Cores and Connectivity in Sparse Random Graphs

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Abstract

We consider k-connectivity in a sparse random graph with a specified degree sequence. When dealing with sparse random graphs, properties that require connectivity are most appropriately phrased in terms of a giant subgraph that satisfies that property since a sparse random graph is generally not connected. Here a giant subgraph is one that contains a constant fraction of the vertices in the original graph.

We obtain a tight threshold on the existence of a giant k-vertex or k-edge connected subgraph for $k \geq 3$ in a sparse random graph drawn from $\mathcal{G}_{n,p}$. Although k-connectivity in $\mathcal{G}_{n,p}$ has been widely studied in the literature, results for higher connectivity have applied mainly to the case when $p \geq \ln n/n$, which makes it likely that the graph is connected, and very little is known about higher connectivity in sparse $\mathcal{G}_{n,p}$.

The k-core of a graph is the maximal induced subgraph with minimum degree k. The key tool in the derivation of our connectivity results is our recent theorem and proof strategy for the giant k-core threshold in sparse random graphs with a specified degree sequence.

A degree sequence exhibits a power law if the number of vertices of degree i is proportional to $\frac{1}{i^\beta}$, for a suitable constant β. We establish that for every $k \geq 3$, a random power-law graph has a giant k-core if $2 < \beta < 3$, and it has no giant 3-core if $\beta \geq 3$. Thus for $\beta \geq 3$ our results establish that a random power-law graph has no giant 3-connected subgraph. For $2 < \beta < 3$ we derive a weaker result, that any $(k-1)$-separator in the k-core, must separate a set of size at most n^c where $c < 1$ depends on β.

Finally, the existence of a giant k-core in a random graph is related in an informal way to the probability that the genealogy tree of a certain branching process contains a perfect infinite $(k-1)$-ary tree. We provide a solution to this latter problem in terms of probability generating functions. This result is tangential to the rest of our paper, but may be of independent interest.

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1 Introduction

We consider vertex- and edge-connectivity in graphs that are drawn uniformly at random conditional on a given degree distribution $\mathcal{D}$ that satisfies certain properties. We consider degree distributions that are *1-smooth* (roughly speaking this means that the graph is sparse, i.e., has a number of edges linear in the number of vertices), that have maximum degree $o(n^{1/3})$, and that have the second moment of the degree distribution bounded by either $O(n^{1/2-\epsilon})$ or by a constant (this latter property is called 2-smoothness). This model includes sparse graphs with Poisson degree distributions which are related to the classical random graphs $\mathcal{G}_{n,p}$ and $\mathcal{G}_{n,m}$, as well as random 'power law' graphs, which are governed by parameters α and β: a random power-law graph is chosen uniformly from all graphs with $y = \left\lfloor e^{\alpha}/x^{\beta} \right\rfloor$ vertices of degree x [1].

A sparse random graph is generally not connected. Thus questions relating to any property that requires connectivity need to be phrased in terms of the existence of a large subgraph with that property. In this paper we consider the existence of a *giant* subgraph with a given property, where a giant subgraph is one that contains a constant fraction of the vertices in the original graph.

We give a sharp threshold for the existence of a giant k-edge connected or k-vertex connected subgraph in sparse $\mathcal{G}_{n,p}$ for $k \geq 3$. Although vertex connectivity in $\mathcal{G}_{n,p}$ has been widely studied, most of these results have applied to the case when $p \geq \ln n/n$, which makes it likely that the graph is connected, and very little is known about higher connectivity in sparse $\mathcal{G}_{n,p}$.

The k-core of a graph is the maximal induced subgraph with minimum degree k. In a recent unpublished manuscript [6], we found conditions under which the k-core of a random graph with a 1-smooth degree sequence almost surely contains a constant fraction of the graph's vertices. The results from [6] are key tools used in the derivation of our results on vertex- and edge-connectivity. In particular, for $\mathcal{G}_{n,p}$ and other 2-smooth random graphs we show that the giant k-core of such a graph, if it exists, is almost surely k-connected.

