LOCAL BALANCE, ROBUSTNESS, POISSON DEPARTURES AND THE PRODUCT FORM IN QUEUEING NETWORKS*

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ABSTRACT

This paper uses the concept of local balance and continuous state analysis to study (a) robustness, (b) poisson departures and (c) the product form for equilibrium state probabilities in queueing networks.

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SECTION I

Introduction

A great deal of research on queueing network models has appeared in the literature[1,2,3,4]. In this paper we study a class of queueing networks with arbitrary differentiable service distributions. This paper studies three aspects of queueing systems:

1) Robustness. Many systems are robust in the sense that they are somewhat insensitive to service distributions: their equilibrium behavior is a function of the mean service time but is largely independent of higher order moments. In this paper we study robustness in queueing systems in some detail. In particular we develop sufficient conditions for a system to be perfectly robust or distribution independent. Processor-sharing and last come first served preemptive resume disciplines are used to illustrate the theorems on distribution independence[4]. The service distributions are not restricted to having rational Laplace transforms.

2) Poisson Departure Processes.

Reich[5] and others showed that departures from a queue with Poisson arrivals and independent exponential service formed a Poisson process 6. In this paper sufficient conditions for a system to have Poisson departures are presented; the Processor-Sharing and Last Come First Served Preemptive Resume disciplines satisfy this condition. Muntz [6] has also obtained sufficient conditions for Poisson departure processes.

3) Product Form of Equilibrium State Probabilities

In some networks, the equilibrium state probability of the network is the product of terms: where each term is concerned with the 'state' of some particular queue in the network. Jackson [7], Gordon and Newell [8], Baskett, Chandy, Muntz and Palacios-Gomez [4]and others have studied networks with the product form. In this paper we present sufficient conditions for the equilibrium state probability of a network to have the product form; this result extends the class of networks known to have this property.

The common thread underlying these investigations is the property of "local balance" [2,4, 9]. We show that if a single service system with Poisson arrivals is in local balance then (a) departures from the system occur in a Poisson manner and (b) a network of such systems will have the product form of solution. We also show that if a single service system with Poisson arrivals is in detailed local balance then the system is departure independent. The concept of local balance is quite a powerful one and finds application in other areas. All the key results of this paper stem from local balance.

The work on networks reported in [4,7,8] uses discrete-state continuous-transition Markov processes. In this paper we use continuous-state analysis obviating the restriction of service distributions to those which can be represented as a sequence of exponential stages.

The structure of this paper is as follows. Section 2 leads into the general approach by considering the First Come First Served (FCFS), Processor Sharing (PS) and the Last Come First Served Preemptive Resume (LCFSPR) disciplines. A key feature is that the differential equation for the state probability density as a function of time is continuous in the variables representing remaining service times for the individual customers. Using this form allows consideration of arbitrary differentiable service time distributions. The concept of distribution-independence is introduced here and is shown to apply to PS and LCFS. Although FCFS is not distribution independent, a tractable form is derived for the exponential case.

The main results appear in Section 3, which describes a general class of service disciplines and constructs the general differential equation for the probability density given an arbitrary differentiable service time distribution and assuming Poisson arrivals. The differential equation is then used to characterize several properties a discipline might have at steady state, including local balance, detailed local balance, and immediate service, and the beneficial effects of these properties are studied. Section 3 considers single service queues; section 4 applies these results to networks of queues. This section extends and clarifies earlier work on local balance in networks . The conclusions are presented in Section 5.

SECTION II

Processor Sharing, Last Come First Served

Preemptive Resume and First Come First Served Disciplines

In this section we set up differential equations which describe the behavior of a single server with customers arriving in a time-invariant Poisson manner. We first analyze the PS, LCFSPR and FCFS disciplines, with arbitrary differentiable service distributions. Equilibrium solutions for the differential equation are developed for the PS and LCFSPR case. We next develop differential equations for a class of networks which includes PS, LCFSPR and FCFS. We characterize a class of disciplines for which the equilibrium probabilities are "distribution-independent"; for this class the equilibrium probability that there are n customers in the queue depends only on the mean value of the service time and is otherwise independent of the service distribution. Equilibrium probabilities for "distribution-independent" disciplines are presented. We show that PS and LCFSPR are distribution-independent disciplines. We emphasize at this point that we only consider differentiable distributions.

We shall use the notation $f_x(\ )$ for the probability density function of a random variable, x, i.e., $f_x(x_0) \cdot \Delta x_0$ is the probability that $x_0 \leq x < x_0 + \Delta x_0$. Let u be the service time with probability density function $f_u(\ )$. Let λ be the arrival rate.

A customer's service rate may be a function of the number of customers in the queue. We assume that the service rate when there are N customers in the queue is h(N) times the rate when there is only one customer in the queue, where h(N) is any function which satisfies the conditions $h(0) = 0$, $h(1) = 1$, and $h(N) \geq 0$ for $N = 2, 3, 4, \ldots$. We refer to h( ) as a <u>rate function</u>. We now discuss PS, LCFSPR and FCFS in detail.

<u>PS</u> is the limiting case of a no-overhead round robin fixed quantum discipline as the quantum tends to zero[2,4,9]. Let N be the number of

customers in a queue. Every customer in the queue is processed in parallel; each customer gets an equal "share" of the server (processor) and is served at 1/N times the rate at which the single customer would be served if he were the only customer in the queue. In other words, for a PS queue every customer in the queue receives service in parallel and the rate function for each customer is $h(N) = 1/N$, for $N = 1, 2, 3, \ldots$

The states of a system are defined such that the future behavior of a system given its current state is independent of the past behavior of the system. The state of the PS system is completely specified by N, the number of customers in the queue, and the remaining service time for each customer. The state of the system at any given time may be represented by $S = (0)$ if the queue is empty, and $S = (N, \{T_1, \ldots, T_N\})$ where $N > 0$ and $T_1 > 0, \ldots, T_N > 0$ are the remaining service times for the customers in the queue. Note that $\{T_1, \ldots, T_N\}$ is an <u>unordered</u> set. We assume $T_i \neq T_j$ if $i \neq j$ since the event $T_i = T_j$ has measure zero.

Let $Q(t, N, \{T_1, \ldots, T_N\}) \cdot \Delta T_1 \cdot \ldots \cdot \Delta T_N$ be the probability that the system is in some state $(N \{t_1, \ldots, t_N\})$ at time t, where $T_i < t_i \leq T_i + \Delta T_i$ for $i = 1, \ldots, N$. Let $Q(t, 0)$ be the probability that the system is in state (0) at time t. Let $Q(t, N \{T_1, \ldots, T_N\})$ be the probability density of state $(N \{T_1, \ldots, T_N\})$ at time t.

The system can be in state $(N, \{T_1, \ldots, T_N\})$ at time $t + \Delta t$ if and only if one of the following conditions holds (neglecting second order terms):

(a) The system was in state $(N \{T_1 + h(N) \cdot \Delta t, \ldots, T_N + h(N) \cdot \Delta t\})$ at time t and no customer enters the system in the interval $[t, t + \Delta t]$. Each customer in the system gets $h(N) \cdot \Delta t$ units of processing time in the interval $[t, t + \Delta t]$. Hence if a customer has $T_i + h(N) \cdot \Delta t$ units of service time remaining at time t, he will have T_i units of service time remaining at $t + \Delta t$.

