On the existence of equiangular tight frames

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Abstract

In a recent paper, Holmes and Paulsen established a necessary condition for the existence of an N-vector equiangular tight frame in a d-dimensional real Euclidean space. This article develops much stronger necessary conditions using a combination of field theory and graph theory. This investigation rules out many possibilities admitted by the work of Holmes and Paulsen. Using a new one-to-one correspondence between equivalence classes of real equiangular tight frames and strongly regular graphs of a certain type, it has been verified that a real equiangular tight frame exists for each pair (d, N) with $N \leq 100$ that meets the new conditions. The arguments also extend to deliver novel necessary conditions for the existence of equiangular tight frames whose Gram matrices have entries drawn from a discrete set of complex numbers.

1 Introduction

Suppose that one constructs a set of N lines that pass through the origin of the Euclidean space $\mathbb{R}^d$. We assume that $N > d$ to avoid trivial cases. The j-th line may be viewed as the linear span of a unit vector $\boldsymbol{s}_j$, and the absolute inner product $|\langle \boldsymbol{s}_j, \boldsymbol{s}_k \rangle|$ may be interpreted as the cosine of the acute angle between the j-th and k-th lines. It can be shown [12, 13] that

$$\max_{j \neq k} |\langle \boldsymbol{s}_j, \boldsymbol{s}_k \rangle| \quad \geq \quad \sqrt{\frac{N-d}{d\,(N-1)}}. \tag{1}$$

In words, it is impossible for every pair of lines to meet at an arbitrarily large angle. The same bound holds for N unit vectors in the complex Euclidean space $\mathbb{C}^d$.

When the bound (1) is met, the matrix formed from the column vectors $\boldsymbol{s}_1, \dots, \boldsymbol{s}_N$ has a very special structure.

Definition 1.1 *Let S be a $d \times N$ matrix with unit-norm columns. The matrix S is called an* equiangular tight frame *if*

1. *the absolute inner product between each pair of columns is identical, and*
2. *it satisfies the equation $\mathsf{S}\mathsf{S}^* = (N/d)\,\mathsf{I}$, where I is the $d \times d$ identity matrix.*

The first condition enforces equiangularity, while the second ensures that the matrix is a tight frame [11]. If the matrix S has real entries, it is called a real equiangular tight frame*.*

In fact, a $d \times N$ matrix with unit-norm columns is an equiangular tight frame *if and only if* the absolute inner products between its columns all meet the bound (1). See [10] for an easy proof.

Equiangular tight frames are somewhat rare. Indeed, Holmes and Paulsen [5] have shown that a real equiangular tight frame

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can exist only if

$$(N-2d)\sqrt{\frac{N-1}{d\,(N-d)}} \quad \in \quad \mathbb{Z}. \tag{2}$$

In this article, we shall strengthen condition (2) significantly. The following theorem summarizes our results in the real case.

Theorem A *Suppose that $1 < d < N-1$. When $N \neq 2d$, a necessary condition for the existence of a real equiangular tight frame is that*

$$\sqrt{\frac{d\,(N-1)}{N-d}} \;\equiv\; \sqrt{\frac{(N-d)\,(N-1)}{d}} \;\equiv\; 1 \pmod 2$$

When $N = 2d$, it is necessary that d be an odd number and that $(N-1)$ equal the sum of two squares.

This theorem forbids many of the possibilities admitted by (2). On the other hand, we have been able to establish that a real equiangular tight frame actually does exist for each pair (d, N) that meets our conditions, where N ranges up to 100. See Tables 1 and 2 in section 7 for details. In consequence, we conjecture that the conditions of Theorem A may be sufficient as well.

We provide two proofs for the condition when $N \neq 2d$. The first is based on field theory. This method of proof generalizes to deliver necessary conditions on complex equiangular tight frames. We consider the case where the inner products between columns of the frame are (scaled) roots of unity. The second proof is based on a new one-to-one correspondence between real equiangular tight frames and strongly regular graphs with a certain parameter set. The graph-theoretic approach yields the results for the case $N = 2d$.

A second type of necessary condition has also appeared in the literature [12]. A real equiangular tight frame can exist only if $N \leq \frac{1}{2}\,d\,(d+1)$, and a complex equiangular tight frame can exist only if $N \leq d^2$. In Section 6, we offer a new matrix-theoretic proof of these upper bounds.

Equiangular tight frames first arose in discrete geometry [12]. More recently, they have found applications in signal processing, communications, and coding theory [3, 10]. As a specific example, Holmes and Paulsen have shown that an equiangular tight frame provides an error correction code that is robust against two erasures [5]. In wireless communication, tight frames have been studied in the context of constructing capacity achieving signature sequences for multiuser communication systems [13]. Equiangular tight frames achieve the capacity of a Gaussian channel because of the tightness condition, and they satisfy an interference invariance property due to their equiangularity [4]. Interference between users is measured by the modulus of the inner product between their signatures, which are simply the columns of the frame. Equiangular tight frames solve the problem of providing signatures that see the same interference from every other signature.