The threshold for giant k-core was solved earlier by Pittel, Spencer, and Wormald [14] for random graphs drawn from $\mathcal{G}_{n,p}$, using a fairly involved proof. Recently the k-core problem has been studied for random hypergraphs in [13]. Very recently we have been informed that results in an unpublished manuscript [4] on random hypergraphs also provide results similar to those in our unpublished k-core paper [6].

We also consider power-law graphs. A degree sequence exhibits a power law if the number of vertices of degree i is proportional to $\frac{1}{i^{\beta}}$, for a suitable constant β. Random graphs with power law degree sequences are of some interest, since graphs that occur in the real world, including the web graph, phone-call graphs, networks of molecules, and networks of social interaction, often exhibit a power law degree sequence. Although it is unlikely that real-world graphs are accurately modeled by random graphs, or for that matter, by graphs which precisely obey a power law, it is nevertheless of interest to know what can be said about random graphs that obey degree sequence similar to several real-world graphs.

For $\beta > 2$ the power-law degree sequence is 1-smooth, and for $\beta > 3$ it is 2-smooth as well. We apply our k-core theorem from [6] to establish that for every constant $k \geq 3$ a random power-law graph almost surely contains a giant k-core if $2 < \beta < 3$, and it almost surely does not contain a giant 3-core if $\beta \geq 3$. We use these results to derive some results on k-connectivity in random power-law graphs.

Finally, the k-core threshold is a key tool in our results, and the existence of a giant k-core in a random graph is related in an informal way to the probability that the genealogy tree of a certain branching process [7] contains a perfect infinite $(k-1)$-ary tree. In this paper we provide a solution to the latter problem in terms of probability generating functions.

The rest of this paper is organized as follows. In section 2 we give some basic definitions and describe the configuration model (CM) for generating a random fixed-degree sequence graph. In section 3 we prove our theorem on the existence of an infinite complete r-ary subtree of a branching tree. We also describe in this section an informal connection between this problem and the k-core threshold, and state our k-core theorem (Theorem 3.2) from [6]. The complete proof of Theorem 3.2 has been included in the appendix for ease of reference since neither [6] nor [4] has been published. In section 4 we apply Theorem 3.2 to obtain the giant k-core thresholds for power-law distributions and for distributions with all convergent moments; this latter result allows us to rederive the giant k-core threshold for $\mathcal{G}_{n,p}$, which was first proved in [14]. Finally in section 5 we present our results on k-connectivity in random fixed-degree sequence graphs.

2 Preliminaries

2.1 Random Graph Definitions

We begin by providing definitions for random graphs with fixed degree sequences (see Molloy and Reed [11, 12]). A sequence $D = \{d_1, d_2, \ldots, d_n\}$ is *graphical* if the set Ω_D of (labelled) graphs with degree sequence D (i.e. such that the degree of the i'th vertex is d_i) is nonempty. If D is a graphical sequence, let $G(D)$ denote a uniformly distributed random element of Ω_D. Thus $G(D)$ is a *random graph with degree sequence D*.

An *asymptotic degree sequence $\mathcal{D}$* is an infinite sequence $D_1, D_2, \ldots$, where each $D_n = \{d_{n,1}, \ldots, d_{n,n}\}$ is a graphical sequence of length n. A *random graph with asymptotic degree sequence $\mathcal{D}$*, denoted by $G(\mathcal{D})$, is a sequence of random graphs $G(D_n)$. The random graph $G(\mathcal{D})$ has a property P *asymptotically almost surely* (*a.a.s.*) if the probability that $G(D_n)$ has property P converges to 1 as $n \to \infty$; $G(\mathcal{D})$ does not have property P *with exponentially high probability* (*w.e.h.p.*) if the probability that $G(D_n)$ has property P is $c^{-\Omega(n^\epsilon)}$, for some $c > 1$ and $\epsilon > 0$.