(b) The system is in state $(N + 1, \{h(N + 1) \cdot \tau, T_1 + h(N + 1) \cdot \Delta t, .., T_N + h(N + 1) \cdot \Delta t\})$ at time t, where $0 \leq \tau < \Delta t$, and no customers enter the queue in the interval $[t, t + \Delta t]$. The customer with a remaining service time of $h(N + 1) \cdot \tau$ will leave the queue at time $t + \tau$, leaving N customers with remaining service times $T_1, T_2, .., T_N$.

(c) The system is in state $(N - 1, \{T_1 + h(N - 1) \cdot \Delta t, .., T_{j-1} + h(N - 1) \cdot \Delta t, T_{j+1} + h(N - 1) \cdot \Delta t, .., T_N + h(N - 1) \cdot \Delta t\})$ at time t and a single customer enters the queue in the interval $[t, t + \Delta t]$, and this customer has a service time of T_j units.

Neglecting second-order terms, we have

$$\begin{aligned} Q(t + \Delta t, N, \{T_1, .., T_N\}) &= Q(t, N, \{T_1 + h(N) \cdot \Delta t, .., T_N + h(N) \cdot \Delta t\})(1 - \lambda \cdot \Delta t) \\ &+ Q(t, N + 1, \{0+, T_1, .., T_N\}) \cdot h(N + 1) \cdot \Delta t \\ &+ (\lambda \cdot \Delta t) \cdot f_u(T_j) \cdot Q(t, N - 1, \{T_1, .., T_{j-1}, T_{j+1}, .., T_N\}) \end{aligned} \tag{1}$$

Expanding the first term on the right side of this equation in Taylor series about the variables T_i, $i = 1, .., N$, and neglecting second-order terms we get:

$$\begin{aligned} \frac{d}{dt} Q(t, N, \{T_1, .., T_N\}) &= \sum_{j=1}^{N} h(N) \cdot \frac{\partial}{\partial T_j} Q(t, N, \{T_1, .., T_N\}) \\ &- \lambda \cdot Q(t, N, \{T_1, .., T_N\}) \\ &+ h(N + 1) \cdot Q(t, N + 1, \{0^+, T_1, .., T_N\}) \\ &+ \sum_{j=1}^{N} \lambda \cdot f_u(T_j) \cdot Q(t, N-1, \{T_1, .., T_{j-1}, T_{j+1} .., T_N\}) \end{aligned} \quad ($$

For N = 0 we have

$$\frac{d}{dt} Q(t, 0) = - \lambda \cdot Q(t, 0) + Q(t, 1, \{0\}) \tag{3}$$

Recall that u is the service time. Let v be the time remaining from a point of random entry into the random variable u. In other words v is the remaining service time if customers are preempted at random times. It is well known that v and u are related by the equations

$$f_v(t) = \mu . P(u \geqslant t) \text{ and } f_u(t) = -\frac{1}{\mu} . \frac{d}{dt} f_v(t) \tag{4}$$

where $P(u \geqslant t)$ is the probability that u is greater than or equal to t. Note that

$$f_v(0) = \mu . P(u \geqslant 0) = \mu \tag{5}$$

We shall denote the equilibrium value of $Q(t, N, \{T_1, .., T_N\})$ by $Q(\infty, N, \{T_1, .., T_N\})$. It is easy to verify that

Lemma 1

$$Q(\infty, N, \{T_1, .., T_N\}) = \frac{1}{G.H(N)} \cdot \left(\frac{\lambda}{\mu}\right)^N . f_v(T_1). \dots f_v(T_N) \tag{6}$$

and $Q(\infty, 0) = 1/G$, where G is a normalizing constant and H(N) is defined recursively as

$$H(1) = 1$$
$$H(N) = H(N - 1) . h(N) \tag{7}$$

For the special case of PS we have $h(N) = \frac{1}{N}$ and hence $H(N) = \frac{1}{N!}$

Proof: Differentiating (4) we see

$$\frac{d}{dt} f_v(t) = -\mu . f_u(t)$$

Consider the first term on the right-hand side of the differential equation. By direct substitution of equation (6) we get

$$h(N) \cdot \frac{\partial}{\partial T_j} Q(\infty, N, \{T_1, \ldots, T_N\}) = \lambda f_u(T_j) \cdot Q(\infty, N-1\{T_1, \ldots, T_{j-1}, T_{j+1}, \ldots, T_N\}) \tag{8}$$

This term cancels with the last term in the differential equation.

Now consider the third term on the right-hand side of the differential equation. Recollecting that $f_v(0) = \mu$, and by direct substitution of equation (6) we get:

$$h(N+1) \cdot Q(\infty, N+1, \{0, T_1, \ldots, T_N\}) = \lambda \cdot Q(\infty, N\{T_1, \ldots, T_N\}) \tag{9}$$

This cancels with the second term. Hence equation (4) implies that:

$$\frac{d}{dt} Q(\infty, N, T_1, \ldots, T_N) = 0 \text{ all states } (N\{T_1, \ldots, T_N\})$$

Similarly:

$$\frac{d}{dt} Q(\infty, 0) = 0$$

Lemma 2

Departures of customers from the PS queue forms a Poisson process.

Proof: Let $R(N, \{T_1, \ldots, T_N\}) \cdot \Delta t \cdot \Delta T_1 \ldots \Delta T_N$ be the probability that a customer finishes service and departs from the system in some interval $[t, t+\Delta t]$ at steady-state causing the system to transit to state $(N\{T_1, \ldots, T_N\})$ immediately after the departure. A customer can finish service in the interval $[t, t+\Delta t]$ causing the system to transit into state $(N\{T_1, \ldots, T_N\})$ if and only if the system was in one of the states $(N+1, \{h(N+1) \cdot \tau, T_1 + h(N+1) \cdot \Delta t, \ldots, T_N + h(N+1) \cdot \Delta t\})$ where $0 \leq \tau \leq \Delta t$. Hence by direct substitution

$$R(N\{T_1, \ldots, T_N\}) = Q(\infty, N+1, \{0+, T_1, \ldots, T_N\}) \cdot h(N+1) \cdot$$
$$= \lambda \cdot Q(\infty, N\{T_1, \ldots, T_N\}). \tag{10}$$

In otherwords, $R(N\{T_1, \ldots, T_N\})$ is the third term in the differential equation. The product $\lambda \cdot Q(\infty, N\{T_1, \ldots, T_N\})$ on the right hand side of equation (10) implies that at equilibrium the probability of a customer leaving the system in any interval $[t, t + \Delta t]$ is independent of the state of the system immediately after the departure. Therefore, departures in non-overlapping intervals are independent and the probability of a departure in any interval of length Δt is $\lambda \cdot \Delta t$. Hence departures form a time-invariant Poisson process with rate λ.

Distribution Independent Disciplines

Consider a single server queue, with customers arriving in a time-invariant Poisson process. The arrival rate is independent of the size of the queue. Consider an external observer who can only observe the number of customers in the queue and the departure of customers from the queue. (This observer cannot differentiate between the individual customers in the queue). Let the server use a service discipline 'd'. Consider the class C of all differentiable service distributions with a given (fixed) mean, say $1/\mu$. If the equilibrium behavior of the system as observed by the external observer is independent of the service distribution, provided the service distribution is a member of class C, then the service discipline d is said to be a DISD (distribution independent service discipline).

Lemma 3

PS is distribution independent.