A word about notation. We denote the $d \times N$ frame matrix by S, the identity matrix by I, and the all-ones matrix by J. The dimension of I and J should be clear from context.

2 An Algebraic Lemma

The first proof relies on some basic results from field theory. A standard textbook for this material is [8]. For the sake of completeness, we shall review the essential definitions. Readers who are familiar with this material may wish to skip to Lemma 2.5.

A polynomial whose coefficients are drawn from a subfield $\mathbb{F}$ of the complex numbers is referred to as a *polynomial over* $\mathbb{F}$. The complex number α is *algebraic* over $\mathbb{F}$ if it is the root of some polynomial over $\mathbb{F}$. An *algebraic integer* is the root of a monic polynomial with integer coefficients.

Fact 2.1 *The algebraic integers form a ring, i.e., they are closed under addition and multiplication.*

Fact 2.2 *The roots of a monic polynomial over the algebraic integers remain algebraic integers.*

The *minimal polynomial* of α over $\mathbb{F}$ is the (unique) lowest degree monic polynomial over $\mathbb{F}$ that contains α among its roots.

Fact 2.3 *A minimal polynomial over $\mathbb{F}$ has simple roots.*

Two numbers that have the same minimal polynomial over $\mathbb{F}$ are called *algebraic conjugates* over $\mathbb{F}$.

Fact 2.4 *Suppose that α and β are algebraic conjugates over $\mathbb{F}$. If p is a polynomial over $\mathbb{F}$ that has α as a root with multiplicity m, then β is also a root of p with multiplicity m.*

With these facts at hand, we may prove the following lemma.

Lemma 2.5 *Let A be a real symmetric matrix whose entries are algebraic integers. Then the eigenvalues of A are real algebraic integers.*

In addition, assume that the entries of A belong to a subfield $\mathbb{F}$ of the complex numbers. If A has an eigenvalue α whose multiplicity is different from that of the other eigenvalues then α also belongs to $\mathbb{F}$.

Proof: The matrix A is real symmetric, hence its eigenvalues are real numbers. By definition, an eigenvalue of A is a root of the characteristic polynomial $x \mapsto \det(x\,\mathsf{I} - \mathsf{A})$. Since the entries of A are algebraic integers, Fact 2.1 implies that the characteristic polynomial is a monic polynomial with algebraic integer coefficients. Then Fact 2.2 shows that the eigenvalues of A are algebraic integers.

Assume that the entries of A belong to $\mathbb{F}$. Thus, the eigenvalues of A are algebraic over $\mathbb{F}$. Since α has a different multiplicity from the other eigenvalues of A, Fact 2.4 precludes the possibility that α might have any algebraic conjugates over $\mathbb{F}$. Applying Fact 2.3, we see that the minimal polynomial of α over $\mathbb{Q}$ is linear. Thus, α belongs to $\mathbb{F}$. □

This type of field-theoretic argument appears frequently in the analysis of integer matrices. A similar argument was used by Lemmens and Seidel in their study of equiangular lines [9].

3 Real Equiangular Tight Frames

Suppose that S is a $d \times N$ real equiangular tight frame, and denote by α the (unique) absolute inner product between its columns. That is,

$$\alpha \quad = \quad |\langle \boldsymbol{s}_j, \boldsymbol{s}_k \rangle| \quad = \quad \sqrt{\frac{N-d}{d\,(N-1)}} \quad \text{for all } j \neq k. \tag{3}$$

Next, we construct the matrix

$$\mathsf{A} \quad = \quad \frac{1}{\alpha}\,(\mathsf{S}^*\mathsf{S} - \mathsf{I}). \tag{4}$$

This matrix is symmetric; it has a zero diagonal; and its off-diagonal entries are all ± 1. Since an equiangular tight frame satisfies the equation $\mathsf{S}\mathsf{S}^* = (N/d)\,\mathsf{I}$, it follows that the two distinct eigenvalues of A are

$$\lambda_1 \quad = \quad -\frac{1}{\alpha} \qquad \text{and} \qquad \lambda_2 \quad = \quad \frac{N-d}{d\,\alpha} \tag{5}$$

with respective multiplicities $(N-d)$ and d.

Theorem 3.1 *Assume that $N \neq 2d$. If S is a real equiangular tight frame, then λ_1 and λ_2 are integers.*

Proof: Since $N \neq 2d$, the two eigenvalues of A have different multiplicities. The entries of A are integers, so Lemma 2.5 implies that λ_1 and λ_2 are rational algebraic integers. It is well known that the only algebraic integers among the rationals are the ordinary integers. □

An immediate corollary is the necessary condition of Holmes and Paulsen.