For any degree sequence D and any $k > 0$, we define the *kth moment of D*

$$M_k(D) = \frac{1}{n}\sum_{i=1}^{n} d_i^k. \tag{1}$$

An asymptotic degree sequence $\mathcal{D}$ is *k-smooth* if if there exists a sequence of real numbers $\lambda_0, \lambda_1, \ldots$ such that

$$\text{Condition S1:} \quad \lim_{n\to\infty} \frac{|\{j : d_j = i\}|}{n} = \lambda_i \text{ for all } i, \text{ and} \tag{2}$$

$$\text{Condition S2:} \quad \lim_{n\to\infty} M_k(D_n) = \sum_{i=0}^{\infty} i^k \lambda_i < \infty. \tag{3}$$

The sequence λ_i is the *limiting degree distribution* of $\mathcal{D}$. Throughout this paper, whenever a property of random graph with degree sequence D is described asymptotically, it is assumed that D is part of a 1-smooth asymptotic degree sequence.

If $\mathcal{D}' = \{D'_1, D'_2, \ldots\}$ is a sequence of random (degree) sequences, we say $\mathcal{D}$ is a.a.s. k-smooth if the convergences described by conditions S1 and S2 occur in probability; that is, if for every $\epsilon > 0$

$$P\left[\left|\frac{|\{j : d_j = i\}|}{|D'_n|} - \lambda_i\right| > \epsilon\right] \to 0 \tag{4}$$

and

$$P\left[\left|M_k(D_n) - \sum_{i=0}^{\infty} i^k \lambda_i\right| > \epsilon\right] \to 0. \tag{5}$$

Similarly, $\mathcal{D}'$ is k-smooth w.e.h.p. if these probabilities are exponentially small.

2.2 The Configuration Model.

It is difficult to directly examine random graphs with given degree sequences, so instead we use the *configuration model (or 'CM')* introduced by Bollobás [2]. For a degree sequence D, consider a set of n vertices and $\sum_i d_i$ *endpoints*, and assign d_i endpoints to the vertex v_i. Now choose a perfect matching of the endpoints uniformly at random, and for each pair of matched endpoints, draw an edge connecting the corresponding vertices.

This procedure generates a graph with degree sequence D; however, the graph may contain loops and/or multiple edges. We shall abuse notation and refer to such a random (multi-)graph as a *random graph with degree sequence D generated by the configuration model.* Definitions for asymptotic degree sequences generalize to the CM in the obvious way.

Under certain circumstances results about random graphs generated by the CM hold in general for random graphs with the same degree sequence [11, 12]. It is easy to see that every simple graph with degree sequence D occurs with the same probability using the CM. A result of McKay and Wormald [10] implies that if the maximum degree of a degree sequence is $o(M_1(D)^{1/3})$, then a random configuration produces a simple graph with probability $e^{-O\left(\frac{M_2(D)^2}{M_1(D)^2}\right)}$. If an asymptotic degree sequence $\mathcal{D}$ is 2-smooth, then this probability is $\Theta(1)$, and a.a.s. and w.e.h.p. results for the CM clearly generalize to random graphs in general. If $\mathcal{D}$ is 1-smooth, then a result which holds with probability $1 - e^{-\omega(M_2(D)^2)}$ using the CM implies an a.a.s. result for general random graphs. Furthermore, if $M_2(D) = O(N^\epsilon)$, then a result which holds with probability $1 - e^{-\Omega(n^{\epsilon+\epsilon'})}$ using the CM implies a w.e.h.p. result for general random graphs.

3 Random Graphs and Branching Processes.

3.1 A Theorem on Branching Processes.

A *branching tree* based on a probability distribution $\{\mu_i\}$ on the non-negative integers is a recursively defined random tree, in which the degree of the root vertex is distributed according to $\{\mu_i\}$, and each child is the root is an independent branching tree based on the same distribution. A *branching process* is a random process $X_0, X_1, X_2, \ldots$, where X_i counts the number of vertices at depth i in a branching tree. A branching tree can also be referred to as the *geneology tree* of the corresponding branching process.

In this section we answer the question of when a branching process generates a infinite complete k-ary tree with positive probability. In the next section, following Pittel, Wormald, and Spencer [14], we shall intuitively argue that a random graph with a fixed degree sequence locally resembles a branching tree, and that the presence of a giant $(k+1)$-core in a random graph is related to the possibility that a branching tree contains an infinite k-ary subtree (note that this result is tangential to results for fixed-degree random graphs, which are obtained through different methods).