Proof: We showed earlier that (eqn. 6):

$$Q(\infty, N\{T_1, \ldots, T_N\}) = \frac{1}{G} \cdot \frac{1}{H(N)} \cdot \left(\frac{\lambda}{\mu}\right)^N \cdot f_v(T_1), \ldots, f_v(T_N)$$

$$= \frac{1}{G} \cdot \left(\frac{\lambda}{\mu}\right)^N \cdot N!\ f_v(T_1) \cdots f_v(T_N)$$

since $h(n) = 1/n$ and hence $H(N) = 1/(N!)$

Let the probability that there are N customers in the queue at time t be $Q(t, N)$. Then

$$Q(\infty, N) = \frac{1}{G} \cdot \left(\frac{\lambda}{\mu}\right)^N \tag{11}$$

and the number of customers in the queue is independent of the remaining service times for all customers.

LCFSPR

The Last Come First Served Preemptive Resume discipline is a priority preemptive resume discipline in which the priority of a customer is the time at which he enters the queue. At most one customer receives service at any instant of time and he is the customer with the highest priority (that is, the customer who arrived last). A customer with a higher priority preempts customers of lower priority. Service is resumed from the point of preemption.

The customers who arrive for service at a LCFSPR server are (effectively) placed in a stack. Only the topmost customer in a stack may receive service. When a new customer arrives he is immediately pushed onto the top of the stack and when a customer departs the next highest customer in the stack begins to receive service.

The state of an LCFSPR server may be represented as either S = (0) if the queue is empty or $S = (N, T_1, \ldots, T_N)$ where $N \geq 1$ if the queue is not empty. T_i is the remaining service time for the ith customer where we use the convention that the first customer is on top of the stack and the Nth customer is at the bottom. Note that in LCFSPR the customers in the queue are ordered whereas in PS the customers are unordered. Hence in LCFSPR the state $(N = 2, T_1 = 1, T_2 = 2)$ is different from the state $(N = 2, T_1 = 2, T_2 = 1)$ since the ith customer receives service before the $(i + 1)$th. Let h(N) be the rate function, with h(0) = 0 and h(1) = 1. When there are N customers in the queue, N > 0, the single customer being served is served at a rate h(N) times the rate at which he would be served if he were the only customer in the queue.

Let $Q(t, N, T_1, \ldots, T_N) \cdot \Delta T_1 \cdot \ldots \cdot \Delta T_N$ be the probability that the system is in any one of the states $(N, t_1, \ldots, t_N)$ where $T_i \leq t_i \leq T_i + \Delta T_i$, $i = 1, \ldots, N$, and let Q(t,0) be the probability that the system is in state (0) at time t. The system can be in state $(N, T_1, \ldots, T_N)$, N > 0 at time $t + \Delta t$ if and only if one of three conditions hold:

(a) The system is in state $(N, T_1 + h(N)\Delta t, T_2, \ldots, T_N)$ at time $t + \Delta t$ and no customer enters the system in the interval $[t, t + \Delta t]$. The customer on top of the stack gets $h(N) \cdot \Delta t$ units of service time in the interval $[t, t + \Delta t]$, and thus his remaining service time is reduced by $h(N) \cdot \Delta t$ units. The other customers in the stack do not get service in the interval $[t, t + \Delta t]$ and hence their remaining service times are unchanged.

(b) The system is in state $(N + 1, h(N + 1)\tau, T_1, \ldots, T_N)$ at time t where $0 \leq \tau \leq \Delta t$, and no customer enters the system in the interval $[t, t + \Delta t]$. The customer on top of the stack will get $h(N + 1) \cdot \tau$ units of service time in τ units of real time and hence he will leave the system in the interval $[t, t + \Delta t]$.

(c) The system is in state $(N-1, T_2 + h(N-1) \cdot \Delta t, T_3, \ldots, T_N)$ at time t and a new customer enters the system in the interval $[t, t + \Delta t]$, and the service time for this new customer is t_1 units where $T_1 \leq t_1 \leq T_1 + \Delta T_1$. This new customer enters the top of the stack causing the state of the system to change to $(N, T_1, \ldots, T_N)$.

A similar set of conditions may be obtained for state (0). The differential equation for LCFSPR is obtained in a manner analogous to that for PS. We get, for $N > 0$

$$\frac{d}{dt} Q(t,N,T_1,\ldots,T_N) = h(N) \cdot \frac{\partial}{\partial T_1} Q(t,N,T_1,\ldots,T_N) - \lambda Q(t,N,T_1,\ldots,T_N)$$
$$+ h(N + 1) \cdot Q(t, N + 1, 0+, T_1, \ldots, T_N)$$
$$+ \lambda \cdot f_u(T_1) \cdot Q(t, N-1, T_2, \ldots, T_N)$$

For $N = 0$ we have

$$\frac{d}{dt} Q(t,0) = -\lambda Q(t,0) + Q(t,1,0+)$$

The equilibrium solutions are the same as for PS:

<u>Lemma 4</u>

$$Q(\infty,N,T_1,\ldots,T_N) = \frac{1}{G} \cdot \frac{1}{H(N)} \cdot \left(\frac{\lambda}{\mu}\right)^N \cdot f_v(T_1), \ldots, f_v(T_N)$$

$$Q(\infty,0) = \frac{1}{G}$$

where G is a normalization constant. All terms have the same meaning as in PS.

Proof: The solution can be verified in the same manner as for PS.

Lemma 5

The departures from the LCFSPR queue form a Poisson process.

Proof: Same as for PS.

Lemma 6

LCFSPR is distribution independent.

Proof: Recall that Q(t,N) is the probability that there are N customers in the queue at time t. Then

$$Q(\infty,N) = \frac{1}{G} \cdot \frac{1}{H(N)} \cdot \left(\frac{\lambda}{\mu}\right)^N, \quad N = 0,1,2,\ldots$$

Hence the equilibrium probability that there are N customers in the queue depends on the mean service time and is otherwise independent of the service distribution. Since the departure process is also independent of the service distribution it follows that LCFSPR is distribution independent.

FCFS

The FCFS discipline is well known. The state of the system may be represented as S = (0) if the queue is empty and $S = (N,T_1,\ldots,T_N)$ for $N > 0$ where T_i is the remaining service time of the ith customer, $i = 1,\ldots,N$. By convention the ith customer gets service before the $(i+1)$th, $i = 1,\ldots,N-1$. A new customer enters the tail-end of the queue, i.e., becomes the $(N+1)$th customer. Only the single customer at the head of the queue receives service at any one time. We define $Q(t,N,T_1,\ldots,T_N)$ and obtain differential equations in the usual manner:

$$\frac{d}{dt} Q(t,N,T_1,\ldots,T_N) = h(N)\cdot \frac{\partial}{\partial T_1} Q(t,N,T_1,\ldots,T_N) - \lambda Q(t,N,T_1,\ldots,T_N)$$
$$+ h(N+1)\cdot Q(t,N+1,0+,T_1,\ldots,T_N)$$
$$+ \lambda \cdot f_u(T_N)\cdot Q(t,N-1,T_1,\ldots,T_{N-1})$$

Notice that the last term on the right-hand side is the only term that is different for LCFSPR and FCFS.

It is well known that FCFS is not distribution independent. Morse [10] and others have obtained equilibrium and transient solutions for the special case where $f_u(t) = f_v(t)$ all t, i.e. the service distribution is exponential. In this special case the solution is

$$Q(t, N, T_1, \ldots, T_N) = \frac{1}{G}\left(\frac{\lambda}{\mu}\right)^N \cdot f_v(T_1) \ldots f_v(T_N)$$

$$\text{where} \quad G = \frac{1}{(1 - \frac{\lambda}{\mu})}$$

SECTION III

General Single Service Systems

__Generalized Service Disciplines.__

We next derive differential equations which describe the behavior of a class of systems including queues with FCFS, LCFSPR and PS disciplines. In this section we discuss only Poisson arrival, single service systems; networks of queues are discussed in the next section. We then define the concept of "local balance" which is central to this paper, and show that local balance implies that departures from the system occur in a Poisson manner. We next study distribution independence and develop sufficient conditions for a system to be distribution independent. We begin this section with definitions of terms which are necessary to describe general systems.

