for i - 1, 2, $(J_1 \cup J_2 \cup \ldots \cup J_j) \cap T_i$, consists of arcs which fall in the three collections (red, white, and blue) mentioned above. The boundary points of these arcs can again be divided into six subsets

A_i, B_i, C_i, A'_i, B'_i, C'_i ,

where all arcs in the red, (p_{i1},q_{i1}), collection go from A_i to A'_i, etc. (The case we are interested in is when there are just two curves J_1 and J_2, i.e., j = 2. But the process we use sometimes results in the other cases which are discarded.)

Example: $N = 4$, $a_1 = 2$, $b_1 = 1$, $c_1 = 1$, $a_2 = 1$, $b_2 = 1$, $c_2 = 2$, $s = 0$ (The "shift", in going from γ_1 to γ_2).

Here there are two curves (j = 2), J_1 and J_2. (See p.).

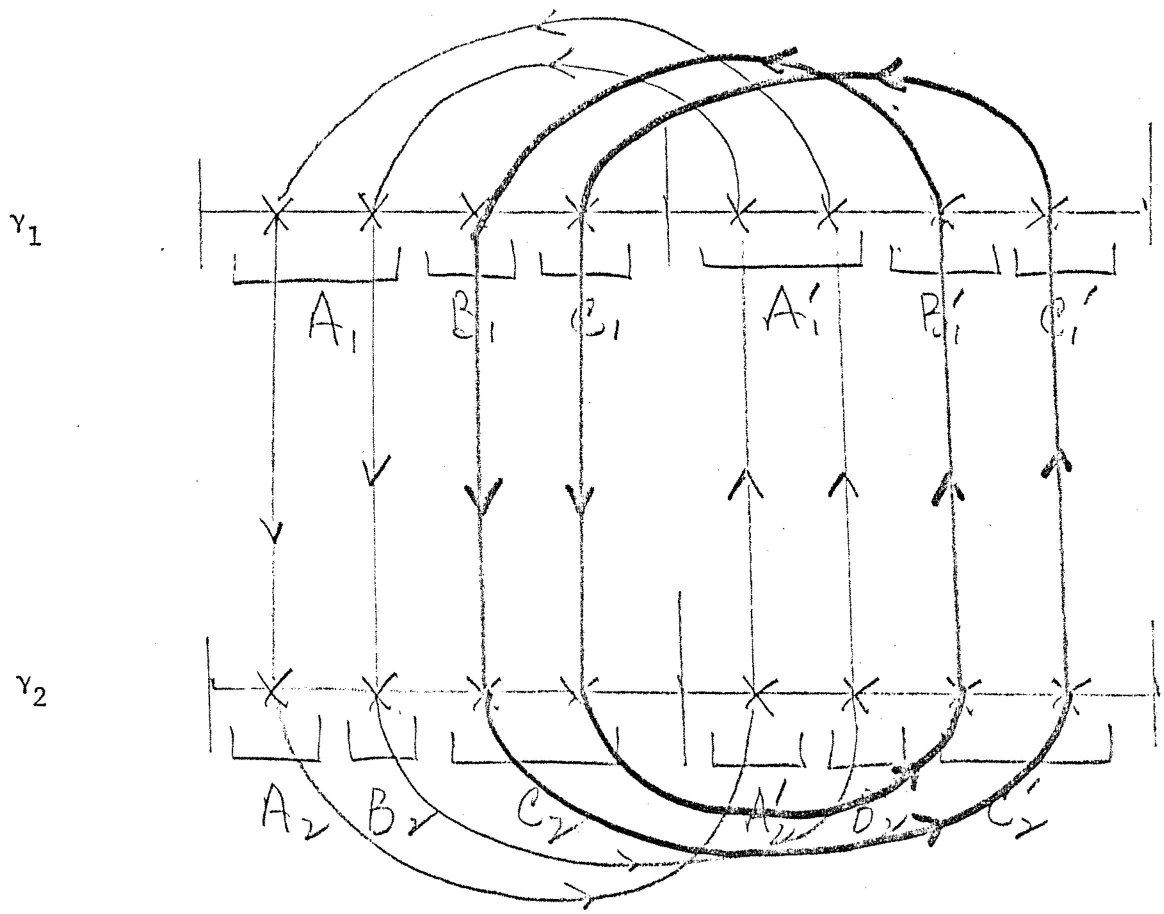
Example: $N = 8$, $a = 3$, $b = 2$, $c = 3$, $a = 4$, $b = 3$, $c = 1$, $s = 2$.

Here again there are two curves, J_1 and J_2. (See p.).