Given a probability distribution $\{\mu_i\}$, the *probability generating function (p.g.f.)* [7, 8] for $\{\mu_i\}$ is defined as $g(q) = \sum_{i=0}^{\infty} \mu_i q^i$. The p.g.f. is a central tool in the theory of branching processes [7]. In particular, a classical result states that extinction probability of a branching process (that is, the probability that $X_i = 0$ for all but finitely many i) is given by the smallest fixed point of g in $[0, 1]$.

Now, for each integer $r \geq 0$, define the function

$$f_r(q) = \sum_{i=0}^{r} \frac{(1-q)^i}{i!} g^{(i)}(q), \tag{6}$$

where $g^{(i)}$ is the i'th derivative of the p.g.f. g. Note that $f_r(q)$ is the r'th order Taylor approximation of $g(1)$ about q, so $f_0(q) = g(q)$, $f_1(q) = g(q) + (1-q)g'(q)$, and so on.

The extinction of a branching process is in fact the complement of the event that the corresponding branching tree contains an infinite 1-ary subtree; the classical result cited above states that the probability of this event is determined by the smallest fixed point of f_0. Using similar techniques, we obtain the following generalization.

Theorem 3.1 *Let q_r be the smallest fixed point of the function f_r (as defined in equation 6) in the interval $[0,1]$. Then the probability that a branching tree based on the probability distribution $\{\mu_i\}$ contains an infinite perfect $(r+1)$-ary tree is $1-q_r$.*

Proof. Let X be a random variable with distribution $\{\mu_i\}$. For any $0 \le q \le 1$, let Z_q be a random variable with distribution

$$P(Z_q = i) = \sum_{j=i}^{\infty} \mu_j \cdot \binom{j}{i} q^{j-i}(1-q)^i.$$

Note that

$$\begin{aligned} P(Z_q \le r) &= \sum_{i=0}^{r}\sum_{j=i}^{\infty} \mu_j \cdot \binom{j}{i} q^{j-i}(1-q)^i \\ &= \sum_{i=0}^{r} E\left[\binom{X}{i} q^{X-i}(1-q)^i\right] \\ &= \sum_{i=0}^{r} \frac{(1-q)^i}{i!} E\left[\frac{X!}{(X-i)!} q^{X-i}\right] \\ &= \sum_{i=0}^{r} \frac{(1-q)^i}{i!} g^{(i)}(q) \\ &= f_r(q). \end{aligned}$$

Now, consider a branching tree based on distribution $\{\mu_i\}$ First, we calculate the probability that the root tree has at least $r+1$ children. Since Z_0 has the same distribution as X, then $P(X > r) = 1 - f_r(0)$. In order to produce an $(r+1)$-ary tree of depth 2, then the root must have at least $r+1$ children, each of whom produce $r+1$ grandchildren. Each child has probability $1 - f_r(0)$ of producing at least $(r+1)$ grandchildren, thus the number of such children is a random variable with distribution $Z_{f_r(0)}$. Accordingly, the probability of producing an $(r+1)$-ary tree of depth 2 is $1 - f_r(f_r(0))$.

In general, producing an $(r+1)$-ary tree of depth d is equivalent to having at least $(r+1)$ children who produce $(r+1)$-ary trees of depth $d-1$. Thus, we inductively conclude that the probability of an $(r+1)$-ary tree of depth d is $1 - f_r^{[d]}(0)$, where $f_r^{[d]}$ is the d'th iterate of f_r. Since $f_r(q)' \ge 0$ in the interval $[0,1]$ and $f_r(1) = 1$, then $f_r^{[d]}(0)$ approaches the lowest fixed point of f_r as $d \to \infty$. ∎

3.2 Relation to the k-core of a Random Graph.

Let $\mathcal{D}$ be a 1-smooth asymptotic degree sequence with limiting degree distribution $\{\lambda_i\}$. We define the *residual degree distribution* $\{\mu_j\}$ of $\mathcal{D}$ by

$$\mu_j = \frac{(j+1)\lambda_{j+1}}{\sum_{i=1}^{\infty} i\lambda_i}. \tag{7}$$

Since $\mathcal{D}$ is 1-smooth, $\sum_i i\lambda_i$ converges, and the residual degree distribution is well defined.

