TOWARD AN ALGEBRAIC THEORY OF FUZZY RELATIONAL SYSTEMS*

by

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I. Introduction

Ever since Zadeh introduced the concept of fuzzy set in 1965 [14], numerous theoretical results and applications of fuzzy sets and relations have been reported. One of the common weaknesses of literatures on fuzzy sets, in our opinion, is that most proofs are tedious and long. One of the purposes of this paper is to provide an algebraic framework for the manipulation of fuzzy sets and relations. Furthermore, by investigating property or structure preserving relations between fuzzy relational systems, we have initiated the development of a formal tool which seems to be quite powerful in modeling systems processing soft information.

In section III, algebraic properties of fuzzy relations on a set are investigated. Our investigation brings out quite naturally the important resolution identity due to Zadeh [16].

Our concern in section IV lies in the concepts of structure preserving relations between systems. Results parallel to the nonfuzzy case are obtained here.

In section V, we consider applications to automata theory, clustering analysis and Information networks using the results developed in sections III and IV.

II. Preliminaries

We will provide in this section basic terminologies and notations which are necessary for the understanding of subsequent results.

A <u>fuzzy (binary) relation</u> R from a set X to a set Y is a fuzzy subset of $X \times Y$ characterized by a membership function $\mu_R: X \times Y \to [0,1]$. For each $x \in X$ and $y \in Y$, $\mu_R(x,y)$ is referred to as the <u>strength</u> of the relation between x and y. If $X = Y$, then we say R is a fuzzy relation on X. As in the case of nonfuzzy binary relations, every fuzzy relation R on X can be represented by a <u>fuzzy graph</u> consisting of a set of vertices X such that there is a weighted arc connecting each pair of vertices x_i, x_j and the weight on the arc (x_i, x_j) is $\mu_R(x_i, x_j)$. Equivalently, R can be represented by a <u>fuzzy matrix</u>, M_R, whose $(i,j)^{th}$ entry is $\mu_R(x_i, x_j)$.

In the following definitions, the symbols $\vee$ and $\wedge$ stand for max and min, respectively.

Let R and S be two fuzzy relations from X to Y. R is said to be <u>contained</u> in S, in symbols, $R \subseteq S$, if $\mu_R(x,y) \leq \mu_S(x,y)$, for all $(x,y) \in X \times Y$. The <u>union</u> of R and S, denoted by $R \cup S$, is defined by $\mu_{R \cup S} = \mu_R \vee \mu_S$. The <u>intersection</u> of R and S, denoted by $R \cap S$, is defined by $\mu_{R \cap S} = \mu_R \wedge \mu_S$. The <u>complement</u> of R, denoted by $\bar{R}$, is defined by $\mu_{\bar{R}} = 1 - \mu_R$. The <u>inverse</u> of R, denoted by R^{-1}, is a fuzzy relation from Y to X defined by $\mu_{R^{-1}} = \mu_R$.

If R and S are fuzzy relations from X to Y and from Y to Z, respectively, then the <u>composition</u> of R and S, denoted by $R \circ S$ (or simply by RS), is a fuzzy relation from X to Z defined by

$$\mu_{R \circ S}(x,z) = \bigvee_y [\mu_R(x,y) \wedge \mu_S(y,z)], \quad x \in X, \quad z \in Z.$$

The n-fold composition $\underbrace{R \circ R \circ \ldots \circ R}_{n \text{ times}}$ is denoted by R^n.

We assume here that basic operations for fuzzy matrices are Max and Min, respectively. Note that $M_{RS} = M_R M_S$

We may extend the composition operation to that between a relation and a subset by creating two relations corresponding to each subset Y of a set X as follows. Define $\vec{Y}$ to be a relation from $\{1\}$ to X such that

$$\mu_{\vec{Y}}(1,x) = \begin{cases} 1, & \text{if } x \in Y \\ 0, & \text{otherwise} \end{cases}$$

Similarly, we define $\overleftarrow{Y}$ as a relation from X to $\{1\}$ such that

$$\mu_{\overleftarrow{Y}}(x,1) = \begin{cases} 1, & \text{if } x \in Y \\ 0, & \text{otherwise} \end{cases}$$

In terms of matrix terminology, $M_{\vec{Y}}$ is a Boolean row vector whose i^{th} entry is 1 if $x_i \in Y$, and is 0, otherwise. Similarly, $M_{\overleftarrow{Y}}$ is a Boolean column vector. It is easily seen that if R is a fuzzy relation from X to Y and X' and Y' are subsets of X and Y, respectively, then $\vec{X}'R$ and $R\overleftarrow{Y}'$ are defined.

The following identities and inequality follow directly from these definitions.

a) $(R^{-1})^{-1} = R$

b) $(RS)^{-1} = S^{-1}R^{-1}$

c) $R(S \cup T) = RS \cup RT$

d) $(RS)T = R(ST)$

e) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

f) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$

g) $(\bar{R})^{-1} = \overline{(R^{-1})}$

h) $R(S \cap T) \subseteq RS \cap RT$

Inequality (h) cannot be written as an identity in general.