Corollary 3.2 (Holmes–Paulsen [5]) *A real equiangular tight frame can exist only when*

$$(N-2d)\sqrt{\frac{N-1}{d\,(N-d)}} \quad \in \quad \mathbb{Z}.$$

Proof: Introducing the value of α from (3), we see that the expression in the statement of the corollary equals $(\lambda_1 + \lambda_2)$. Since λ_1 and λ_2 are integers, the result follows instantly. □

In the next theorem we establish stricter conditions on λ_1 and λ_2.

Theorem 3.3 *Assume that $N \neq 2d$, and exclude the degenerate cases $d = 1$ and $d = N-1$. If S is a real equiangular tight frame, then λ_1 and λ_2 are both odd integers. That is,*

$$\sqrt{\frac{d\,(N-1)}{N-d}} \quad \equiv \quad \sqrt{\frac{(N-d)\,(N-1)}{d}} \quad \equiv \quad 1 \pmod 2.$$

When $d = N-1$, the unique tight frame (modulo rotations) is always equiangular [10]. We shall attend to the case $N = 2d$ in Section 5. Our proof adapts an argument of P. M. Neumann quoted in [9].

Proof: Let us form a new matrix whose entries all equal zero or one:

$$\mathsf{M} \quad = \quad \tfrac{1}{2}\,(\mathsf{J} - \mathsf{I} - \mathsf{A})$$

where the symbol J denotes a conformal matrix of ones. We have ruled out the possibility that $d = N-1$, so the eigenvalue λ_1 of A has geometric multiplicity at least two. In consequence, the $(N-1)$-dimensional null space of J must intersect the invariant subspace of A associated with λ_1. Any vector in this intersection is an eigenvector of M with eigenvalue $\mu_1 = -\frac{1}{2}\,(1+\lambda_1)$. A similar argument establishes that $\mu_2 = -\frac{1}{2}\,(1+\lambda_2)$ is an eigenvalue of M.

Theorem 3.1 establishes that λ_1 and λ_2 are integers, so μ_1 and μ_2 must be rational numbers. The entries of M are integers, so Lemma 2.5 proves that the eigenvalues of M are algebraic integers. We conclude that μ_1 and μ_2 are ordinary integers. □

Theorem 3.3 is much stronger than Corollary 3.2. Indeed, there are many pairs (d, N) that are excluded by Theorem 3.3 but not by Corollary 3.2. For example, when $d = 3$ and $N = 9$, then $\lambda_1 = -2$ and $\lambda_2 = 4$. As another example, when $d = 10$ and $N = 25$, $(\lambda_1 + \lambda_2)$ is an integer but λ_1 and λ_2 are not odd integers. See Table 2 for more examples.

4 Complex Equiangular Tight Frames

It is also natural to study equiangular tight frames whose entries are complex. The experiments in [11] indicate that complex equiangular tight frames must satisfy integrality conditions like Theorem 3.3, but no such conditions are presently available. We have used field theory to develop strong constraints on equiangular tight frames for which the inner products between columns are scaled roots of unity. This type of equiangular tight frame can arise in electrical engineering applications when the entries of the frame matrix are restricted to be roots of unity.

Our starting point is the auxiliary matrix

$$A = \frac{1}{\alpha}(S^* S - \mathrm{I}).$$

As before, the eigenvalues of A are given by (5). If we restrict the off-diagonal entries of A, then we may apply Lemma 2.5 to prove that the eigenvalues of A must lie in a prescribed set. The following fact is fundamental [7, Thm. I.10.4].

Fact 4.1 *Suppose that ζ_p is a primitive p-th root of unity. The ring of algebraic integers in the field $\mathbb{Q}(\zeta_p)$ coincides with the ring $\mathbb{Z}[\zeta_p]$.*

Note that $\mathbb{Q}(\zeta_p)$ denotes the smallest field extending $\mathbb{Q}$ that contains ζ_p, while $\mathbb{Z}[\zeta_p]$ is the smallest ring extending $\mathbb{Z}$ that contains ζ_p. A general theorem falls from Fact 4.1 and Lemma 2.5.

Theorem 4.2 *Suppose that the off-diagonal entries of the symmetric A are p-th roots of unity. Then the eigenvalues of A belong to the ring $\mathbb{Z}[\zeta_p] \cap \mathbb{R}$.*

We shall provide several examples that demonstrate how to use Theorem 4.2 to obtain specific conditions for several different types of equiangular tight frames.

Gaussian Integers Suppose that the off-diagonal entries of A belong to the set $\{\pm 1, \pm \mathrm{i}\}$. This situation can occur when the entries of the frame matrix are drawn from the set $\{\pm d^{-1/2}, \pm \mathrm{i}\, d^{-1/2}\}$ and the pair (d, N) is suitably restricted. Theorem 4.2 shows that the eigenvalues of A must be ordinary integers. Using equations (3) and (5), we discover the necessary conditions

$$\sqrt{\frac{d\,(N-1)}{N-d}} \in \mathbb{Z} \quad \text{and} \quad \sqrt{\frac{(N-d)\,(N-1)}{d}} \in \mathbb{Z}.$$

This is the same condition we obtained in Theorem 3.1.