__Servers__ An arriving customer has a service time from a given service distribution. However, there may be one or more service stations at which customers are processed. These stations may have different service rates. Consider a customer with a remaining service time of T units being served at the ith station when there are N customers in the station. After Δt units of real (or clock) time have elapsed this customer will have a remaining service time of $T - r_{i,N} \cdot \Delta t$ units. Note $r_{i,N} \geq 0$ may vary from station to station and may also depend on the number of customers in the system. In some systems a customer may switch from station to station until his service is complete. We will find it convenient to think of an infinite ordered sequence of stations 1, 2, 3,... with generic index i; some of the service stations may have $r_{i,N} = 0$ for all N--in other words some stations may be merely positions in which customers await service. Thus a single server FCFS system may be thought of as having an infinite number of stations with only one station--station 1--serving customers with $r_{1,\,N} = 1$ for all $N > 0$, and all other stations having $r_{i,\,N} = 0$ for all N. Whenever a customer departs from the first station the waiting customers move up one station. We will always assume that there can be at most one customer per station.

For processor-sharing we can have an infinite number of stations with $r_{i,\,N} = 1/N$ for $N > 0$ and $i = 1, 2, .., N$.

<u>States.</u> Suppose there are N customers with remaining service times $T_1, .., T_N$. Suppose the customer with remaining service time T_j is in station s_j, $j = 1, .., N$. The state of the system may be represented as the 2N-tuple $(T_1, s_1; T_2, s_2; ..; T_N, s_N)$. For brevity we shall let s represent the mapping $j \rightarrow s_j$ from the set of integers 1, .., N into the stations; we may then abbreviate the state to $s(T_1, .., T_N)$. We shall sometimes use the short-form S for $s(T_1, .., T_N)$.

<u>Transition due to an Arrival.</u> We shall use the notation $(T_1, .., \hat{T}_i, .., T_N)$ for the set $(T_1, .. T_{i-1}, T_{i+1}, .. T_N)$. In other words, the hat above T_i indicates that it is to be deleted from the set. Let A be the state $A = a(T_1, .., \hat{T}_i, .., T_N)$ and let S be $s(T_1, .., T_N)$. Let $P(A, T_i, S)$ be the probability that the system transits from state A to S when a customer with remaining service time T_i arrives. For instance in FCFS if s and a are the mappings: $j \rightarrow j$ (i.e. the customer with remaining service time T_j is in station j) and customers are only served in station 1, then $P(A, T, S) = 1$ if and only if $A = a(T_1, .., T_{N-1}, \hat{T}_N)$ and $T = T_N$.

<u>Transition due to a Departure.</u> Let B_i be the state $b_i(T_1, .., T_N, T)$ with $T = 0+$ and where the customer with $T = 0$ remaining service time is in station i. Let S be defined as before. Let $P(B_i, S)$ be the probability that the system will transit from state B_i to S when the customer who is

about to depart from the ith station finishes service. For instance, in FCFS if B_1 is the state in which the customer being served has 0+ units of service time remaining, and the customer in the ith station has a remaining service time of T_{i-1}, $i = 2, .., N + 1$, and S is the state in which the customer in the ith station has a remaining service time of T_i, $i = 1, .., N$ then $P(B_1, S) = 1$.

The Difference Equation.

Let $Q(t, s(T_1, .., T_N))$ be the density function for the system being in state $S = s(T_1, .., T_N)$ at time t. Let us determine how the system could be in state S at time $t + \Delta t$ in terms of the state at time t. The development is very similar to that for PS and LCFSPR discussed in Section 2.

There are three possibilities:

(1) The system is in state $s(T_1 + r_{s(1),N} \cdot \Delta t, .., T_N + r_{s(N),N} \cdot \Delta t)$ at time t and no customer arrives in the interval $[t, t + \Delta t]$. A customer with remaining service time $T_i + r_{s(i),N} \cdot \Delta t$ is at station s_i, and hence after Δt units of real time have elapsed his remaining service time will be reduced by $r_{s(i),N} \cdot \Delta t$ for $i = 1, .., N$

(2) The system is in state $A = a(T_1, .., \hat{T}_i, .., T_N)$ and a new customer with a service time of T_i arrives and the system then transits into state S.

(3) The system is in state $b_i(T_1, .., T_N, T)$ where the customer with remaining service time T is in station i, and $0 \leq T \leq r_{i,N+1} \cdot \Delta t$; within an interval of Δt the customer with remaining service time T will finish service and the system may transit into state $s(T_1, .., T_N)$.

The difference equation is:

$$Q(t + \Delta t, s(T_1, .., T_N))$$
$$= Q(t, s(T_1 + r_{s(1), N} \cdot \Delta t, .., T_N + r_{s(N),N} \cdot \Delta t)(1 - \lambda \cdot \Delta t)$$
$$+ Q(t, a(T_1 .., \hat{T}_i, .., T_N))(\lambda \cdot \Delta t) \cdot f_u(T_i) \cdot P(A, T_i, S)$$
$$+ Q(t, b_i(T_1, .., T_N, 0)) \cdot (r_{i, N+1} \cdot \Delta t) \cdot P(B_i, S)$$

Expanding $Q(t, s(T_1 + r_{s(1), N} \Delta t, .., T_N + r_{s(N), N} \cdot \Delta t))$ using a Taylor's series expansion we have the following differential equation:

$$\frac{d}{dt} \cdot Q(t, S) = V_1(t, S) + V_2(t, S) + V_3(t, S) + V_4(t, S) \quad (12)$$

$$V_1(t, S) = \sum_{i=1}^{N} \frac{\partial}{\partial T_i} Q(t, S) \cdot r_{s(i), N}$$

$$V_2(t, S) = -\lambda \cdot Q(t, S)$$

$$V_3(t, S) = \sum_{i=1}^{N} \sum_{A} \lambda \cdot f_u(T_i) \cdot Q(t, A) \cdot P(A, T_i, S)$$

This summation is taken over all states such that $P(A, T_i, S) \neq 0$

$$V_4(t, S) = \sum_{i} \sum_{B_i} Q(t, B_i) \cdot P(B_i, S) \cdot r_{i, N+1}$$

The summation is taken over all states B_i such that $P(B_i, S) \neq 0$ i.e. the system can transit from B_i to S due to the departure of a customer from the ith station.

The first term $V_1(t, S)$ is the leave-depart term which is concerned with the net rate at which the system leaves S because customers get serviced. The second term $V_2(t, S)$ is the leave-arrive term which is concerned with the system leaving S due to the arrival of a customer. The first two terms are negative in sign.

The third term $V_3(t, S)$ is the enter-arrive term; it is concerned with the net rate at which the system enters S because customers arrive for service. The fourth and last term is the enter-depart term which is concerned with the system entering S because a customer departs. To summarize, we have

$\frac{d}{dt} Q(t, S)$ = leave-depart term + leave-arrive term + enter-arrive term

+ enter-depart term

The relationship between these four terms is crucial in analyzing distribution-independent and Poisson departure processes.

