# ON THE STEADY STATE PROPERTIES OF NETWORKS OF QUEUES

by

Fernando Palacios Gomez

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Department of Computer Sciences
The University of Texas
Austin, Texas 78712

# ON THE STEADY STATE PROPERTIES OF NETWORKS OF QUEUES

Fernando Palacios Gomez
Department of Computer Science
The University of Texas at Austin
Austin, Texas 78712

ABSTRACT

This paper derives some important properties of the equilibrium behavior of arbitrarily complex networks of queues.

Emphasis is given to networks that have been extensively used in modeling computer systems but the results are also applicable to general queueing problems. No restriction is made on the queueing discipline or the distribution of service requirements of a job in a particular station (server) of the network. Closed networks with an arbitrary number of circulating jobs and open networks fed by external sources where the distribution of interarrival times is arbitrary are both considered.

## INTRODUCTION

Queueing networks have found wide application in the analysis of time-sharing and multiprogramming systems along with applications in other several disciplines. Most queueing models developed so far (1, 2, 3, 8, 9, 11, 12) restrict attention to the analysis of systems under equilibrium conditions. Also in most cases analysis has been restricted to networks containing

a) exponential servers

b) servers whose service distribution has a rational Laplace-Stieltjes transform (1, 3, 11, 12) and where the device is simultaneously shared by all jobs in the queue. This discipline has been called "processor sharing" (15).

c) non-exponential servers that work under Last Come First Served preemptive-resume (3) scheduling disciplines.

Chandy (3) proved that the concept of local balance equations is applicable to these models. In almost every situation that arises in computer systems and in several other disciplines, the goal of the analysis is to determine the average throughput of the entire network and of each of its individual components. It is also the case in most situations that the most important parameters of design are the utilizations (busy time) of each component in the network. This paper shows how these goals can be achieved with a minimum effort and:

a) independently of the way the individual queues are managed (scheduling discipline)

b) the networks considered are arbitrary

c) the probability distributions of the interarrival times of the external sources feeding the networks and also the probability distribution of service time in each queue, are irrelevant except for their expected values (means).

The reader may be surprised of the simplicity and neatness of the solutions offered here to this problem.

The argument used to derive these properties seems to have been used for the first time by Khintchine (10). It states that if a queue is to achieve steady state then the expected number of arrivals per unit time should be equal to the expected number of departures per unit time for that queue.

## Mathematical Analysis

The paper is divided in four independent sections: Open Networks, Saturation Conditions, Throughput Analysis and Cyclic Networks.

Open Networks - Let us consider a general network of queues (see figure 2) where arrivals enter the network from an external source(s) according to some probability distribution. In each server of the network a job is serviced for a period of time depending on its demand for service at this queue. Its service in the queue may depend upon:

a) service demand

b) input process

c) scheduling discipline

The latter case can be appropriately handled in most cases because for the purpose of this analysis the only thing that matters is the average service time of a random job in the server. After a job receives service in the $i^{th}$ queue it chooses with some fixed probability $P_{ij}$ to join the $j^{th}$ queue or it departs from the system with probability $P_{i\infty}$. The network contains any number of queues interconnected in an arbitrary fashion. We will describe the approach by using a Poisson source for simplicity and compactness. It should be borne in mind, however, that this restriction is superfluous and

<u>does not restrict the scope of the results</u>. Assume that the network is being fed by infinite sources of jobs and that the interarrival times are identically distributed independent random variables with a finite mean. Assume that the average number of arrivals per unit time to the $i^{th}$ queue is $W_i$. We follow Chandy's approach (3) of replacing several external poisson sources by a unique Poisson source with mean W in the following form:

For $i = 1, \ldots, n$ define $P_{oi}W = W_i$ then $P_{oi} = W_i/W$ where $W = \sum_{i=1}^{m} W_i$

i.e. when a new job arrives it joins the $i^{th}$ queue with probability $P_{oi}$. Obviously the two arrival mechanisms are equivalent. It is clear that if a queue is in steady state, the expected number of arrivals is equal to the expected number of departures per unit time. In the case under consideration it is also true that the input rate at steady state to any queue is a fraction of W.