Sixth Roots of Unity Assume that the off-diagonal entries of A are sixth roots of unity. Theorem 4.2 implies that the eigenvalues of A are real elements of $\mathbb{Z}[\zeta_6]$ where $\zeta_6 = e^{2\pi i/6}$. The elements of $\mathbb{Z}[\zeta_6]$ can be written as

$$a_0 + a_1\zeta_6 + a_2\zeta_6^2,$$

where a_0, a_1 and a_2 are integers. The expression yields a real number if and only if $a_2 = -a_1$. Thus, using $\zeta_6 - \zeta_6^2 = 1$ we conclude that the real algebraic integers in $\mathbb{Z}[\zeta_6]$ are the ordinary integers. We obtain the same necessary conditions as in Theorem 3.1.

Eighth Roots of Unity Assume that the off-diagonal entries of A are eighth roots of unity. Theorem 4.2 now forces the eigenvalues of A to lie in the ring $\mathbb{Z}[\zeta_8] \cap \mathbb{R}$, where $\zeta_8 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$. The elements of $\mathbb{Z}[\zeta_8]$ can be written as

$$a_0 + a_1\zeta_8 + a_2 i + a_3\zeta_8^3,$$

where a_0, a_1, a_2 and a_3 are integers. If the expression yields a real number then we must have $a_2 + \frac{1}{\sqrt{2}}(a_1 + a_3) = 0$, implying $a_3 = -a_1$ and $a_2 = 0$. Thus, using $\zeta_8 - \zeta_8^3 = \sqrt{2}$, the real elements in $\mathbb{Z}[\zeta_8]$ must be of the form $a_0 + a_1\sqrt{2}$ in other words $\mathbb{Z}[\zeta_8] \cap \mathbb{R} = \mathbb{Z}[\sqrt{2}]$.

According to (3) and (5) the λ_1 and λ_2 eigenvalues are both square roots of rational numbers. It is easy to verify, that either both eigenvalues must be integers, or they both must be an integer multiple of $\sqrt{2}$. Thus, we discover that either

$$\sqrt{\frac{d\,(N-1)}{N-d}} \in \mathbb{Z} \quad \text{and} \quad \sqrt{\frac{(N-d)\,(N-1)}{d}} \in \mathbb{Z},$$

or

$$\sqrt{\frac{d\,(N-1)}{2\,(N-d)}} \in \mathbb{Z} \quad \text{and} \quad \sqrt{\frac{(N-d)\,(N-1)}{2\,d}} \in \mathbb{Z}$$

must be true.

5 The Graph Connection

It was observed in [5,10] that real equiangular tight frames naturally give rise to regular two-graphs and *vice versa*. It is also known that regular two-graphs naturally give rise to strongly regular graphs with certain parameter sets [1, Ch. 4]. In consequence, we may establish a natural correspondence between real equiangular tight frames and strongly regular graphs. The parameters of a strongly regular graph must be integers, which provides additional restrictions on the potential values of d and N.

Definition 5.1 *We say that two real equiangular tight frames S and T with the same dimensions are* equivalent *if*

$$S = Q\,T\,P$$

where Q is a orthogonal matrix, and P is a generalized permutation matrix, which is a permutation matrix whose nonzero entries may equal ± 1.

Observe that the eigenvalues of $\mathsf{S}^*\mathsf{S}$ are identical to those of $\mathsf{T}^*\mathsf{T}$, and that the absolute inner products of the columns of T and S are the same.

Next, we offer a short introduction to the theory of strongly regular graphs, which is drawn from [1]. An *undirected graph* is a (finite) collection of points, called *vertices*, along with a list of vertex pairs, called *edges*. In a *simple graph*, no edge may appear twice and all edges are between distinct vertices. Two vertices are *adjacent* or *neighboring* if the graph contains an edge between them. The *adjacency matrix* of a graph on n ordered vertices is the $n \times n$ matrix whose (j, k) entry equals 1 when the j-th and k-th vertices are adjacent and 0 otherwise.

Definition 5.2 *A* strongly regular graph *with parameters (n, r, s, t) is a simple graph on n vertices for which*

1. *every vertex is adjacent to r others,*
2. *two adjacent vertices have s neighbors in common, and*
3. *two nonadjacent vertices have t neighbors in common.*

From this definition, we exclude graphs in which no vertices are adjacent or all vertices are adjacent. By a simple edge counting argument it can be shown that the parameters of a strongly regular graph are not independent,

$$r\,(r - s - 1) = (n - r - 1)\,t. \tag{6}$$

It can also be shown that a matrix M is the adjacency matrix of a strongly regular graph with parameters (n, r, s, t) if and only if

$$\mathsf{M}^2 \quad = \quad r\,\mathsf{I} + s\,\mathsf{M} + t\,(\mathsf{J} - \mathsf{I} - \mathsf{M}) \tag{7}$$

where J is the matrix of ones [1].