A system is said to be in equilibrium if $\frac{d}{dt} Q(t, S) = 0$ for all states S. We shall use the notation $Q(\infty, S)$ for the equilibrium value of $Q(t, S)$,

$$\text{i.e.} \quad Q(\infty, S) = \lim_{t \to \infty} Q(t, S)$$

and of course $\frac{d}{dt} Q(\infty, S) = 0$ for all S.

<u>Departure- Independence</u>. A system is said to be departure independent if at equilibrium

leave-arrive term + enter-depart term = 0

$$\text{i.e.} \quad -\lambda \cdot Q(\infty, S) + \sum_i \sum_{B_i} Q(\infty, B_i) \cdot P(B_i, S) \cdot r_{i,N+1} = 0$$

We shall show that if a system is departure independent then departures from the system form a Poisson process.

<u>Local Balance</u>. A single service system is said to be in local balance if at equilibrium the rate at which it enters a state S due to the arrival of a customer is equal to the rate at which it leaves S due to a customer getting serviced.

i.e. leave-depart term + enter-arrive term = 0

$$\text{i.e.} \quad \sum_{i=1}^{N} \frac{\partial}{\partial T_i} Q(\infty, S) \cdot r_{s(i), N}$$

$$+ \sum_{i=1}^{N} \sum_{A} \lambda \cdot f_u(T_i) \cdot Q(\infty, A) \cdot P(A, T_i, S) = 0$$

<u>Theorem 1</u>: A single service system is in local balance if and only if it is departure-independent.

<u>Proof:</u> At equilibrium we have

(leave-arrive term + enter-depart term)+(leave-depart term + enter-arrive term) = 0

Clearly, at equilibrium:

leave-arrive term + enter-depart term = 0

if and only if

leave-depart term + enter-arrive term = 0

Theorem 2: If a single service system is departure independent then departures from the system occur in a Poisson manner; i.e. the probability that a departure occurs in any incremental interval of time Δt is $\lambda\Delta t$ independent of earlier departures.

Proof: See Appendix A.

Immediate Service. A system is an immediate service system if every arriving customer who enters the system begins to receive service immediately. It is evident that PS and LCFSPR are immediate service systems whereas FCFS is not.

Detailed Local Balance. A system is said to be in detailed local balance if

$$\frac{\partial}{\partial T_i} Q(\infty,S) \cdot r_{s(i),N} + \lambda \cdot f_u(T_i) \cdot \sum_A Q(\infty,A) \cdot P(A, T_i S) = 0 \text{ for all states } S \text{ and all } i, \quad (14)$$

The rate at which the system enters a state S due to the arrival of a customer in a given station s(i) is equal to the rate at which the system leaves S due to the customer in s(i) getting service. Local balance is concerned with entries and departures into a queue; detailed Local balance is concerned with entries and departures from the individual stations which constitute the queue.

Lemma:7 If a system is in detailed local balance then it is an immediate service system.

Proof: See Appendix B.

Lemma:8 If a system is in detailed local balance then it is in local balance.

Proof: Follows by summing eqn (14) over i going from 1 to N.

Theorem 3: Consider a single-service system described by differential eqn. (12). Let the probability density function for the service time be $g(T) = \mu e^{-\mu T}$ for $T \geq 0$, i.e., exponential. Let the system satisfy the following two conditions when the service time probability density function is g(T):

(1) the system satisfies detailed local balance and

(2) the equilibrium solution is of the form:

$$Q(\infty, S) = P_s \cdot g(T_1) \ldots g(T_N) \quad (15)$$

where P_s is a function of the set of stations which are occupied but is independent of the remaining service times of customers in these stations.

Then the system is distribution-independent.

Proof: See Appendix C.

This theorem provides a simple sufficient condition for distribution independence. Assume that the service time for a system is exponentially distributed (or assume any other convenient distribution). If the system with the assumed service distribution satisfies the two conditions then the system is distribution independent. It is easy to verify that LCFSPR and PS satisfy the two conditions when the service time is an exponential random variable.

We now present a somewhat unusual distribution-independent system to show that distribution independent systems apart from PS and LCFSPR exist and to demonstrate how the above theorem may be used. Consider a two server system where the server in position 1 serves with $r_{1,N} = 1$, $N>0$ and the server in position 2 with $r_{2,N} = \lambda_2$. All other positions are merely places at which customers await service. If a customer enters the system when both positions 1 and 2 are occupied, he enters position i with probability p_i and preempts the customer being served in that position, $i = 1, 2$. If a customer enters the system when both positions 1 and 2 are empty, he enters either position with probability 1/2. If one of the positions is empty and the other busy, the entering customer begins to receive service at the empty position. Whenever one of positions 1 and 2 become free one of the preempted customers is selected at random for resumption of service at that position; if there are no preempted customers the vacated position remains empty. It is easy to verify that this is an immediate service system and that the two distribution independence conditions are satisfied when the service distribution is exponential. It is also straightforward to extend this to the case where there are n servers with rates $\lambda_1, \ldots \lambda_n$.

SECTION IV

NETWORKS OF QUEUES

A queueing network consists of queues, their servers and paths connecting the various queues together. A queueing network may be defined in terms of a directed graph with a set of vertices and edges; each vertex represents a single service sub-system (queue and server combination), while an edge represents a path a customer may take as he finishes service at one system and enters another. Fig. 1 is an example of a queueing network. We restrict attention to single service subsystems which when fed by a Poisson source can be described by differential equation (12).

Let a network contain M subsystems labeled 1, 2, .., M. Let p_{ij} be the "branching probability" that a customer who finishes service at the ith system joins the jth subsystem, i, j = 1, .., M; we assume that the probability of a customer branching to the jth subsystem is independent of the state of the network.

In an <u>open</u> network customers may enter the network from an external source labeled 0 (zero) and may leave the network and go to an external sink which is given the same label 0. The source generates customers in a Poisson manner with rate λ. Let p_{0i} be the probability that a customer generated by the source directly joins the ith subsystem, i = 1, .., M. Let p_{i0} be the probability that a customer who finishes service at the ith subsystem leaves the system. We may assume $p_{00} = 0$. Then $\sum_{j=0}^{M} p_{ij} = 1$ for i = 0, 1, .., M.

In a closed network customers do not enter or leave the network. A constant number (N) of customers cycle through the network endlessly. A closed network does not have a source or a sink. For a closed network $\sum_{j=1}^{M} p_{ij} = 1$ for i = 1, .., M. An example of a closed network is shown in Fig. 2.

We will only discuss open networks here. The extension to closed networks is straightforward. The set of states of an open network is the Cartesian product of the states of the M subsystems which constitute the network. In this section we set up differential equations which describe the behavior of a network. We show that if each of the M subsystems which constitute the network is in local balance when fed with a Poisson source, then the equilibrium state density functions for the network have the product form 8!.

The Differential Equation

We shall use the generic indices k and ℓ to refer to the subsystems of the network and we shall use a subscript to refer to a particular subsystem in the network. When there are two subscripts the first will refer to a subsystem in the network and the second to a particular customer or service station within the subsystem.