Several basic properties of fuzzy relations on a set are listed in the following:

R is <u>ε-determinate</u> iff for each $x \in X$, there exists at most one $y \in Y$ such that $\mu_R(x,y) \geq \varepsilon$.

R is <u>ε-productive</u> iff for each $x \in X$ there exists at least one $y \in Y$ such that $\mu_R(x,y) \geq \varepsilon$.

R is an <u>ε-function</u> iff it is both ε-determinate and ε-productive.

R is <u>ε-onto</u> or <u>ε-subjective</u> iff for each $y \in Y$, there exists an $x \in X$ such that $\mu_R(x,y) \geq \varepsilon$.

R is <u>ε-injective</u> (or <u>ε-into</u>) iff it is an ε-function and R^{-1} is ε-determinate.

R is <u>ε-one-to-one</u> iff R and R^{-1} are both ε-functions. In the case $\varepsilon = 1$, we will simply drop the prefix 1- in the definition. Thus, a 1-function is simply a function.

We note that R is ε-subjective if and only if R^{-1} is ε-productive.

Let R be a fuzzy relation on a set X. We define the following notions:

1. R is <u>ε-reflexive</u> iff $(\forall x \in X)\ [\mu_R(x,x) \geq \varepsilon]$. A 1-reflexive relation will simply be referred to as a <u>reflexive</u> relation.

2. R is <u>symmetric</u> iff $\mu_R(x,y) = \mu_R(y,x)$ for all x, y in X.

3. R is <u>weakly reflexive</u> iff for all x, y in X,

$$\mu_R(x,y) = \varepsilon \rightarrow \mu_R(x,x) \geq \varepsilon.$$

4. R is <u>transitive</u> iff $R \supseteq R \circ R$.

III. An Algebra of fuzzy Relations

In this section we will show that the concept of similarity relation introduced by Zadeh [16] is derivable in much the same way as equivalence relation. Furthermore, through this derivation, the resolution identity [16] is brought out quite naturally.

Lemma 1 If R is a fuzzy relation from X to Y, then the relation RR^{-1} is weakly reflexive, symmetric, and transitive.

Proof: i) $\mu_{RR^{-1}}(x,x') = \bigvee_y [\mu_R(x,y) \wedge \mu_{R^{-1}}(y,x')]$

$$\leq \bigvee_y [\mu_R(x,y) \wedge \mu_R(x,y)] = \mu_{RR^{-1}}(x,x)$$

Hence, RR^{-1} is weakly reflexive.

ii) $(RR^{-1})^{-1} = RR^{-1}$. Hence, RR^{-1} is symmetric.

iii) $\mu_{(RR^{-1})^2}(x,x') = \bigvee_{x''} [\mu_{RR^{-1}}(x,x'') \wedge \mu_{RR^{-1}}(x'', x')]$

$$\leq \bigvee_{x''} \bigvee_y [\mu_R(x,y) \wedge \mu_{R^{-1}}(y,x'') \wedge \mu_R(x'',y) \wedge \mu_{R^{-1}}(y,x')]$$

$$\leq \bigvee_y [\mu_R(x,y) \wedge \mu_R(x',y)] = \mu_{RR^{-1}}(x,x')$$

Hence, RR^{-1} is transitive. ||

Let R now be a weakly reflexive, symmetric and transitive relation on X. Define a fimily of non-fuzzy sets F^R as follows:

$$F^R = \{K \subseteq X \mid (\exists 0 < \varepsilon \leq 1)[(\forall x \in X)[x \in K \leftrightarrows (\forall x' \in K)[\mu_R(x,x') \geq \varepsilon]]\} \quad (1)$$

We note that if we let

$$F^R_\varepsilon = \{K \subseteq X \mid (\forall x \in X)[x \in K \leftrightarrows (\forall x' \in K)[\mu_R(x,x') \geq \varepsilon]]\} \quad (2)$$

then we see that $\varepsilon_1 \leq \varepsilon_2 \Rightarrow F^R_{\varepsilon_2} \precsim F^R_{\varepsilon_1}$, where "$\precsim$" denotes covering relation i.e. every element in $F^R_{\varepsilon_2}$ is a subset of an element in $F^R_{\varepsilon_1}$.

A subset J of X is called <u>ε-complete</u> with respect to R iff $(\forall x,x' \in J)[\mu_R(x,x') \geq \varepsilon]$. A <u>maximal</u> ε-complete set is one which is not properly contained in any other ε-complete set.

<u>Lemma 2.</u> F^R is the family of all maximal ε-complete sets with respect to R for $0 \leq \varepsilon \leq 1$.