With this background, we may establish the connection between real equiangular tight frames and strongly regular graphs.

Theorem 5.3 *There is a one-to-one correspondence between the equivalence classes of $d \times N$ real equiangular tight frames and strongly regular graphs with parameters $(N-1, 2t, s, t)$, where*

$$s = \frac{N - 3\,\sigma - 6}{4} \quad , \quad t = \frac{N - \sigma - 2}{4}$$

and

$$\sigma \quad = \quad (N - 2d)\sqrt{\frac{N-1}{d\,(N-d)}}.$$

Proof: Suppose that S is a $d \times N$ real equiangular tight frame. Once again, define the matrix

$$\mathsf{A} \quad = \quad \frac{1}{\alpha}\,(\mathsf{S}^*\mathsf{S} - \mathsf{I}).$$

We have shown that A has exactly two distinct eigenvalues whose product is $-(N-1)$ and whose sum is σ. Therefore, A must satisfy the quadratic equation

$$\mathsf{A}^2 \quad = \quad \sigma\,\mathsf{A} + (N-1)\,\mathsf{I} \tag{8}$$

From this matrix A, we shall construct the adjacency matrix of a strongly regular graph.

Without loss of generality, assume that the off-diagonal entries in the first row and column of A all equal one. One may achieve this standardization by negating the columns of S that have a negative inner product with the first column. Let A_1 denote the $(N-1) \times (N-1)$ matrix obtained by deleting the first row and column of A. Examining the first row and column of (8), we see that

$$\mathsf{J}\,\mathsf{A}_1 \quad = \quad \mathsf{A}_1\,\mathsf{J} \quad = \quad \sigma\,\mathsf{J}. \tag{9}$$

Equation (8) also implies

$$\mathsf{A}_1^2 \quad = \quad \sigma\,\mathsf{A}_1 + (N-1)\,\mathsf{I} - \mathsf{J}. \tag{10}$$

Next, we define a matrix M whose entries all equal zero or one.

$$\mathsf{M} \quad = \quad \tfrac{1}{2}\,(\mathsf{J} - \mathsf{I} - \mathsf{A}_1).$$

Square both sides to obtain

$$4\,\mathsf{M}^2 \quad = \quad (N-1)\,\mathsf{J} + \mathsf{I} + \mathsf{A}_1^2 - 2\,\mathsf{J} + 2\mathsf{A}_1 - \mathsf{J}\,\mathsf{A}_1 - \mathsf{A}_1\mathsf{J}.$$

Use (9) and (10), and $\mathsf{A}_1 = \mathsf{J} - \mathsf{I} - 2\mathsf{M}$ to establish that

$$4\,\mathsf{M}^2 = -2\,(\sigma + 2)\,\mathsf{M} + (N - \sigma - 2)\,\mathsf{I} + (N - \sigma - 2)\,\mathsf{J}. \tag{11}$$

Rearrange (11) to obtain

$$\mathsf{M}^2 = \frac{N - \sigma - 2}{2}\,\mathsf{I} + \frac{N - 3\sigma - 6}{4}\,\mathsf{M} + \frac{N - \sigma - 2}{4}\,(\mathsf{J} - \mathsf{I} - \mathsf{M}). \tag{12}$$

On comparison with (7), we discover that M is potentially the adjacency matrix of a strongly regular graph with parameters

$$\left(N-1, \quad \frac{N - \sigma - 2}{2}, \quad \frac{N - 3\,\sigma - 6}{4}, \quad \frac{N - \sigma - 2}{4}\right).$$

We need only check that the graph parameters are integers. Suppose that an off-diagonal entry of M equals zero. By examining the right-hand side of (11), we see that the corresponding entry of $4\,\mathsf{M}^2$ must equal $(N - \sigma - 2)$. If an off-diagonal entry of M equals one, the corresponding entry of $4\,\mathsf{M}^2$ equals $(N - 3\,\sigma - 6)$. Since M is an integer matrix, each entry of $4\,\mathsf{M}^2$ is divisible by four. This observation completes the argument.

Conversely, we must demonstrate that each strongly regular graph on n vertices can be associated with a unique equivalence class of real equiangular tight frames with $N = n + 1$ vectors.

This argument simply reverses the construction of a strongly regular graph from a real equiangular tight frame.

