An inductive solution to a domino tiling problem.

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Abstract:

An inductive method of covering a mutilated chessboard (with two cells of *opposite* colors removed) with 2 x 1 dominoes is shown.

Problem Description:

Consider a matrix that has an even number of rows - say 2 x M - and an arbitrary number of columns N, N > 1. The cells of such a matrix can be colored black or white, where each black cell has only white cells as neighbors, and vice versa; a chessboard obeys such a coloring. An arbitrary white cell and a black cell have been designated open; the remaining cells are designated closed.

A domino is a device that can cover two adjacent cells, i.e. cells in the same row or same column. The problem is to show dominoes can be placed on the given matrix in such a way that a single domino covers every closed cell and no open cell is covered at all.

It is instructive to note that a domino covers two cells of opposite colors. Therefore, if two black cells (or two white cells) are left open, then no covering is possible because among the closed cells there will be more whites than blacks. Also, if the number of columns, N, is equal to 1 then there is no covering in general, for instance with 4 rows when the cells in the second and third rows are designated open, there is no possible covering.

A related problem can be found in page 47 of "Mathematics for the Young and Old", *Halmos*, Mathematical Association of America. In the solution provided, the method shown was non-inductive and was specific to the standard 8 x 8 chessboard.

An inductive strategy for covering:

We give an inductive proof, based on the number of rows, for the existence of a covering. First, we establish an elementary result.

Definition: A path is a sequence of cells such that adjacent cells in the path are neighbors. An even path has an even number of cells.

From the definition, adjacent cells in a path belong to the same row or same column, so they can be covered by a domino. Since an even path has an even number of cells, all its cells can be covered by an appropriate placement of dominoes, a fact that we state formally below.

Lemma (Even path): Any even path can be covered.
Note: All the cells of a matrix that has an even number of cells are on an even path.

Theorem: A matrix with 2 x M rows, M ≥ 1 and N columns N> 1, in which there is an open black cell and open white cell can be covered

Proof: Proof is by induction on M.

Case M=1: The closed cells form two even paths, one possibly empty (see Figures 1 and 2). From the lemma, each even path can be covered; therefore, the matrix can be covered as required.

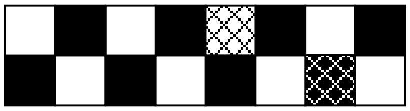


FIGURE 1: Two non-empty even paths.

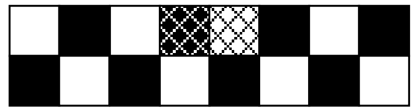
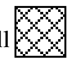


FIGURE 2: One empty and one non-empty even path.

 Open black cell  Open White cell

Case M>1: Divide the matrix into two parts, the upper two rows that we call <u>upper</u> and the remaining portion that we call <u>lower</u>. If both the open cells belong to upper then upper can be covered as shown above for M=1, and the cells in lower belong to an even path (see *Note*); therefore, they can be covered.

Similarly if both open cells belong to lower then lower can be covered inductively; and upper, being an even path, can be covered. Hence we need only consider the case when one open cell - say a black one - belongs to upper and the other open cell belongs to lower.

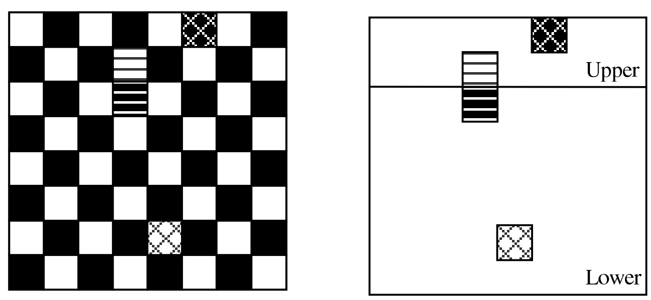
Pick an arbitrary white cell in the second row and call it <u>half-open</u>; Since N > 1 such a choice is always possible. Its adjacent black cell in lower is also designated <u>half-open</u>; See Figure 3.

Open black cell.

Open white cell.

Half-open black cell.

Half-open white cell.



If we regard each half-open cell as open, upper can be covered (see the case M=1) and lower can be covered inductively. Next cover the two half-open cells using a single domino.

<u>Acknowledgement:</u>

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