__Example.__ $N = 4,\ A = 2,\ b = 1,\ c = 1$

$A' = 1,\ b' = 1,\ c' = 2$

$s = 0$ (the "shift", in going from γ_1 to γ_2)

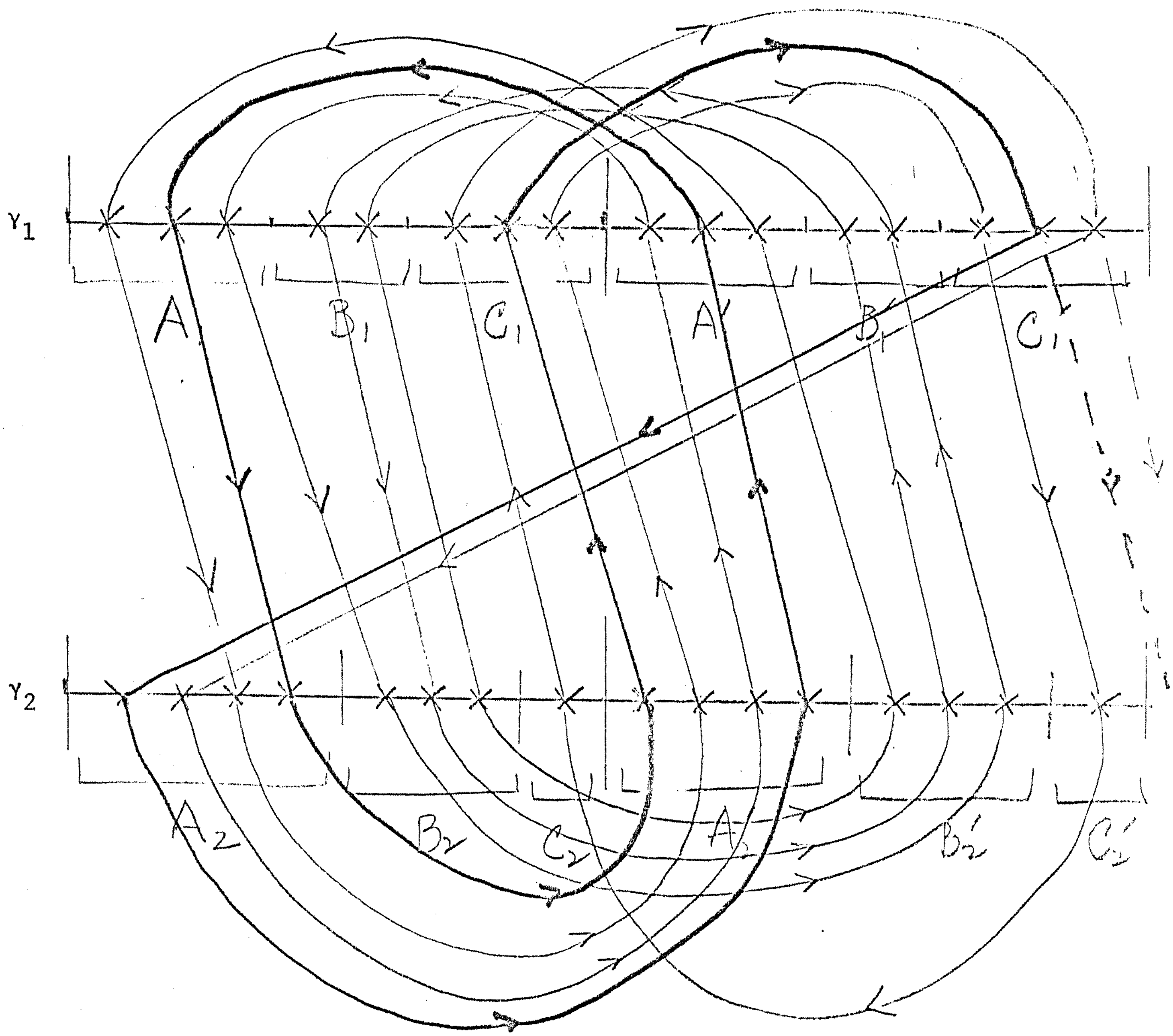


Here there are two curves ($j = 2$), J_1 and J_2.

Example. $N = 8,\ a = 3,\ b = 2,\ c = 3$

$a' = 4,\ b' = 3,\ c' = 1$

$s = 2$ (the "shift", in going from γ_1 to γ_2)



Here there are two curves ($j = 2$), J_1 and J_2.

Every simply connected, compact, 3-manifold, without boundary, and with genus 2, can be represented with curves with the following additional property: J_1 goes longitudinally around T_1 once (in total), and around T_2, zero (in total). More precisely, any curve J_1 might enter or leave any one of the sets A_1, B_1, C_1, and any one of the sets A_2, B_2, C_2; the above condition requires that $S_{11} = S_{22} = +1$, and $S_{12} = S_{21} = 0$, where the sums S_{11}, S_{12}, S_{21}, S_{22}, are defined as follows: If $J_1 \cap T_1$ has

k_1 arcs from A_1 to A'_1, and

k_{-1} arcs from A'_1 to A_1, and

k_2 arcs from B_1 to B'_1, and

k_{-2} arcs from B'_1 to B_1, and

k_3 arcs from C_1 to C'_1, and

k_{-3} arcs from C'_1 to C_1,

then put $S_{11} = (k_1 - k_{-1})p_{11} + (k_2 - k_{-2})p_{12} + (k_3 - k_{-3})p_{13}$.