Denote the input rate to the $i^{th}$ queue by $W \cdot BT(i)$, (this notation is used by Chandy (3) where BT stands for BTERM) define also BT(0) as 1, then at steady state the input (output) rates of all queues should satisfy:

(1) $i = 1, \ldots, n \; W \cdot BT(i) = \sum_{j=0}^{n} W \cdot BT(j) \, P_{ji}$ then the process of finding the input rate to any queue at equilibrium reduces to solving the set of linear equations

(2) $i = 1, \ldots, n \; BT(i) = \sum_{j=0}^{n} BT(j) \, P_{ji}$

In matrix notation and recalling that $BT(0) = 1$

(3) $[BT(1) \; BT(2) \; . \; . \; . \; . \; . \; BT(n)] \; [I - P] = [P_{01} \; P_{02} \; . \; . \; . \; P_{0n}]$

where P is the matrix of $P_{ij}$'s. But notice that this matrix is non-singular because $P_{i\infty} > 0$ for at least one queue; otherwise jobs would accumulate in the network and steady state cannot be achieved. Then a unique solution for the vector $BT = [BT(1) \; BT(2) \; . \; . \; . \; BT(n)]$ exists.

Now denote by $A_i$ the probability that at steady state the $i^{th}$ server is busy, while the server is busy the expected number of customers served per unit time by this server is $A_i/E(S_i)$ where $E(S_i)$ is the expected service time of a random customer in the $i^{th}$ queue.

By Khintchine's (10) argument the following has to be true:

(4) $i=1, \ldots, n \quad A_i/E(S_i) = W \cdot BT(i)$ for if not the queue is not in steady state.

From (4) we have:

(5) $A_i = W \cdot BT(i) \cdot E(S_i) \quad i = 1, \ldots, n$

Saturation Condition - In order for the $i^{th}$ queue not to reach saturation and thus to accomplish any service in a finite it must be true that:

(6) $A_i = W \cdot BT(i) \cdot E(S_i) < 1$

It should be emphasized that the saturation condition for the whole network cannot be established a priori and will depend on its "geometry" (configuration).

Throughput Analysis - Denote by TH(i) the throughput (rate of completions) of the $i^{th}$ server, then:

(7) $TH(i) = A_i \, P_{i\infty}/E(S_i) = W \cdot BT(i) \cdot P_{i\infty} \quad i = 1, \ldots, n$

Naturally the total throughput of the network is W.

In general if the input is not Poisson the method reduces to find a solution to the system of equations:

(8) $RT(i) = \alpha_i + \sum_{j=1}^{n} RT(j) \, P_{ji}$

where RT(i) denotes the steady state input rate to the $i^{th}$ queue and $\alpha_i$ the average rate of the external sources feeding the $i^{th}$ queue.

Here again,

(9) $A_i/E(S_i) = RT(i) \quad i = 1, \ldots, n$

Cyclic Networks - Consider the network of figure 3 where all jobs in the network go through a cycle between the points labeled "source" and "sink". We assume that there are N circulating jobs. The connections between devices are arbitrary as are the service distributions and disciplines in any queue. Assume that at steady state $R_s$ jobs per unit time are passing through the source, then it is obvious that if we define BT(i) as the probability that a job gets to the $i^{th}$ queue in one "cycle" the following is true at equilibrium:

$BT(i) \cdot R_s$ jobs are joining the $i^{th}$ queue per unit time and by Khintchine's argument:

(10) $BT(i) \cdot R_s = A_i/E(S_i)$ $i = 1, \ldots, n$ where $A_i$ and $E(S_i)$ are defined as before.