The state of the network is $(S_1, .., S_M)$ where S_k is the state of the kth subsystem. We shall use the same notation and terminology for the kth subsystem as for the single service systems discussed in the earlier section. Thus $S_k = s_k(T_{k1}, .., T_{kN_k})$, where s_k is a mapping from the customers in subsystem k to the service stations of subsystem k: thus a customer with remaining service time T_{kj} will be in station $s_k(j)$ of the kth subsystem and after Δt units of real time his remaining service time will be reduced to $T_{kj} - r_{k,\, s_k(j),\, N_k} \cdot \Delta t$ where N_k is the number of customers in the kth subsystem. For brevity we shall write $r_{k,\, s(j)}$ instead of $r_{k,\, s_k(j),\, N_k}$. The relation between states $B_{k,\, i}$ and B_i, and that between A_k and A are obvious. Let $f_{k,\, u}(\ )$ be the service density function in the kth subsystem. For compactness in expression we shall use the following notation: let $\psi = (S_1, .., S_M)$ be a state of the network and let $\psi(X_k, Y_\ell, ..)$ refer to the state of the network obtained by replacing the kth state S_k by X_k and the ℓth state S_ℓ by Y_ℓ, ..., in ψ. Hence

$$\psi(A_k) = (S_1, \ldots, S_{k-1}, A_k, S_{k+1}, \ldots, S_M)$$

$$\psi(B_{ki}) = (S_1, \ldots, S_{k-1}, B_{ki}, S_{k+1}, \ldots, S_M)$$

$$\psi(B_{ki}, A_\ell) = (S_1, \ldots, S_{k-1}, B_{ki}, S_{k+1}, \ldots, S_{\ell-1}, A_\ell, S_{\ell+1}, \ldots, S_M)$$

The differential equation is derived in a manner identical to that of the previous section. The notation of this section is analogous to that of the previous section.

$$\frac{d}{dt} Q(t,\psi) = \sum_k V_{k1}(t,\psi) + V_2(t,\psi) + \sum_k V_{k3}(t,\psi) + V_4(t,\psi)$$

where

$$V_{k1}(t,\psi) = \sum_j \frac{d}{dT_{kj}} Q(t,\psi) \cdot r_{k,\, s_k(j), N_k}$$

$$V_2(t,\psi) = -\lambda \cdot Q(t,\psi)$$

$$V_{k3}(t,\psi) = \sum_j \sum_{A_k} \sum_{\ell=1,}^{M} \sum_i \sum_{B_{\ell i}} Q(t,\psi(B_{\ell i}, A_k)) \cdot r_{\ell,\, i, N_\ell+1}$$

$$\cdot P(B_{\ell i}, S_\ell) \cdot p_{\ell k} \cdot f_{k,u}(T_{kj}) \cdot P(A_k, T_{kj}, S_k)$$

$$+ \sum_{A_k} \sum_j \lambda \cdot p_{0k} \cdot f_{k,u}(T_{kj}) \cdot Q(t,\psi(A_k)) \cdot P(A_k, T_{kj}, S_k)$$

$$V_4(t,\psi) = \sum_k \sum_i \sum_{B_{ki}} Q(t,\psi(B_{ki})) \cdot r_{kj, N_k+1} P(B_{ki}, S_k) \cdot p_{k0}$$

$V_{k1}(t,\psi)$ is the net rate at which the system transits from state ψ because customers get processed at subsystem k. Hence $\sum_k V_{k1}(t,\psi)$ is the net rate at which the network transits from state ψ because customers get processed at some subsystem: it is the leave-depart term.

$V_2(t,\psi)$ is the rate at which the network transits from state ψ because a customer arrives for service; it is the leave-arrive term.

$V_{k3}(t,\psi)$ has two terms. The first is concerned with the rate at which the network transits from state $\psi(B_{\ell i}, A_k)$ to ψ because the customer in the ith station of subsystem ℓ finishes service causing the state of the ℓth subsystem to change from $B_{\ell i}$ to S_ℓ; the customer who finishes service in subsystem ℓ enters subsystem k with a service time requirement of T_{kj}, causing the state of the kth subsystem to change from A_k to S_k. This term is summed over all ℓ, $B_{\ell i}$, A_k, T_{kj} etc. and represents the rate at which the network enters ψ because a customer finishes service at some subsystem and then arrives for service at the kth subsystem.

The second term is concerned with the rate at which the network transits from state $\psi(A_k)$ to state ψ because a customer enters the network from outside and directly enters the kth subsystem with a service time of T_{kj} causing the state of the kth subsystem to change from A_k to S_k. This term is summed over all A_k and T_{kj}. It is the rate at which the network enters state ψ because a customer enters the system from outside and directly enters subsystem k. $\sum_k V_{k3}(t,\psi)$ is the net rate at which the system enters ψ because a customer arrives for service at some subsystem of the network: it may be thought of as the enter-arrive term.

$V_4(t,\psi)$ is concerned with the rate at which the network transits from state $\psi(B_{ki})$ to state ψ because the customer in the ith station of subsystem k finishes service and then leaves the network; the departure of this customer causes the state of the kth subsystem to change from B_{ki} to S_k. This term is summed over all k and B_{ki}. It is the enter-depart term because it is the net rate at which the network enters state ψ, due to the departure of a customer from the network. Once again we have

$$\frac{d}{dt} Q(t,\psi) = \text{leave-depart term} + \text{leave-arrive term} + \text{enter-arrive term} + \text{enter-depart term.}$$

Definitions. A network is said to be in local balance if at equilibrium

$$V_{1,k}(\alpha,\psi) + V_{3,k}(\alpha,\psi) = 0 \text{ for all states } \psi \text{ and all } k.$$

Local balance states that the rate at which the network enters ψ due to a customer arriving at queue k is equal to the rate at which it leaves ψ due to a customer departing k.

Associated with each subsystem k is a number y_k, and associated with the Poisson source is a number y_0 where the set $y_0, y_1, .., y_M$ satisfies the equations

$$y_k = \sum_{\ell=0}^{M} y_\ell \cdot P_{\ell k} \text{ for } k = 1, .., M \text{ and } y_0 = 1$$

Theorem 4: If each of the M subsystems which constitute a network are in local balance when fed with a Poisson source then

(a) the network is in local balance and

(b) the equilibrium probability density function for the states of the network have the product form

$$Q(\alpha, S_1, .., S_M) = \prod_{k=1}^{M} q_k(S_k)$$

In particular: $q_k(S_k)$ is the equilibrium probability density function for the kth subsystem when fed with a Poisson source with rate $\lambda \cdot y_k$.

Proof: See Appendix D.

SECTION V

Conclusions

We have used local balance and continuous state analysis to study robustness, Poisson departure processes and the product form of equilibrium state probability of networks. We have studied systems with arbitrary differentiable service distributions; this class includes the class of distributions with rational Laplace transforms. A simple sufficient condition to determine whether a system is distribution independent was presented. It is easy to verify that processor - sharing, last-come first served preemptive resume and infinite parallel servers and some other disciplines satisfy this condition. It was shown that a system which is in local balance when fed with a Poisson source has Poisson departures and that a network of such systems has the product form.

APPENDIX A

<u>Theorem 2</u>: If a single service system is departure independent then departures from the system occur in a Poisson manner.

<u>Proof</u>: Let $R(S) . \Delta t . \Delta T_1 \ldots \Delta T_N$ be the probability that, at equilibrium, a departure will take place in an interval of length Δt and that the state of the system immediately at the end of the interval is in the incremental hypercube around S. The only way a departure can take place leaving the system in state S is if the system is in some state B_i (with N + 1 customers) at the beginning of the interval and if the departure of a customer causes the system to transit to state S. In other words $R(S) . \Delta t$ is the third term in the difference equation and $R(S) = V_4(\infty, S)$.