Suppose that M is the $n \times n$ adjacency matrix of a strongly regular graph with parameters $(n, 2t, s, t)$. Together (7) and (6) imply that the adjacency matrix satisfies (11) with $\sigma = N - 4t - 2$. Define the block matrix

$$A \quad = \quad \left[\begin{array}{c|c} 0 & \mathbf{1}^T \\ \hline \mathbf{1} & \mathsf{J} - \mathsf{I} - 2\,M \end{array} \right]. \tag{13}$$

Clearly, A is symmetric; it has a zero diagonal; and its off-diagonal entries all equal ± 1. Using (11), one may check that A satisfies the quadratic equation (8). Therefore, A has two non-zero eigenvalues, which we label λ_1 and λ_2. The trace of A is zero, so we may assume that $\lambda_1 < 0 < \lambda_2$. Moreover, if d denotes the multiplicity of λ_2, then $(N-d)\,\lambda_1 + d\,\lambda_2 = 0$. Both eigenvalues of A satisfy (8), so their product equals $-(N-1)$. Combining these facts, we determine that

$$\lambda_1 = -\sqrt{\frac{d\,(N-1)}{N-d}} \qquad \text{and} \qquad \lambda_2 = \sqrt{\frac{(N-d)\,(N-1)}{d}}.$$

The quadratic equation (8) also implies that $\sigma = \lambda_1 + \lambda_2$, which yields the value of σ stated in the theorem.

Finally, we construct the matrix

$$G \quad = \quad -\frac{1}{\lambda_1}\,A + \mathsf{I}.$$

It has a unit diagonal, and its off-diagonal entries have magnitude identically equal to $-1/\lambda_1$. Its two eigenvalues are $\frac{N}{d}$ with multiplicity d and zero with multiplicity $N - d$. Therefore, we may factor $G = S^* S$, where S is a $d \times N$ real equiangular tight frame. $\square$

Our construction of an adjacency matrix from a real equiangular tight frame is related to the concept of switching among regular two-graphs. In fact, the argument can be modified to establish the connection between regular two-graphs and strongly regular graphs. The interested reader may consult [1] for more information on these graph-theoretic results.

By definition, the parameters of a strongly regular graph must be integers. When $N \neq 2d$, one may reproduce Theorem 3.3 using Theorem 5.3 and the fact that the parameters of a strongly regular graph are integers. When $N = 2d$, the integrality of the graph parameters implies that d is odd, which is a new result.

Strongly regular graphs are further classified as Type I or Type II [1]. When $N = 2d$, it can be shown that the construction in Theorem 5.3 always yields a Type I graph, while the case $N \neq 2d$ leads to a Type II graph.

Theorem 5.4 *Suppose that $N = 2d$. A $d \times N$ real equiangular tight frame can exist only if $(N-1)$ is the sum of two squares.*

Proof: The strongly regular graph corresponding to the frame is a Type I strongly regular graph. Therefore $n = N - 1$ must be the sum of two squares [1, Thm. 2.18]. $\square$

We end this section with an explicit example demonstrating the creation of a real equiangular tight frame from a suitable strongly regular graph. We start with the strongly regular graph with parameters: $(15, 6, 1, 3)$ which is depicted on Figure 1. The incidence matrix M of this graph is:

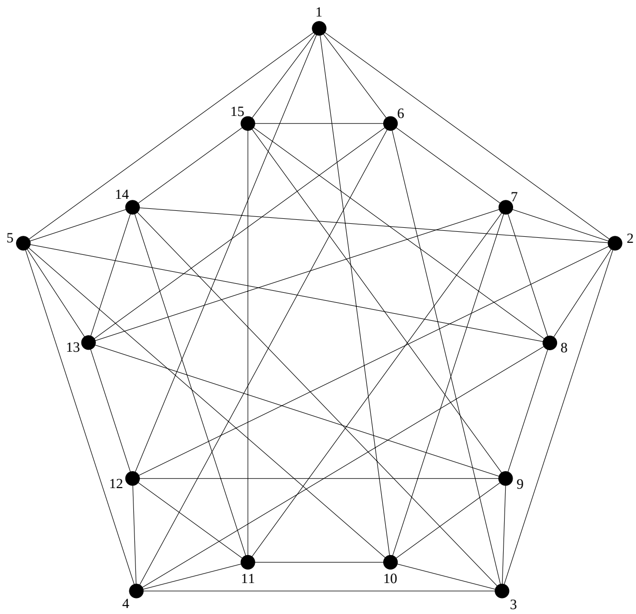

Figure 1: Strongly regular graph with parameters $(15, 6, 1, 3)$.

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0
\end{pmatrix}.$$

Following the notation used in the proof of Theorem 5.3, $N = 16$, $t = 3$ and we set $\sigma = N - 4t - 2 = 2$. We construct matrix A according to (13) and we can verify that this matrix satisfies

$A^2 = 2A + 15\mathsf{I}$. The eigenvalues are equal to $\lambda_1 = -3$ and $\lambda_2 = 5$, the latter with multiplicity $d = 6$. Finally, the matrix $G = -\frac{1}{3}A + \mathsf{I}$, factors as $S^T S$ (e.g. by the spectral decomposition) providing 16 frame vectors of $\mathbb{R}^6$. Below we show the frame vectors as the row vectors for typesetting reasons.