Pittel, Spencer, and Wormald [14] noted, using the graph model $\mathcal{G}_{n,m}$, that a giant k-core in a random graph relates to the probability of finding an infinite $(k-1)$-ary subtree of a Poisson branching tree. For an arbitrary degree sequence D, the presence of a giant k-core relates the probability of finding an infinite $(k-1)$-ary subtree in a branching tree based on the residual distribution $\{\mu_i\}$. However, in both cases, the argument is incomplete and it is not clear that the link to the branching process can be made sufficiently rigorous to produce a simple proof of the k-core result.

Consider a single endpoint s in the random graph $G(D)$ generated by the CM, and let us examine the structure of $G(D)$ in a neighborhood of s. According to the CM, s is matched to an endpoint t whcih is chosen uniformly from the set of all endpoints (other than s).

If s is matched to an endpoint t which is assigned to a vertex v, we define the *residual degree* of s to be a the number of endpoints assigned to v other then t; hence the residual degree of s is one less than the degree of v. Since, asymptotically, the fraction of vertices in $G(D)$ with degree i is λ_i, and vertex v of degree i has i chances of matching to s, we deduce that the residual degree of s is a random variable distributed according to the residual distribution

$$\left\{\mu_i = \frac{(i+1)\lambda_{i+1}}{\sum_i \lambda_i}\right\}.$$

Next, consider the all of the endpoints assigned to v other than t. By a similar informal argument, the residual degrees of these endpoints will be almost independent, and almost identically distributed. If proceed to examine larger neighborhoods of s, and if we ignore

1. the possibility of small cycles around s, and

2. slight changes in the effective residual distribution caused by the fact that the same endpoint cannot be matched twice (i.e. we are sampling without replacment),

then the graph in a small neighborhood of s will have the structure of a branching tree based on the residual distribution. Eventually, the factors we have ignored will become significant, and even in small neighborhoods, we have not precisely quantified the extent to this analogy is valid. Nevertheless, it is heuristically useful to imagine (or hope) that, the graph $G(D)$ locally resembles a branching tree.

We now consider the question of a giant k-core in the random graph $G(D)$. Pittel, Spencer, and Wormald [14] noted, using the graph model $\mathcal{G}_{n,m}$, that a giant k-core in a random graph relates to the probability of finding an infinite $(k-1)$-ary subtree of a Poisson branching tree. The following informal argument is taken from [14].

Choose any vertex v in $G(D)$, and let us attempt to determine whether or not v is in the k-core of $G(D)$. Clearly, v must have degree at least k to be part of the k-core. Furthermore, v must have at least k endpoints each of whom have residual degree at least $k-1$, and these $k-1$ neighbors must in turn have k endpoints of residual degree at least $k-1$ and so on. If we assume that residual degrees are i.i.d. random variables, then in order for v to be in the k-core of $G(D)$, v must have k endpoints which generate branching trees containing a complete $(k-1)$-ary tree.

Of course, this argument is incomplete. As pointed out in [14], it is not clear that the link to the branching process can be made entirely rigorous; in particular, we have only argued that producing a complete $(k-1)$-ary branching tree with positive probability is necessary for a giant k-core. We are not aware of an equally simple argument that this condition is sufficient. Further, the assumption that the residual degrees are i.i.d. random variables is not accurate. Thus, the branching process argument should be treated as an intuitive explanation or perhaps as a guess at the true solution.

If we now use the μ_i as defined in equation 7, the informal connection between the existence of a k-core in G and the existence of an infinite $(k-1)$-ary subtree in the branching process gives an informal justification of the following Theorem 3.2 from [6].