But this implies that if $K \neq L$

$$(11) \quad \frac{A_K}{A_L} = \frac{BT(K)\ E(S_K)}{BT(L)\ E(S_L)}$$

this is a strong generalization of a conservation law first derived by BUZEN (2) for a simple central server exponential network (2, 11, 12). This law says that knowing the utilization of any server in the network then we can find the others with equation (11).

Mean Flow time in a Cyclic Network with a Central Server

For this section we consider cyclic networks where the "source" is clearly identified with one queue (see Figure 3).

Denote by E(T) the mean flow time of a job through the network i.e. the expected time taken by a job in a cycle between the source and the sink.

Then by the steady state argument used by Khintchine (9), Adiri (7) Kleinrock and others, the expected arrival rate per unit time to the source in this network is NE(T), but this arrival rate has to be identical to the departure rate of the central server (source) then $N/E(T) = A_o/E(S_o)$ where the subscript o refers to the queue chosen as source in the cyclic network. Then,

$$E(T) = NE(S_o)/A_o \tag{12}$$

Here again we only need to know one utilization to derive several important parameters of the network under stationary conditions.

The same approach is applicable to strongly connected graphs of queues.

## Conclusions and Extensions

A set of important results concerning queueing networks was presented. These results permit in most applications of importance to determine the throughput of any arbitrarily complex open network. For closed networks with a finite number of customers a conservation law and an expression for the mean flow time of a job were derived in terms of one parameter of the network.

It should be emphasized that no assumption whatsoever was made regarding arrival patterns, scheduling disciplines or service distributions. All these results were derived using a well-known property of any queueing situation under stationary behavior.

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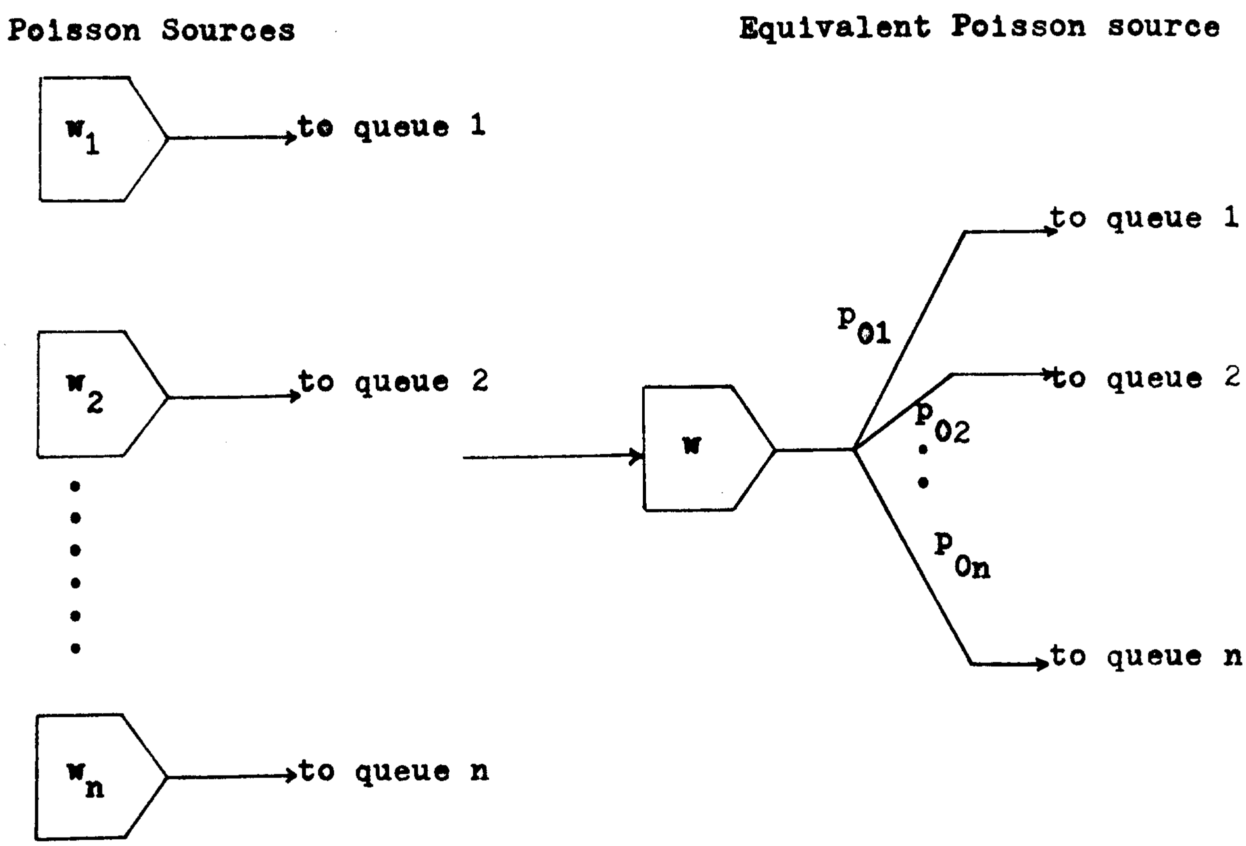