Proof: Let $K \in F^R$ and $x \in K$. Then there exists $0 < \varepsilon \leq 1$ such that $x' \in K$, $\mu_R(x,x') \geq \varepsilon$ by (1). Hence K is complete. Next, consider an ε-complete set J which is not maximal. This means that there exists a maximal ε-complete set J' such that $J \subsetneq J'$, which implies that there exists $x' \in J' - J$. But since J' is ε-complete, we conclude that for each x in J, $\mu_R(x',x) \geq \varepsilon$. Hence, by (1) we must conclude that $x' \in J$, a contradiction. Hence, J must also be maximal. ||

<u>Lemma 3</u>. Whenever $\mu_R(x,x') > 0$, there is some ε-complete set $K \in F^R$ such that $\{x,x'\} \subseteq K$.

Proof: If $x = x'$, then $\{x\}$ is certainly ε-complete for $\varepsilon = \mu_R(x,x)$. Otherwise, if $x \neq x'$, then since $\mu_R(x,x') = \mu_R(x',x)$ by symmetry, and $\mu_R(x,x) \geq \mu_R(x,x')$ and $\mu_R(x',x') \geq \mu_R(x,x')$ by weak reflexivity, we see that $\{x,x'\}$ is ε-complete, where $\varepsilon = \mu_R(x,x')$. Thus, $\{x,x'\}$ is contained in some ε-complete set C. Denote by C_ε the family of all ε-complete sets C' which include C. Then C is partially ordered under set inclusion and hence satisfies the condition of Zorn's lemma. Therefore we conclude from Zorn's lemma that C_ε has a maximal element K. This element is also maximal in the family of all ε-complete sets since any sets including K must also include C. Hence, $K \in F^R$ by lemma 2, and the proof is completed. ||

It should be remarked here that sometimes a subclass of F^R, satisfying condition of lemma 3, will cover the set X. For example, let R be the fuzzy relation on X = {a,b,c,d,e,f} given by the following matrix.

$$\begin{bmatrix} 1 & .3 & .4 & 0 & .4 & .3 \\ .3 & 1 & .2 & .3 & 0 & .3 \\ .4 & .2 & 1 & .3 & .5 & 0 \\ 0 & .3 & .3 & 1 & 0 & 0 \\ .4 & 0 & .5 & 0 & 1 & 0 \\ .3 & .4 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We see that the family three maximal complete sets {a,b,f}, {b,c,d} and {a,c,e} satisfy the condition of lemma 3 but it does not contain the maximal complete set {a,b,c}.

Let O and I denote two special relations on a set X such that for all x,x' in X,

$$\mu_O(x,x') = 0 \ , \ \mu_I(x,x') = 1.$$

Lemma 4. If $R \neq O$ is a weakly reflexive and symmetric relation on X, then there exists a set Y and a fuzzy relation S from X to Y such that $R = SS^{-1}$.

Proof: Denote by Y the set $\{K^* \mid K \in F^R\}$, we define a fuzzy relation S from X to Y as follows:

$$\mu_S(x,K^*) = \begin{cases} \alpha, & \text{if } x \in K \text{ and } \alpha \text{ is the largest number for which } K \in F^R_\alpha. \\ 0 & \text{otherwise.} \end{cases} \qquad (3)$$

If $\mu_R(x,x') = \alpha > 0$, then by lemma 3, there is an α-complete set $K \in F^R$ such that $\{x,x'\} \subseteq K$. Since

$$\mu_{SS^{-1}}(x,x') = \bigvee_{K^*} [\mu_S(x,K^*) \wedge \mu_S(x',K^*)] \geq \alpha = \mu_R(x,x'),$$

we conclude that $R \subseteq SS^{-1}$.

Suppose now that $\mu_{SS^{-1}}(x,x') = \beta$. Then there exists a $K^* \varepsilon F_\beta$ such that $\mu_S(x,K^*) = \mu_S(x',K^*)$. This means that $\{x,x'\} \subseteq K$ and hence $\mu_R(x,x') \geq \beta$. Therefore, $SS^{-1} \subseteq R$. ||

Combining lemmas 1 and 4, we have the following result.

Theorem 1. A fuzzy relation $R \neq 0$ on a set X is weakly reflexive and symmetric iff there is a set Y and a fuzzy relation S from X to Y such that $R = SS^{-1}$.

In the sequel, we shall use the notation φ_R to denote the relation S defined in (3).

Definition 1. A cover C on a set X is a family of subsets X_i, $i \varepsilon I$, of X such that $\bigcup_{i \varepsilon I} X_i = X$.

Lemma 5. If R is an ε-reflexive relation on X then φ_R is ε-productive and for each $\varepsilon' \leq \varepsilon$, F^R_ε is a cover of X.

Proof: Since for each $x \varepsilon X$, $\mu_R(x,x) \geq \varepsilon$, and because $\{x\}$ is ε-complete, there is some K in $F^R_{\varepsilon'}$ $(\varepsilon' \leq \varepsilon)$ such that $x \varepsilon K$. Hence, $F^R_{\varepsilon'}$ is a cover. Also, by definition of φ_R, $x \varepsilon K$ implies that $\mu_{\varphi_R}(x,K^*) \geq \varepsilon$ which implies that φ_R is ε-productive. ||

In the sequel, we shall use the term productive for 1-productive etc.