Theorem 3.2 *[6] Let $\mathcal{D}$ be a 1-smooth asymptotic degree sequence with maximum degree in D_n being $o(n^{1/3})$ and with residual degree distribution $\{\mu_i\}$. Then*

1. *If there exists a value q in the interval $[0,1)$ such that $f_{k-2}(q) < q$ then there exists a constant $C > 0$ such that the k-core of $G(\mathcal{D})$ contains at least Cn vertices w.e.h.p.*
2. *If $f_{k-2}(q) > q$ for all $q \in [0,1)$ then for every $C > 0$, then the k-core of $G(\mathcal{D})$ has less than Cn vertices w.e.h.p.*

The formal proof of Theorem 3.2 in [6] (and in the appendix for reference) uses an algorithm adapted from [14] which, at each time step, removes an edge incident on a vertex of degree less than k, and continues until there are no longer any vertices of nonzero degree less than k. The remaining edges and vertices will be the (possibly empty) k-core of the original graph. This algorithm is incorporated within the CM in a natural way. In particular, the algorithm in [6] chooses the random matching used by the CM while the algorithm executes, exposing edges only as they are needed. When the algorithm terminates, the k-core will remain unexposed, and thus a corollary to Theorem 3.2 is that the k-core of a random graph with asymptotic degree sequence $\mathcal{D}$ is itself a random graph with a different 1-smooth asymptotic degree sequence and a limiting degree distribution which can be calculated from the limiting distribution of $\mathcal{D}$. We state this as a corollary, which we will use in section 5.

Corollary 3.3 *Let $\mathcal{D}$ be a 1-smooth asymptotic degree sequence with maximum degree in D_n being $o(n^{1/3})$. Then the giant k-core of $G(\mathcal{D})$, if it exists, is $G(\mathcal{D}')$, where $\mathcal{D}'$ is a 1-smooth asymptotic degree sequence a.a.s. If $\mathcal{D}$ is 2-smooth, then $\mathcal{D}'$ is a.a.s. 2-smooth as well.*

4 k-cores in $\mathcal{G}_{n,p}$ and Random Power-law Graphs

Using Theorem 3.2 and the results in [11, 12] for the a.a.s. presence of a giant component it is not difficult to determine that a random graph G with a 1-smooth degree sequence has a giant 2-core a.a.s. if and only if it has a giant component a.a.s. (see end of Appendix).

For $k > 2$ the conditions necessary for a giant k-core are less easily verified, since it is not necessarily true that f_{k-2} will have all positive derivatives. In this section, we consider the case where all of the moments of the distribution $\{\mu_i\}$ are convergent, and the case of power-law graphs.

4.1 Distributions with All Convergent Moments.

Let X be a random variable with distribution $\{\mu_i\}$. By assumption, $g^{(i)}(1) = E[X(X-1)\cdots(X-i+1)] = \nu_i$, the i'th factorial moment of the distribution $\{\mu_i\}$, is finite for all i. This allows us to write

$$g(q) \;=\; \sum_{i=0}^{\infty} \frac{(q-1)^i}{i!} g^{(i)}(1) \;=\; \sum_{i=0}^{\infty} \frac{(q-1)^i}{i!} \nu_i.$$

We can now express f_r as a power series in $(q-1)$

$$\begin{aligned} f_r(q) &= \sum_{i=0}^{r} \frac{(1-q)^i}{i!} g^{(i)}(q) \\ &= \sum_{i=0}^{r} \frac{(1-q)^i}{i!} \sum_{j=i}^{\infty} \frac{j!}{j-i!} \frac{(q-1)^{j-i}}{j!} \nu_j \\ &= \sum_{j=0}^{\infty} \frac{(q-1)^j}{j!} \nu_j \sum_{i=0}^{r} (-1)^i \binom{j}{i} \\ &= 1 + \sum_{j=r+1}^{\infty} \frac{(q-1)^j}{j!} \nu_j (-1)^r \binom{j-1}{r}, \end{aligned}$$

where the last step uses the binomial identity $\sum_{i=0}^{r}(-1)^i\binom{j}{i} = (-1)^r\binom{j-1}{r}$ for $j > 0$.