Departure independence implies that

$$V_4(\infty, S) = V_3(\infty, S) \quad \text{and hence:}$$

$$R(S) = \lambda . Q(\infty, S)$$

Hence the probability that there is a departure in an interval Δt and that the state of the system at the end of the interval is S is:

$$R(S) . \Delta t . \Delta T_1 \ldots \Delta T_N = (\lambda \Delta t) . Q(\infty, S) . \Delta T_1 \ldots \Delta T_N$$

Hence at equilibrium, the probability of a departure in an interval Δt is independent of the state of the system at the end of the interval. Therefore, at equilibrium, the probability of a departure in any interval Δt is $\lambda \Delta t$ independent of earlier departures. This implies that departures form a Poisson process.

APPENDIX B

Lemma: If a system is in detailed local balance then it is an immediate service system.

Proof: We shall prove this lemma by contradiction. We show that the assumption of a system in detailed local balance which is not an immediate service system results in a contradiction. If a system is not an immediate service system then there exists some state S such that the system can enter S due to a customer entering some station s(i), but the customer in s(i) does not receive service. Hence the system cannot leave S because the customer in s(i) receives service. But a system is in detailed local balance only if the rate at which it enters a state S due to the arrival of a customer in station s(i) is equal to the rate at which it leaves S because the customer in s(i) gets serviced. Hence a system which is not an immediate service system cannot be in detailed local balance.

More formally, a system which is not an immediate service must have some case where a customer with service time T_i arrives causing the system to enter S and where this arriving customer enters station s(i) but does not begin service immediately. Hence

$$r_{s(i), N} = 0$$

Since the system can transit into S due to the arriving customer we have

$$\lambda \cdot f_u(T_i) \cdot \sum_A Q(\infty, A) \cdot P(A, T_i, S) > 0$$

This implies that eqn.(14) cannot be satisfied.

APPENDIX C

<u>Theorem 3</u>: Consider a single service system described by differential equation (12). Let the probability density function for the service time be $g(T) = \mu e^{-\mu t}$ for $T \geq 0$, i.e., exponential. Let the system satisfy the following two conditions when the service time probability density function is $g(T)$:

(1) The system satisfies detailed local balance and

(2) The equilibrium solution is of the form

$$Q(\sigma, S) = P_s \cdot g(T_1) \ldots g(T_N) \tag{15}$$

where P_s is a function of the set of stations which are occupied but is <u>independent</u> of the remaining service times of customers in these stations. Then the system is distribution independent.

<u>Proof</u>: We have $\frac{d}{dt} g(T) = -\mu g(T)$

Substituting (15) in (14) and simplifying we have

$$-\mu \cdot P_s \cdot r_{s(i),N} + \lambda \sum_A P_a \cdot P(A, T_i, S) = 0 \tag{16}$$

Similarly, substituting (15) in (13) and recalling that

(a) $B_i = b_i(T_1, .., T_N, 0)$

and (b) $g(0) = \mu$

we have

$$-\lambda \cdot P_s + \mu \sum_i \sum_{B_i} P_{b_i} \cdot P(B_i, S) . r_{i, N+1} = 0 \tag{17}$$

Thus if the service time is an exponential random variable and if the two conditions of the theorem are satisfied eqns (16) and (17) follow. Note that eqns (16) and (17) are invariant to service distribution.

Now suppose the service time is an arbitrary random variable with a differential distribution. Let the service time have density $f_u(T)$ and let the remaining service time from a point of random entry have density $f_v(T)$. We shall show that in this case, eqns (16) and (17) imply that the equilibrium state probabilities are of the form:

$$Q(\infty, S) = P_s \cdot f_v(T_1) \;.....\; f_v(T_N) = P_s \cdot \prod_i f_v(T_i)$$

Multiplying eqn (16) by $f_u(T_i) \cdot \prod_{j \neq i} f_v(T_j)$

we have

$$(-\mu .\, f_u(T_i)) \cdot \prod_{j \neq i} f_v(T_j) \cdot P_s \cdot r_{s(i),\, N}$$

$$+ \lambda \cdot f_u(T_i) \sum_A P_a \prod_{j \neq i} f_v(T_j) \cdot P(A, T_i, S) = 0$$

Recall that $A = a(T_1, \ldots, \hat{T}_i, \ldots, T_N)$ and hence

$Q(\infty, A) = P_a \cdot \prod_{j \neq i} f_v(T_j)$. Recall also $\frac{d}{dt} \cdot f_v(T) = -\mu f_u(T)$

Hence eqn (14) is satisfied which in turn implies that for any differentiable distribution we have

leave-depart term + enter-arrive term = 0

Multiplying eqn (17) by $\prod_i f_v(T_i)$ and recalling that $f_v(0) = \mu$ and that

$$Q(\infty, B_i) = P_{b_i} \cdot \prod_{i=1}^{N} f_v(T_i) \cdot f_v(0)$$

we see that equation (13) is satisfied by any differentiable distribution

Hence we have

leave-arrive term + enter-depart term = 0

Hence $\frac{d}{dt} Q(\infty, S)$ = leave-depart term + leave-arrive term + enter-arrive term + enter-depart term = 0

APPENDIX D

Theorem 4: If each of the M subsystems which constitute a network are in local balance when fed with a Poisson source then

(a) the network is in local balance and

(b) the equilibrium probability density function for the states of the network have the product form

$$Q(\omega, S_1, \ldots, S_M) = \prod_{k=1}^{M} q_k(S_k)$$

In particular: $q_k(S_k)$ is the equilibrium probability density function for the kth subsystem when fed with a Poisson source with rate $\lambda \cdot y_k$.

Proof: Let $q_k(S_k)$ be the equilibrium state probability density function for the kth subsystem when fed with a Poisson source with rate $\lambda \cdot y_k$. Then if the kth subsystem is in local balance when fed with a Poisson source, it is also distribution independent and the following equations hold:

distribution independence

$$-\lambda \cdot y_k \cdot q_k(S_k) + \sum_{i} \sum_{B_{ki}} q_k(B_{ki}) \cdot P(B_{ki}, S_k) \cdot r_{k,\, i,\, N_k+1} = 0 \qquad (18)$$

local balance

$$\sum_{j=1}^{N_k} \frac{\partial}{\partial T_{kj}} q_k(S_k) \cdot r_{k,\, s(j),\, N_k} + \sum_{j} \sum_{A_k} \lambda y_k\, f_{k,u}(T_{k,j})$$

$$\cdot\, q_k(A_k) \cdot P(A_k, T_{kj}, S_k) = 0 \qquad (19)$$

Substituting eqn (18) in $V_4(\infty, \Psi)$ of the network eqns and assuming $Q(\infty, \Psi) = \prod_{k=1}^{M} q_k(S_k)$ we have

$$V_4(\infty, \psi) = \sum_{k=1}^{M} - \lambda \cdot y_k \quad Q(\infty, \psi) \cdot p_{k0}$$

$$= \lambda \cdot Q(\infty, \psi)$$

since $\sum_{k=1}^{M} y_k \cdot p_{k0} = y_0 = 1$

Hence $V_2(\infty, \psi) + V_4(\infty, \psi) = 0$

By the same argument

$$\sum_{i} \sum_{B_{\ell i}} Q(\infty, \psi(B_{\ell i}, A_k)) \cdot r_{\ell, i, N_\ell + 1} \cdot P(B_{\ell i}, S_\ell) \cdot$$

$$= \lambda \cdot y_\ell \cdot Q(\infty, \psi(A_k))$$

Substituting in the eqn for $V_{k,3}(\infty, \psi)$ we get

$$V_{k,3}(\infty, \psi) = \sum_j \sum_{A_k} \lambda \cdot f_{k,u}(T_{k,j}) \cdot Q(\infty, \psi(A_k)) \cdot P(A_k, T_{kj}, S_k) \sum_{\ell=0}^{M} y \cdot p_{\ell k}$$