Corollary 1. If R is reflexive, then φ_R is productive and each F^R_ε $(0 < \varepsilon \leq 1)$ is a cover of X.

The following result is a consequence of lemma 4 and corollary 1.

Corollary 2. R is reflexive and symmetric relation on X iff there is a set Y and a productive fuzzy relation S from X to Y such that $R = SS^{-1}$.

Lemma 6. Let R be a weakly reflexive, symmetric and transitive relation on X, and let φ_R^ε denote the relation φ_R whose range is restricted to F_ε^R. Then for each $0 < \varepsilon \leq 1$, φ_R^ε is ε-determinate and elements of F_ε^R are pairwise disjoint.

Proof: Let K and K' be two not necessarily distinct elements of F_ε^R and assume that $K \cap K'' \neq \emptyset$. For any $q_1 \in K \cap K'$, we have $\mu_R(q,q_1) \geq \varepsilon$, for all q in K and $\mu_R(q_1,q') \geq \varepsilon$, for all q' in K'. Since R is transitive, we see that $\mu_R(q,q') \geq \varepsilon$, for $q \in K$, and $q' \in K'$. Since R is weakly reflexive and symmetric, we conclude that $K \cup K'$ is ε-complete. However, since K and K' are maximal ε-complete, we must conclude that $K = K'$. Hence, $K \neq K' \rightarrow K \cap K' = \emptyset$. Now since $\mu_{\varphi_R^\varepsilon}(x,K^*) = \varepsilon$, and since x cannot belong to any other sets in F^R, φ_R^ε is determinate. ||

<u>Definition 2</u>. A <u>similarity</u> relation R on X is a fuzzy relation on X which is reflexive, symmetric and transitive. R is called an <u>ε-similarity</u> relation if it is ε-reflexive for some $0 < \varepsilon \leq 1$.

Since clearly reflexivity implies weak reflexivity, we have the following consequence of lemmas 5 and 6.

<u>Corollary 3</u>. If R is a similarity relation on X, then for each $0 < \varepsilon \leq 1$, F_ε is a partition on X.

Note that corollary 3 says that every similarity relation R admits a resolution $\bigcup_\alpha \alpha R_\alpha$, where R is the equivalence relation induced by the partition F_α^R. Indeed, it was pointed out by Zadeh [16] that if the R_α, $0 < \alpha \leq 1$, are a nested sequence of distinct equivalence relations on X, with $\alpha_1 > \alpha_2 \Longleftrightarrow R_{\alpha_1} \subseteq R_{\alpha_2}$, R_1 is nonempty and domain of R_1 is equal to domain of R_2, then $R = \bigcup_\alpha R_\alpha$ is a similarity relation on X.

The following result, which is a straight forward consequence of theorem 1 and corollary 3, offers another characterization of similarity relation.

Theorem 2. A relation R is a ε-similarity $(0 < \varepsilon \leq 1)$ relation on a set X iff there is another set Y and an ε-function f from X to Y such that $R = ff^{-1}$.

IV. Fuzzy Relational Systems

It was pointed out in the introduction that it is our contention here that specific structural properties of a system are expressible in terms of a family of fuzzy relations of finite ranks on a set. The purpose of this section is to investigate properties of information or structure preserving relations between systems having similar structures.

Definition 3. A fuzzy relation System (FRS) A is a triple $[X,\Sigma,f]$, where X is a set, Σ is a finite alphabet, and f maps Σ to the family of fuzzy relations on X. The image of $a \in \Sigma$ under f will be denoted by f_a.

Two FRS $A = [X,\Sigma,f]$ and $B = [Y,\Omega,g]$ are said to be similar systems iff $\Sigma = \Omega$.

The concepts of a structural preserving relation is familiar such as the concepts of homomorphism in algebra and logic, homeomorphism in topology and continuity in analysis. We shall extend a few of these familiar notions in the following.

Let $A = [X,\Sigma,f]$ and $B = [Y,\Sigma,g]$ be two similar FRS. A fuzzy relation σ from X to Y is called a structural preserving relation between A and B if, for each $a \in \Sigma$, it satisfies any of the following properties:

a) $f_a^{-1}\sigma g_a \subseteq \sigma$

b) $\sigma^{-1} f_a \subseteq g_a \sigma^{-1}$

c) $\sigma g_a \subseteq f_a \sigma$

d) $\sigma^{-1} f_a \sigma \subseteq g_a$

e) $f_a \sigma \subseteq \sigma g_a$

f) $\sigma g_a \overleftarrow{Y} \subseteq f_a \overleftarrow{X}$

g) $\sigma^{-1} f_a \overleftarrow{X} \subseteq g_a \overleftarrow{Y}$.

Usually, depending on specific applications, a combination of the above mentioned properties are selected to define the concepts of homomorphism, continuity etc. For our purpose, we shall only investigate the properties of a relation satisfying three of the properties listed above.