Now we write $p = 1 - q$, and note that and finding a fixed point $f_r(1-p) = 1-p$ is equivalent to solving

$$\begin{aligned} 1-p &= 1 + (-1)^r \sum_{j=r+1}^{\infty} \frac{(-p)^j}{j!} \nu_j \binom{j-1}{r} \\ 0 &= p + (-1)^r \sum_{j=r+1}^{\infty} p^j \frac{(-1)^j}{j!} \nu_j \binom{j-1}{r}. \end{aligned}$$

In order to ascertain the presence of a giant k-core, we must find a point whe re $f_{k-2}(q) < q$, or

$$p + (-1)^k \sum_{j=k-1}^{\infty} p^j \frac{(-1)^j}{j!} \nu_j \binom{j-1}{k-2} < 0. \qquad \text{(equation)}$$

4.1.1 Application to $\mathcal{G}_{n,p}$.

As shown by Molloy and Reed [11], the Erdos-Renyi random graph model $\mathcal{G}_{n,p}$ produces a random graph with a Poisson degree distribution, and thus results derived for random graphs whose limiting degree distribution is a Poisson distribution are valid for $\mathcal{G}_{n,p}$. Since a Poisson distribution has all convergent moments, we can re-derive some of the results of Pittel, Spencer, and Wormald [14] regarding the k-core of $\mathcal{G}_{n,m}$.

Consider a random graph whose limiting degree distribution is a Poisson distribution with expected value r, so $\lambda_i = \frac{r^i e^{-r}}{i!}$. Since

$$i\lambda_i = r\frac{r^{i-1}e^{-r}}{(i-1)!},$$

and $\sum i\lambda_i = r$, then

$$\mu_i = \frac{(i+1)\lambda_{i+1}}{\sum_i \lambda_i} = \lambda_i,$$

the residual degree distribution is identical to the limiting degree distribution.

Now, the factorial moments of a Poisson distribution are $\nu_j = r^j$. Using equation 4.1 from the previous discussion, if

$$p + (-1)^k \sum_{j=k-1}^{\infty} (pr)^j \frac{(-1)^j}{j!} \binom{j-1}{k-2} < 0$$

has a solution, then $\mathcal{G}_{n,p}$ with expected degree r has a giant k-core w.e.h.p. Let

$$C_k(x) = (-1)^{k+1} \sum_{j=k-1}^{\infty} x^j \frac{(-1)^j}{j!} \binom{j-1}{k-2},$$

and let $x = pr$. Then we must solve

$$\begin{aligned} x/r - C_k(x) &< 0 \\ x/C_k(x) &< r. \end{aligned}$$

Thus the giant k-core threshold for $\mathcal{G}_{n,p}$ occurs at

$$\min \frac{x}{C_k(x)}.$$

This is exactly the threshold derived in [14].

4.2 Power Law Graphs

Several massive graphs that occur in the real-world, including the web graph, have degree sequences that obey a power law [9], thus there has been considerable interest in understanding the properties of massive power law graphs. One approach to studying such graphs, introduced by Aiello, Chung, and Lu [1], is to generate random graphs with power law degree sequences.

A degree sequence obeys a power law if the number of vertices of degree i is proportional to $i^{-\beta}$ for some β. If $\beta \leq 2$, this degree sequence is not sparse, but for $\beta > 2$, this power law graph can be characterized by a 1-smooth asymptotic degree sequence with

$$\lambda_i = \frac{1}{\zeta(\beta)} \frac{1}{i^\beta},$$

where $\zeta(\beta) = \sum_{i=1}^{\infty} i^{-\beta}$ is the Riemann Zeta function. The corresponding residual endpoint distribution is

$$\mu_i = \frac{1}{\zeta(\beta-1)} \frac{1}{(i+1)^{\beta-1}}.$$

Since the number of vertices of degree i is approximately $n\lambda_i$, and $\lambda_i = \Theta(i^{-\beta})$, it might be natural to consider the largest degree in a random power law graph to be $\Theta(n^{1/\beta})$. This is the assumption made by [1]. The configuration model and Theorem 3.2 require the maximum degree to be $o(n^{1/3})$, and hence for $\beta < 3$, this power-law graph model would violate the CM maximum degree requirement. However, we may extend our results to the maximum degree bound in the power law model of [1] by the following mechanism. There are $O(n^{1-\delta})$ edges incident on vertices of degree greater than $n^{1/3-\epsilon}$ for some $\delta > 0$ when $\beta > 2$. Consider exposing the edges of such a graph in the CM by first placing the edges of these high degree vertices in any (adversarial) manner while respecting the degree constraints on the other vertices. Then apply the CM mechanism to the graph represented by the residual degrees of the remaining vertices. Since only $O(n^{1-\delta})$ endpoints are affected, the resulting degree distribution is indistinguishable from the starting power-law distribution with respect to smoothness, hence the result in [10] continues to hold for this residual graph.