$$\begin{pmatrix} -0.1436 & 0.0386 & 0.6623 & -0.4633 & -0.3333 & -0.4620 \\ -0.7718 & 0.4187 & 0.2162 & -0.2004 & 0.3333 & -0.1767 \\ 0.6783 & 0.4346 & 0.0137 & -0.1800 & -0.3333 & -0.4554 \\ -0.6362 & -0.2930 & -0.2162 & -0.5925 & -0.3333 & -0.0235 \\ -0.1356 & 0.7116 & 0.4324 & 0.3921 & -0.3333 & -0.1528 \\ 0.1464 & -0.5292 & -0.2162 & -0.1633 & -0.3333 & -0.7169 \\ 0.6282 & -0.3801 & 0.4461 & -0.2630 & 0.3333 & -0.2857 \\ -0.6675 & -0.2522 & 0.2025 & 0.4089 & -0.3333 & -0.4143 \\ -0.1544 & -0.1439 & 0.4461 & -0.6921 & 0.3333 & 0.4077 \\ 0.0108 & 0.1824 & 0.2162 & 0.2288 & 0.3333 & -0.8697 \\ 0.4818 & 0.1491 & 0.6623 & -0.0997 & -0.3333 & 0.4312 \\ 0.0421 & 0.1416 & -0.2025 & -0.7725 & 0.3333 & -0.4789 \\ -0.0501 & -0.8147 & 0.4324 & -0.0829 & -0.3333 & 0.1697 \\ -0.1043 & 0.6708 & 0.0137 & -0.6092 & -0.3333 & 0.2380 \\ -0.1857 & -0.1031 & 0.8648 & 0.3092 & 0.3333 & 0.0169 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -1.0000 & 0.0000 \end{pmatrix}$$

6 Upper Bounds

The literature also contains upper bounds on the number of equiangular lines that can exist in a Euclidean space. The usual proof of these results [2] is not very accessible. We offer an elegant new argument that relies only on matrix theory.

Theorem 6.1 *An upper bound on the number N of equiangular lines that can be constructed in a d-dimensional Euclidean space is*

$$\begin{array}{ll} N \le \frac{1}{2}\, d\,(d+1) & \text{in } \mathbb{R}^d, \text{ and} \\ N \le d^2 & \text{in } \mathbb{C}^d. \end{array} \tag{14}$$

Proof: Suppose that $\{\boldsymbol{s}_j\}$ is a collection of N distinct unit vectors in d-dimensional Euclidean space. Define the $N \times N$ Gram matrix G whose entries are $g_{jk} = \langle \boldsymbol{s}_j, \boldsymbol{s}_k \rangle$. The Gram matrix is conjugate symmetric, and it has rank d. If the vectors represent a set of equiangular lines, then the off-diagonal entries of the Gram matrix all have the same magnitude, say α.

Let the symbol '$\circ$' denote the Hadamard, or elementwise, product of two matrices. Observe that $G \circ G^T$ is a nonnegative matrix with a unit diagonal and with all off-diagonal entries equal to α^2. In symbols,

$$G \circ G^T \quad = \quad \alpha^2\, \mathsf{J} + (1 - \alpha^2)\, \mathsf{I}.$$

Since α lies in the interval $[0, 1)$, one may calculate directly that $G \circ G^T$ has rank N.

It is well known that matrix rank is Hadamard submultiplicative [6]. Therefore,

$$N = \operatorname{rank}(G \circ G^T) \le (\operatorname{rank} G)(\operatorname{rank} G^T) = d^2.$$

This establishes the result in the complex case.

The real case requires a slightly more detailed analysis. Since G is a symmetric matrix of rank d, it may be written as the sum of d rank-one matrices:

$$G \quad = \quad \sum\nolimits_{j=1}^{d} \boldsymbol{u}_j\, \boldsymbol{u}_j^T.$$

Therefore,

$$\begin{aligned} G \circ G^T &= \sum\nolimits_{j,k=1}^{d} (\boldsymbol{u}_j\, \boldsymbol{u}_j^T) \circ (\boldsymbol{u}_k\, \boldsymbol{u}_k^T) \\ &= \sum\nolimits_{j,k=1}^{d} (\boldsymbol{u}_j \circ \boldsymbol{u}_k)\, (\boldsymbol{u}_j \circ \boldsymbol{u}_k)^T \\ &= \sum\nolimits_{j=1}^{d} (\boldsymbol{u}_j \circ \boldsymbol{u}_j)\, (\boldsymbol{u}_j \circ \boldsymbol{u}_j)^T \\ &\quad + 2 \sum\nolimits_{j<k} (\boldsymbol{u}_j \circ \boldsymbol{u}_k)\, (\boldsymbol{u}_j \circ \boldsymbol{u}_k)^T. \end{aligned}$$

It is evident from this expression that the rank of $G \circ G^T$ cannot exceed $\frac{1}{2}\, d\,(d+1)$. The real case follows. This argument can also be adapted to provide a direct proof of the complex case. $\square$

d	N
3	6
5	10
6	16
7	14
7	28
9	18

d	N
13	26
15	30
15	36
19	38
19	76
20	96

d	N
21	42
23	46
25	50
27	54
28	64
31	62

d	N
33	66
41	82
43	86
45	90
45	100
49	98

Table 1: The pairs (d, N) with $N \le 100$ for which a real equiangular tight frame exists. This table excludes the trivial cases $N = d$ and $N = d + 1$ where an equiangular tight frame always exists.