Definition 4. Let $A = [X,\Sigma,f]$ and $B = [Y,\Sigma,g]$ be two similar FRS. A fuzzy relation σ from X to Y is called a generalized congruence (GC) between A and B iff $\forall a \in \Sigma$, σ satisfies properties (a), (f) and (g) given above.

Two special cases of GC are now listed.

1. σ is called a ε-congruence relation if A and B are identical and σ is an ε-similarity relation.

2. σ is called an ε-homomorphism if it is an ε-function.

Theorem 3. If σ and δ are GC between $A = [X,\Sigma,f]$ and $A' = [X',\Sigma,f']$ and between A' and $A'' = [X'',\Sigma,f'']$, respectively, then $\sigma\delta$ is a GC between A and A''.

Proof: For arbitrary $x \in X$ and $x'' \in X''$, we have

$$\mu_{f_a^{-1}\sigma\delta f''_a}(x,x'') = \bigvee_{x' \varepsilon X'} [\mu_{f_a^{-1}\sigma}(x,x') \wedge \mu_{\delta f''_a}(x',x'')]$$

Since $\mu_{f_a^{-1}\sigma}(x,x') = \mu_{\sigma^{-1}f_a}(x',x)$ and the fact that $\sigma^{-1}f_a\overleftarrow{X} \subseteq f'_a\overleftarrow{X'}$,

we conclude that $\bigvee_{x\varepsilon X} \mu_{\sigma^{-1}f_a}(x',x) \leq \bigvee_{y\varepsilon X'} \mu_{f'_a}(x',y)$.

Hence, $\mu_{f_a^{-1}\sigma}(x,x') \leq \mu_{\sigma^{-1}f_a}(x',x) \wedge \bigvee_{y\varepsilon X'} \mu_{f'_a}(x',y)$ we have

$$\mu_{f_a^{-1}\sigma\delta f''_a}(x,x'') \leq \bigvee_{x'\varepsilon X''} \bigvee_{y\varepsilon X''} [\mu_{\sigma^{-1}f_a}(x',x) \wedge \mu_{f'_a}(x',y) \wedge \mu_{\delta f''_a}(x',x'')]$$

$$\leq \bigvee_{y,x'\varepsilon X'} [\mu_{f_a^{-1}\sigma f'_a}(x,y) \wedge \mu_{\delta f''_a}(x',x'')] . \qquad (4)$$

By a similar argument, we see that

$$\mu_{\delta f''_a}(x',x) \leq \mu_{\delta f''_a}(x',x'') \wedge \bigvee_{y\varepsilon X'} \mu_{f'_a}(x',y) .$$

Hence,

$$(4) \leq \bigvee_{y\varepsilon X} [\mu_{f_a^{-1}\sigma f'_a}(x,y) \wedge \mu_{f_a^{-1}\delta f''_a}(y,x'')] \leq \bigvee_{y\varepsilon X'} [\mu_\sigma(x,y) \wedge \mu_\delta(y,x'')] = \mu_{\sigma\delta}(x,x'') .$$

Hence $f_a^{-1}\sigma\delta f''_a \subseteq \sigma\delta$.

Now $\sigma\delta f''_a\overleftarrow{X''} \subseteq \sigma f'_a\overleftarrow{X'} \subseteq f_a\overleftarrow{X}$, and $\delta^{-1}\sigma^{-1}f_a\overleftarrow{X} \subseteq \delta^{-1}f'_a\overleftarrow{X'} \subseteq f''_a\overleftarrow{X''}$.

Therefore, $\sigma\delta$ is indeed a GC between A and A'' . ||

Let $A = [X,\Sigma,f]$ be a FRS. Let Σ^+ denote the set of all finite sequences of elements of Σ. We extend the notation by defining f_w, for $w \varepsilon \Sigma^+$ by the following rule: for $lg(w) > 0$, i.e. $w = a,a_2 \ldots a_n$, then

$$f_w = f_{a_1} f_{a_2} \ldots f_{a_n} .$$

Let F^+ be the set of fuzzy relations which can be represented in the form f_w, $w \varepsilon \Sigma^+$. We have extended f to a mapping from Σ^+ to F^+.

Theorem 4. If σ is a GC between $A = [X,\Sigma,f]$ and $A' = [X',\Sigma,f']$, then for each $w \varepsilon \Sigma^+$,

i) $f_w^{-1}\sigma f_w' \subseteq \sigma$,

ii) $\sigma f_w'\overleftarrow{X}' \subseteq f_w\overleftarrow{X}$, and

iii) $\sigma^{-1}f_w\overleftarrow{X} \subseteq f_w'\overleftarrow{X}'$.