We must also verify that $M_2(D_n) = O(n^{1/2-\epsilon})$. Using the power law model from [1], we find that

$$M_2(D_n) = \sum_{i=0}^{n} \frac{d_i^2}{n} \simeq \sum_{i=0}^{d_{\max}} \Theta\left(\frac{i^2}{i^\beta}\right) \simeq \Theta(d_{\max}^{3-\beta})$$

If $d_{\max} = n^{1/\beta}$, then $M_2(D_n) = n^{\frac{3}{\beta}-1}$, which is $O(n^{1/2-\epsilon})$ for $\beta > 2$.

By Theorem A.2 in the Appendix, the a.a.s existence of a giant 2-core in a random power-law graph is equivalent to the a.a.s. existence of a giant component, which appears if $\beta < 3.47875\ldots$ [1]. For $k \geq 3$, we have the following theorem.

Theorem 4.1 *Let $k \geq 3$ be an integer constant.*

1. *For $\beta \geq 3$, a random power law graph does not have a giant k-core w.e.h.p.*

2. *For $2 < \beta < 3$, a random power law graph has a giant k-core w.e.h.p.*

Proof. Since $f_r(q) = \sum_{i=0}^{r} \frac{(1-q)^i}{i!} g^{(i)}(q)$, we derive

$$\begin{aligned} {f'}_r(q) &= \sum_{i=0}^{r} \frac{(1-q)^i}{i!} g^{(i+1)}(q) - \sum_{i=1}^{r} \frac{(1-q)^{(i-1)}}{(i-1)!} g^{(i)}(q) \\ &= \frac{(1-q)^r}{r!} g^{(r+1)}(q). \\ &= \frac{1}{r!} \frac{g^{(r+1)}(q)}{\sum_{i=0}^{\infty} \binom{i+r-1}{r-1} q^i}. \end{aligned}$$

For a power law graph, the probability generating function of the endpoint distribution is given by

$$g(q) = \frac{1}{\zeta(\beta-1)} \sum_{i=0}^{\infty} \frac{q^i}{(i+1)^{\beta-1}},$$

and thus

$$g^{(r+1)}(q) = \frac{1}{\zeta(\beta-1)} \sum_{i=0}^{\infty} \frac{i(i-1)(i-2)\cdots(i-r)q^{i-r-1}}{(i+1)^{\beta-1}}.$$

Thus, for power law graphs,

$$\begin{aligned} {f'}_r(q) &= \frac{1}{r!} \frac{g^{(r+1)}(q)}{\sum_{i=0}^{\infty} \binom{i+r-1}{r-1} q^i} \\ &= \frac{1}{r!\zeta(\beta-1)} \frac{\sum_{i=0}^{\infty} \frac{i(i-1)(i-2)\cdots(i-r)q^{i-r-1}}{(i+1)^{\beta-1}}}{\sum_{i=0}^{\infty} \binom{i+r-1}{r-1} q^i} \\ &= \frac{1}{r!\zeta(\beta-1)} \frac{\sum_{i=r+1}^{\infty} \frac{i(i-1)(i-2)\cdots(i-r)q^{i-r-1}}{(i+1)^{\beta-1}}}{\sum_{i=r+1}^{\infty} \binom{i-2}{r-1} q^{i-r-1}}. \end{aligned}$$

Now, let $a_i = \frac{i(i-1)(i-2)\cdots(i-r)}{(i+1)^{\beta-1}}$ and let $b_i = r!\binom{i-2}{r-1}$, so

$${f'}_r(q) = \frac{1}{\zeta(\beta-1)} \frac{\sum_{i=r+1}^{\infty} a_i q^{i-r-1}}{\sum_{i=r+1}^{\infty} b_i q^{i-r-1}},$$

and note that

$$\frac{a_i}{b_i} = \frac{\frac{i(i-1)(i-2)\cdots(i-r)}{(i+1)^{\beta-1}}}{r!\binom{i-2}{r-1}} = \frac{i(i-1)}{r(i+1)^{\beta-1}}.$$