Recollect $y_k = \sum_{=0}^{M} y_\ell \cdot p_{\ell k}$ and $y_0 = 1$

Hence $V_{k3}(\infty, \psi) = \sum_j \sum_{A_k} \lambda y_k \, f_{k,u}(T_{k,j}) \cdot Q(\infty, \psi(A_k)) \cdot P(A_k, T_{kj}, S_K)$

Assuming $Q(\infty, \psi) = \prod_\ell q_\ell(S_\ell)$

$$V_{k,3}(\infty, \psi) = \prod_{\ell \neq k} q_\ell(S_\ell) \sum_j \sum_{A_k} \lambda \cdot y_k \cdot f_{k,u}(T_{k,j}) \cdot q_k(A_k) \cdot P(A_k, T_{kj}, S_k)$$

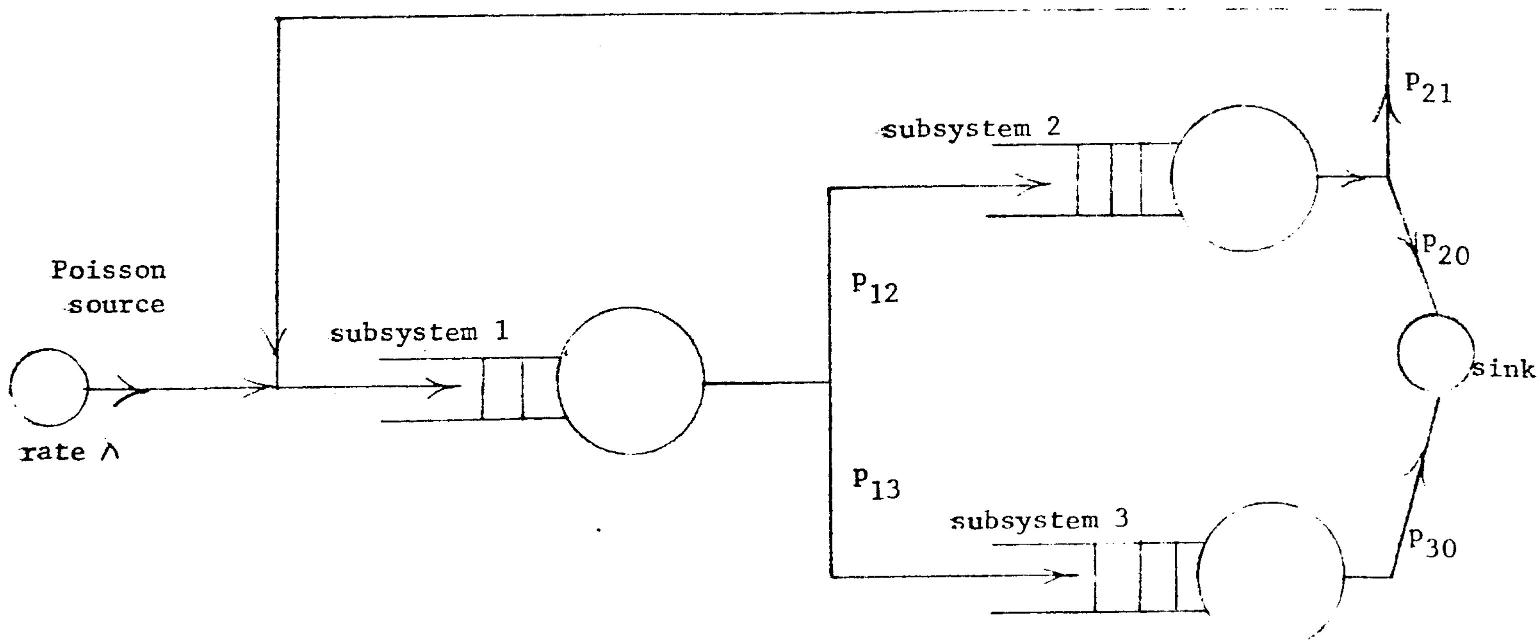
Using eqn(19) we get

$$V_{k,3}(\infty, \psi) = - \prod_{\ell \neq 1} q_\ell(S_\ell) \cdot \sum_j \frac{\partial}{\partial T_{kj}} q_k(S_k) \cdot r_{k, s_k(j), N_k}$$

$$= \sum_j \frac{\partial}{\partial T_{kj}} Q(\infty, \psi) \cdot r_{k, s_k(j), N_k}$$

Hence $V_{k,1}(\infty, \psi) + V_{k,3}(\infty, \psi) = 0$

Hence the product form of solution satisfied the network local balance equations, and hence satisfies equilibrium conditions.

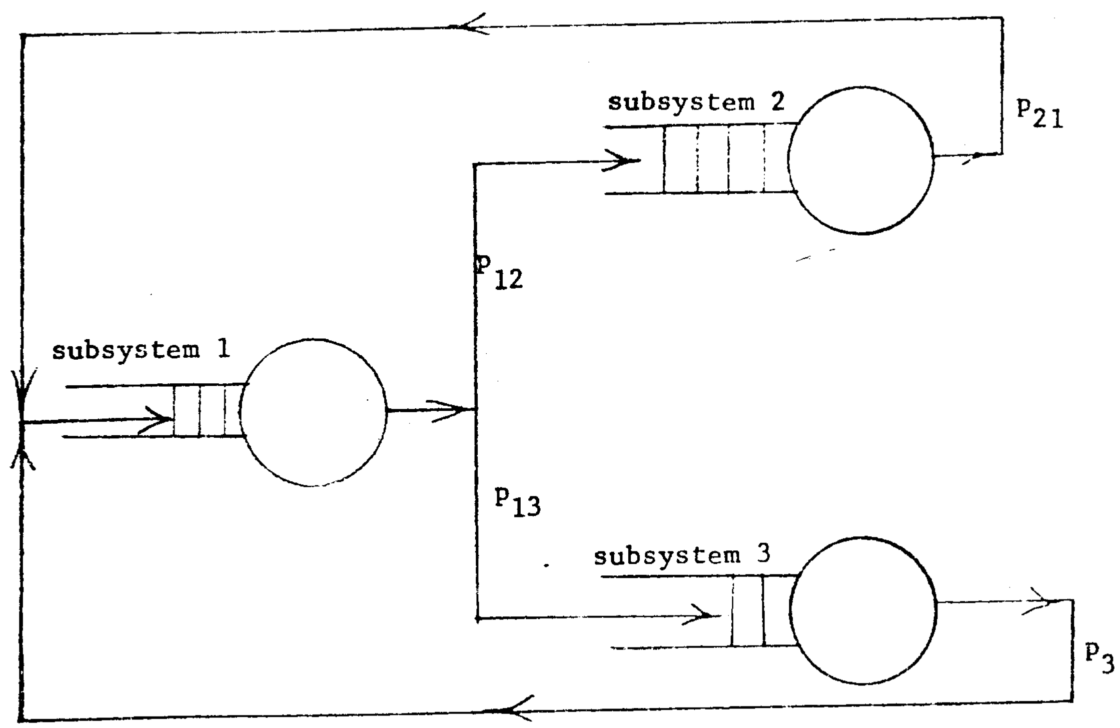


AN OPEN QUEUEING NETWORK

Figute 1.

Figure 2.

A CLOSED QUEUEING NETWORK



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