And if $(J_1 \cap T_2)$ has

k_1 arcs from A_2 to A'_2, and

k_{-1} arcs from A'_2 to A_2, and

etc.,

put $S_{12} = (k_1 - k_{-1})p_{21} + (k_2 - k_{-2})p_{22} + (k_3 - k_{-3})p_{23}$.

Similarly for S_{21} and S_{22}. Right.

In addition to this requirement that

$$S_{11} = S_{22} = \pm 1 , \quad S_{12} = S_{21} = 0 ,$$

we hope to show that, without loss of generality, that the curve J_1 meet all three colors in T_1 and J_2 meet all three colors in T_2. (For otherwise, the conjecture would be trivially true).

These are severe constraints, of course, but they do not yet guarantee that we have a manifold satisfying the property that M^3 is simply connected. To check for simple connectivity we need to know that the group

$$\{a,b:\ \underbrace{a^{p_1} b^{-l_2} a^{-l_1} b^{p_2} a^{l_1} \ldots = 1}_{\text{condition on } J_1},\ \underbrace{a^{-l_1} b^{l_2} a^{p_1} b^{-l_2} \ldots = 1}_{\text{condition on } J_2}\}$$

is the trivial group. (Each p and l, in the above is equal to ± some P_{ij}. The a corresponds to T_1 and the b to T_2).

Let us recall here, the definition of the fundamental group in the case of a 3-manifold of genus 2 (i.e., the case we are considering). Let H be a handlebody of genus 2, with holes H_1 in T_1 and H_2 in T_2 .

We place on Bd H around H_1 and H_2 two simple loops a and b, with orientation.

Each curve J, like the one pictured below, corresponds to an element of the fundamental group $\pi,(M^3)$.

In the above example J has 3 arcs on each side. Starting from the point labeled 0 we go twice around T_1, so the word that J represents is π_1 (the handlebody) has the form

$a^2b^1a^{-1}b^{-2}a^{-1}b$

in general a curve J has the form

(1) $a^{P_{1i_1}} b^{P_{2i_2}} a^{P_{1i_3}} b^{P_{2i_4}} \ldots a^{P_{1i_k}} b^{P_{2i_{(k+1)}}}$.

The two curves on Bd H yield the two relators for $\pi_1(M^3)$. So

$$\pi_1(M^3) = \{a,b: (\quad) = 1, (\quad) = 1\}$$

where ( ) are filled in by strings corresponding to J_1 and J_2 as in (1).

As was mentioned earlier we wish to construct examples of compact, simply connected, connected, 3-manifolds, without boundary, and use these examples to make conjectures that might help prove or disprove the Poincare conjecture. We will try to do this, for the genus 2 case, using our new representation. This is done by producing two curves, J_1 and J_2.

Recall that

N = the number of arcs on T_1 (= the number of arcs on T_2)

$a_1 = |A_1|$ = the number of "red" arcs on T_1 ,

$b_1 = |B_1|$ = the number of "white" arcs on T_1 ,

$c_1 = |C_1|$ = the number of "blue" arcs on T_1 ,

$a_2 = |A_2|$ = the number of "red" arcs on T_2 ,

$b_2 = |B_2|$ = the number of "white" arcs on T_2 ,

$c_2 = |C_2|$ = the number of "blue" arcs on T_2 ,

$N = a_1 + b_1 + c_1 = a_2 + b_2 + c_2$.

Let s = the "shift", as the curves cross the cylinder Cy from γ_1 to γ_2 (See examples in). Note that when we select values for (n, a_1, b_2, a_2, b_2, s), this determines the values of c_1 and $-c_2$.

Given a particular set of values for the parameters N, a_1, b_1, ... s, a computer calculation can be made to determine the number j of the curves

$$J_1, J_2, \ldots, J_j$$

on Bd H, and cylindrical information of each, such as sequencing through the sets A_1, B_1, C_1, A'_1, B'_1, C'_1, A_2, B_2, C_2, A'_2, B'_2, C'_2 (See Fig.). This behavior of these J's on the cylinder is independent of their precise behavior on the tori, which is determined by the parameters (p_{ij}, q_{ij}), $i = 1, 2$; $j = 1, 3$.

We propose the following procedure:

PROCEDURE:

1. Select values for (N, a_1, b_1, a_2, b_2, s) for which $j = 2$,

Not complete

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