Proof: The theorem clearly holds in case $w \varepsilon \Sigma$. Now use induction on the length of w. Assuming that (i), (ii), and (iii) hold for some $w \varepsilon \Sigma^+$ and let $a \varepsilon \Sigma$.

i) $f_{wa}^{-1}\sigma f_{wa} = f_a^{-1}f_w^{-1}\sigma f_w f_a \subseteq f_a^{-1}\sigma f_a \subseteq \sigma$.

ii) $\mu_{\sigma f_w' f_a'\overleftarrow{X}'}(x,1) = \bigvee_{x'} [\mu_{\sigma f_w'}(x,x') \wedge \mu_{f_a'\overleftarrow{X}'}(x,1)]$

Now, since $\sigma f_w'\overleftarrow{X}' \subseteq f_w\overleftarrow{X}$ by inductive hypothesis,

$$\mu_{\sigma f_w'}(x,x') \leq \bigvee_{x'\varepsilon X'} \mu_{\sigma f_w'}(x,x') \leq \bigvee_{x''\varepsilon X} \mu_{f_w}(x,x'') \tag{5}$$

Hence, $\mu_{\sigma f_w' f_a'\overleftarrow{X}'}(x,1) \leq \bigvee_{x'} [\bigvee_{x\varepsilon X} [\bigvee_{x''\varepsilon X'} \mu_{f_w}(x,x'') \wedge \bigvee_{x\varepsilon X'} \mu_{\sigma f_w'}(x,x')]] \wedge \mu_{f_a'\overleftarrow{X}'}(x,1)]$

$$= \bigvee_{x'\varepsilon X'} \bigvee_{x''\varepsilon X} [\mu_{f_w^{-1}\sigma f_w'}(x'',x') \wedge \mu_{f_a'\overleftarrow{X}'}(x',1)]$$

$$\leq \bigvee_{x'} \bigvee_{x''\varepsilon X} [\mu_{\sigma}(x'',x') \wedge \mu_{f_a'\overleftarrow{X}'}(x,1)]$$

$$= \bigvee_{x'\varepsilon X} \mu_{\sigma f_a'\overleftarrow{X}'}(x'') \leq \bigvee_{x'\varepsilon X} \mu_{f_a\overleftarrow{X}}(x'') \tag{6}$$

By (5) and (6), we conclude that

$$\mu_{\sigma f_{wa}'\overleftarrow{X}'}(x,1) = \mu_{\sigma f_a'\overleftarrow{X}'}(x,1) \leq \left(\bigvee_{x''\varepsilon X} \mu_{f_w}(x,x'')\right) \wedge \left(\bigvee_{x''\varepsilon X} \mu_{f_a\overleftarrow{X}}(x'')\right)$$

$$\leq \mu_{f_w f_a\overleftarrow{X}}(x,1) = \mu_{f_{wa}\overleftarrow{X}}(x,1)$$

Hence (ii) holds. Similarly, (iii) holds. ||

Let $A = [X,\Sigma,f]$ be a fuzzy relational system. We define an operator on the set of fuzzy relations as follows:

$$\lambda(\sigma) = \bigcup_{a\varepsilon\Sigma} f_a^{-1}\sigma f_a$$

We will also use the notation $\lambda^0(\sigma) = \sigma$, and for $k > 0$,

$\lambda^k(\sigma) = \lambda^{k-1}(\lambda(\sigma))$, and $\lambda^*(\sigma) = \bigcup_{k\geq 0} \lambda^k(\sigma)$.

Lemma 8. Let $A = [X,\Sigma,f]$ be a fuzzy relational system, and let R, R_i and S be fuzzy relations on X. Then

i) $\lambda^*(\bigcup_{i\varepsilon I} R_i) = \bigcup_{i\varepsilon I} \lambda^*(R_i)$.

ii) $R \subseteq S \rightarrow \lambda^*(R) \subseteq \lambda^*(S)$.

iii) $\lambda^*(\lambda^*(R)) = \lambda^*(R)$.

iv) $\lambda^*(R)$ is the minimum of all fuzzy relations S on X such that $R \subseteq S$ and $\lambda(S) \subseteq S$.

Proof: We will only prove statement (iv) here since statements (i), (ii) and (iii) are quite straight forward.

To prove (iv), it is quite clear that $\lambda^*(R)$ satisfies the two given conditions, i.e. $R \subseteq \lambda^*(R)$ and $\lambda(\lambda^*(R)) \subseteq \lambda^*(R)$. Let S be another relation satisfying these two conditions, then $\lambda(R) \subseteq \lambda(S) \subseteq S$. Using induction, it is clear that for any $n \geq 0$, $\lambda^n(R) \subseteq \lambda^n(S) \subseteq S$. Hence, $\lambda^*(R) \subseteq S$ and this shows the minimality of $\lambda^*(R)$.

Theorem 5. Let $A = [X,\Sigma,f]$ and $A' = [X',\Sigma,f']$ be two similar fuzzy relational systems, and σ a fuzzy relation between X and X'. Then there is a unique minimum fuzzy relation $m(\sigma)$ between X and X' such that (i) $\sigma \subseteq M(\sigma)$ and (ii) $\bigcup_{a\varepsilon\Sigma} f_a^{-1} m(\sigma) f'_a \subseteq m(\sigma)$. Furthermore, if there is a generalized congruence σ' between A and A' such that $\sigma \subseteq \sigma'$, then $m(\sigma)$ is the minimum such generalized congruence.