$$w = \sum_{i=1}^{n} w_i \qquad w_i = p_{0i} w \; ; \quad p_{0i} = \frac{w_i}{w}$$

Figure 1. Equivalent Poisson Sources

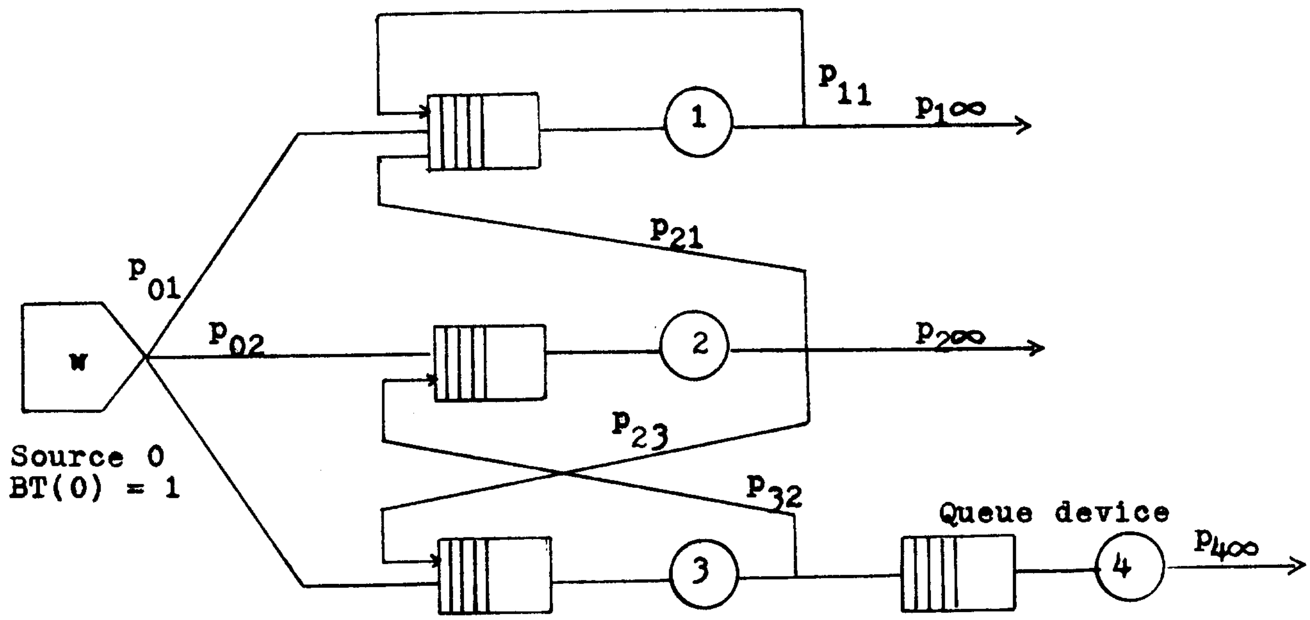
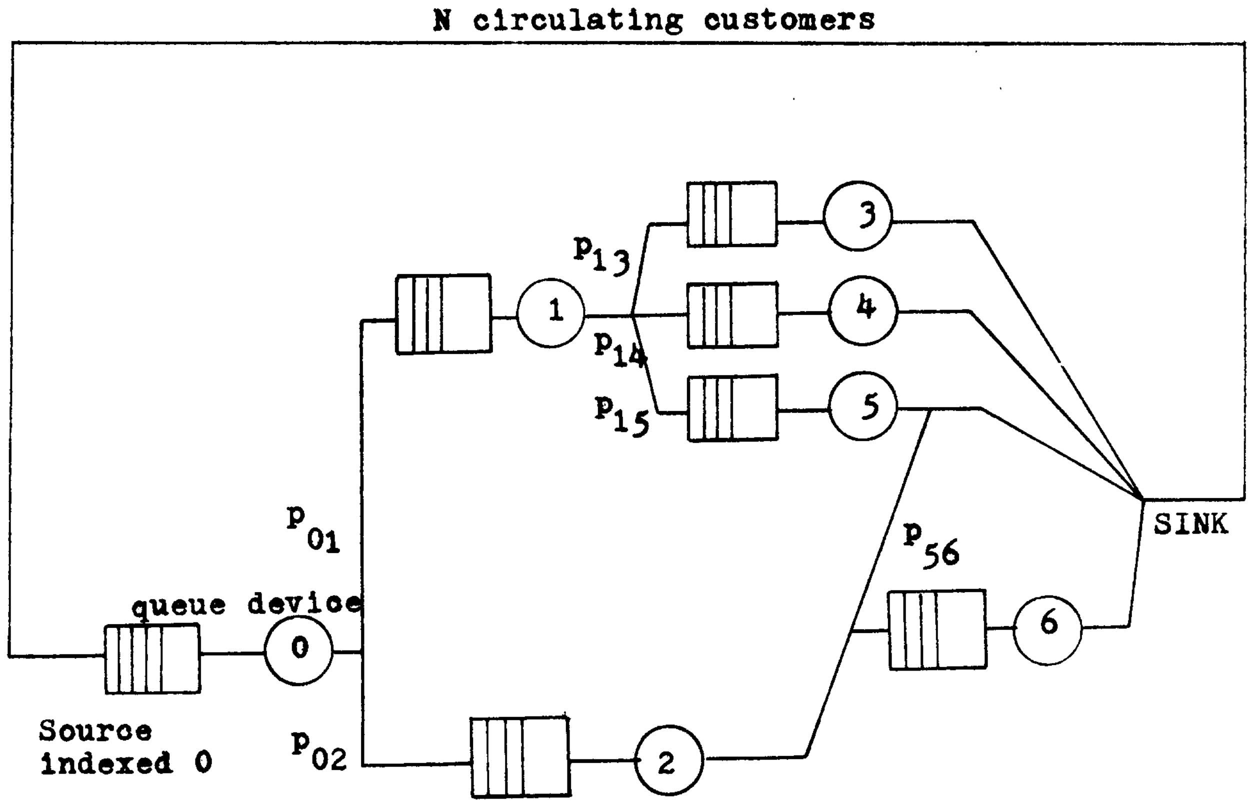



Figure 2. Open Network




| Queue number | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| BT(i) | 1 | $p_{01}$ | $p_{02}$ | $p_{01}p_{13}$ | $p_{01}p_{14}$ | $p_{01}p_{15}$ | $p_{02}+p_{56}p_{01}p_{15}$ |

Figure 3. Cyclic Network

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