7 Consequences of Necessary Conditions

Table 1 lists all pairs (d, N) with $N \le 100$ for which the necessary conditions of Theorem A are in force. Using tables of known strongly regular graphs, we have been able to establish that a real equiangular tight frame actually exists in each of these cases.

To emphasize how much our conditions improve on the results of Holmes and Paulsen, we have tabulated cases where

d	N	$\sigma = \lambda_1 + \lambda_2$	λ_1	λ_2	s	t	Reason
4	8	0	-2.65	2.65	0.50	1.50	P2, P3
6	12	0	-3.32	3.32	1.50	2.50	P2, P3
8	16	0	-3.87	3.87	2.50	3.50	P2, P3
10	20	0	-4.36	4.36	3.50	4.50	P2, P3
10	25	2	-4.00	6.00	3.25	5.25	P1, P2
11	22	0	-4.58	4.58	4.00	5.00	P3
11	33	4	-4.00	8.00	3.75	6.75	P1, P2
12	24	0	-4.80	4.80	4.50	5.50	P2, P3
12	45	7	-4.00	11.00	4.50	9.00	P1, P2
13	65	12	-4.00	16.00	5.75	12.75	P1, P2
14	28	0	-5.20	5.20	5.50	6.50	P2, P3
16	32	0	-5.57	5.57	6.50	7.50	P2, P3
17	34	0	-5.74	5.74	7.00	8.00	P3
17	51	5	-5.00	10.00	7.50	11.00	P1, P2
18	36	0	-5.92	5.92	7.50	8.50	P2, P3
20	40	0	-6.25	6.25	8.50	9.50	P2, P3
21	49	2	-6.00	8.00	9.25	11.25	P1, P2
22	44	0	-6.56	6.56	9.50	10.50	P2, P3
22	55	3	-6.00	9.00	10.00	12.50	P1, P2
24	48	0	-6.86	6.86	10.50	11.50	P2, P3
26	52	0	-7.14	7.14	11.50	12.50	P2, P3
26	91	9	-6.00	15.00	14.50	20.00	P1, P2
28	56	0	-7.42	7.42	12.50	13.50	P2, P3
29	58	0	-7.55	7.55	13.00	14.00	P3, P3
30	60	0	-7.68	7.68	13.50	14.50	P2, P3
32	64	0	-7.94	7.94	14.50	15.50	P2, P3
33	99	7	-7.00	14.00	18.00	22.50	P1, P2
34	68	0	-8.19	8.19	15.50	16.50	P2, P3
35	70	0	-8.31	8.31	16.00	17.00	P3
36	72	0	-8.43	8.43	16.50	17.50	P2, P3
36	81	2	-8.00	10.00	17.25	19.25	P1, P2
37	74	0	-8.54	8.54	17.00	18.00	P3
38	76	0	-8.66	8.66	17.50	18.50	P2, P3
39	78	0	-8.78	8.78	18.00	19.00	P3
40	80	0	-8.89	8.89	18.50	19.50	P2, P3
42	84	0	-9.11	9.11	19.50	20.50	P2, P3
44	88	0	-9.33	9.33	20.50	21.50	P2, P3
46	92	0	-9.54	9.54	21.50	22.50	P2, P3
47	94	0	-9.64	9.64	22.00	23.00	P3
48	96	0	-9.75	9.75	22.50	23.50	P2, P3
50	100	0	-9.95	9.95	23.50	24.50	P2, P3

Table 2: All pairs (d, N) with $N \leq 100$ that meet the condition of Holmes and Paulsen even though no real equiangular tight frame exists (we exclude cases with $N \leq \frac{1}{2}d(d+1)$, see (14)). Holmes and Paulsen's condition requires that σ, the sum of the eigenvalues λ_1 and λ_2 of the derived matrix (4), be an integer. The "Reason" field above indicates why no equiangular tight frame exists using the following legend, P1: $N \neq 2d$, but λ_1 or λ_2 is not odd, P2: The calculated parameters s and t of the strongly regular graph are not integers, P3: $N = 2d$ but $(N-1)$ is not the sum of two squares.

their conditions admit the possibility of a real equiangular tight frame even though none exist. Table 2 lists each pair (d, N) with $N \leq 100$ and $N \leq \frac{1}{2}\,d\,(d+1)$ that meets their condition 2 but fails to satisfy Theorem A. We have calculated the eigenvalues of the derived matrix(4), their sum σ, as well as the parameters s and t of the strongly regular graph. The "Reason" field explains which of our necessary conditions forbids the existence of a real equiangular tight frame.

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