Proof: Let $m(\sigma) = \lambda^*(\sigma)$, then the first part of the theorem follows directly from lemma 8.

Assume now that σ' is a generalized congruence such that $\sigma \subseteq \sigma'$. Since $m(\sigma)$ is minimum over conditions (i) and (ii), we see that $m(\sigma) \subseteq \sigma'$. Therefore,

$$m(\sigma)f'_a\overleftarrow{X'} \subseteq \sigma' f'_a\overleftarrow{X'} \subseteq f_a\overleftarrow{X}.$$

Similarly, $m(\sigma)^{-1}f_a\overleftarrow{X} \subseteq f'_a\overleftarrow{X'}$. Finally, it follows from the definition of $\lambda^*(\sigma)$ that $f_a^{-1}m(\sigma)f'_a \subseteq m(\sigma)$, for each $a \in \Sigma$. Hence $m(\sigma)$ is the minimum generalized congruence containing σ. ||

Lemma 9. If σ is a generalized congruence on a fuzzy relational system $A = [X,\Sigma,f]$ which is symmetric and weakly reflexive, then φ_σ is a GC between A and $A' = [F^\sigma,\Sigma,f']$ such that for each $a \in \Sigma$, the relation f'_a is defined by

$$f'_a = \overline{\varphi_\sigma^{-1}f_a\bar{\varphi}_\sigma} \cap \varphi_\sigma^{-1}f_a\varphi_\sigma .$$

Proof: We need to show that φ_σ has the three properties

i) $f_a^{-1}\varphi_\sigma f'_a \subseteq \varphi_\sigma$,

ii) $\varphi_\sigma f'_a\overleftarrow{F^\sigma} \subseteq f_a\overleftarrow{X}$, and

iii) $\varphi_\sigma^{-1}f_a\overleftarrow{X} \subseteq f'_a\overleftarrow{F^\sigma}$.

To prove (i), we first note that

$$f_a^{-1}\varphi_\sigma f'_a \subseteq f_a^{-1}\varphi_\sigma\varphi_\sigma^{-1}f'_a.$$

Since $\sigma = \varphi_\sigma\varphi_\sigma^{-1}$ is a generalized congruence on A, by theorem 3, we see that $f_a^{-1}\varphi_\sigma f'_a \subseteq f_a^{-1}\sigma f'_a \subseteq \sigma = \varphi_\sigma\varphi_\sigma^{-1} \subseteq \varphi_\sigma$

To prove (ii), we have

$$\varphi_\sigma f'_a \overleftarrow{F}^\sigma \subseteq \varphi_\sigma \varphi_\sigma^{-1} f_a \varphi_\sigma \overleftarrow{F}^\sigma = \sigma f_a \varphi_\sigma \overleftarrow{F}^\sigma \subseteq \sigma f_a \overleftarrow{X} \subseteq f_a \overleftarrow{X},$$

again utilizing the fact that $\sigma = \varphi_\sigma \varphi_\sigma^{-1}$ is a generalized congruence on A.

To prove (iii), let $\mu_{\varphi_\sigma^{-1} f_a \overleftarrow{X}}(K,1) = \varepsilon$, for some $K \in F^\sigma$.

Let $C = \{x \in X \mid \mu_{\sigma^{-1} f_a}(K,x) = \varepsilon\}$.

Clearly, for any two elements x, x' in C, we have

$$\varepsilon \le \mu_{f_a^{-1} \varphi_\sigma \varphi_\sigma^{-1} f_a}(x,x') \le \mu_{\varphi_\sigma \varphi_\sigma^{-1}}(x,x') = \mu_\sigma(x,x')$$

Therefore, C is an ε-complete set with respect to the generalized congruence σ on A. By Zorn's lemma, C is contained in a maximal ε-complete set K' in F^σ. Hence, for each element x in K', we see that $\mu_{\varphi_\sigma}(x,K') \ge \varepsilon$.

It is easily seen that $\mu_{\varphi_\sigma^{-1} f_a \varphi_\sigma}(K,K') \ge \varepsilon$ and $\mu_{\overline{\varphi_\sigma^{-1} f_a \varphi_\sigma}}(K,K') \ge \varepsilon$.

Hence, $\mu_{\varphi_\sigma^{-1} f_a \overleftarrow{X}}(K,1) \le \mu_{f'_a \overleftarrow{F}^\sigma}(K,1)$; i.e. $\varphi_\sigma^{-1} f_a \overleftarrow{X} \subseteq f'_a \overleftarrow{F}^\sigma$. This completes the proof of the theorem. ||

The following result is a consequence of theorem 2 and lemma 9.

Theorem 7. A fuzzy relation σ is an ε-congruence $(0 < \varepsilon \le 1)$ on a fuzzy relational system A if there is a similar fuzzy relational system A' and an ε-function φ_σ such that φ_σ is a generalized congruence between A and A' and $\sigma = \varphi_\sigma \varphi_\sigma^{-1}$.

V. Applications

We will now give several specific applications of the concepts and results introduced in the previous sections. The first application is to automata theory in which we generalize the notions of structural and behavioral equivalence of nondeterministic automata discussed by Yeh [13] to the more general fuzzy automata. The second application concerns with pattern classification which arises naturally in the framework of fuzzy relations. Finally, we discuss briefly the possibility of modeling information networks using fuzzy graphs.

A. Application to Automata Theory

<u>Definition 5</u>. A <u>fuzzy automaton</u> M is a quadruple $[Q,\Sigma,f,F]$, where $[Q,\Sigma,f]$ is a fuzzy relational system in which Q is referred to as a set of <u>states</u>, and $F \subseteq Q$ is called the set of <u>final states</u> of M. Two fuzzy automata $M = [Q,\Sigma,f,F]$ and $N = [Q',\Sigma,f',F']$ are said to be <u>behaviorally equivalent</u> with respect to states $q_o \in Q$ and $q_o' \in Q'$, denoted by $M_{q_o} \sim N_{q_o'}$, iff

$$(\forall x \in \Sigma^*)[\vec{q}_o f_x \overleftarrow{F} = \vec{q}_o' f_x' \overleftarrow{F}'] \tag{7}$$

M and N are called <u>structurally equivalent</u>, denoted by $M \simeq_\sigma N$, iff there exists a generalized congruence σ from $[Q,\Sigma,f]$ to $[Q',\Sigma,f']$ such that $\vec{F}\sigma \subseteq \vec{F}'$ and $\vec{F}'\sigma^{-1} \subseteq \vec{F}$.

<u>Lemma 10</u>. Let $A = [X,\Sigma,f]$ and $B = [Y,\Sigma,g]$ be two similar fuzzy relational systems. If σ is a generalized congruence from A to B, then $a \in \Sigma$,

$$\sigma^{-1} f_a \subseteq g_a \sigma^{-1} \quad \text{and} \quad \sigma g_a \subseteq f_a \sigma.$$

Proof: σ is a GC implies that for each $a \in \Sigma$, $\sigma^{-1} f_a \overleftarrow{X} \subseteq g_a \overleftarrow{Y}$, which in turn implies that for each $y \in Y$, there exists a y^* in Y such that

$$\mu_{\sigma^{-1} f_a \overleftarrow{X}}(y,1) \leq \mu_{g_a}(y,y^*)$$

Hence,

$$(\forall y \in Y)(\exists y^* \in Y)[(\forall x \in X)[\mu_{\sigma^{-1} f_a}(y,x) \leq \mu_{g_a}(y,y^*)] \tag{9}$$

$$\sigma \text{ is a GC} \iff (\forall a \in \Sigma)[f_a^{-1} \sigma g_a \subseteq \sigma]$$

$$\iff (\forall a \in \Sigma)[g_a^{-1} \sigma^{-1} f_a \subseteq \sigma^{-1}] \tag{10}$$

Let y be an arbitrary element of Y and y^* be the corresponding element such that y and y^* satisfies (9), then from (10), we have

$$\mu_{g_a^{-1} \sigma^{-1} f_a}(y^*,x) = \bigvee_{\substack{x' \in X \\ y' \in Y}} [\mu_{g_a^{-1}}(y^*,y') \wedge \mu_{\sigma^{-1}}(y',x') \wedge \mu_{f_a}(x',x)]$$

since $\mu_{\sigma^{-1} f_a}(y,x) \leq \mu_{g_a}(y,y^*)$ by (9)

Thus,

$$\bigvee_{x'} [\mu_{g_a^{-1}}(y^*,y) \wedge \mu_{\sigma^{-1}}(y,x') \wedge \mu_{f_a}(x',x)] = \mu_{\sigma^{-1} f_a}(y,x)$$

Since $\bigvee_{x'} [\mu_{g_a^{-1}}(y^*,y) \wedge \mu_{\sigma^{-1}}(y,x') \wedge \mu_{f_a}(x',x)] \leq \mu_{g_a^{-1} \sigma^{-1} f_a}(y^*,x)$,

we conclude that

$$\mu_{\sigma^{-1} f_a}(y^*,x) \leq \mu_{g_a^{-1} \sigma^{-1} f_a}(y^*,x) \leq \mu_{\sigma^{-1}}(y^*,x) \tag{11}$$

By (9) and (11), we see that for each $y \in Y$

$$\mu_{\sigma^{-1} f_a}(y,x) \leq \mu_{g_a}(y,y^*) \wedge \mu_{\sigma^{-1}}(y^*,x) \leq \bigvee_{y'} [\mu_{g_a}(y,y') \wedge \mu_{\sigma^{-1}}(y',x)]$$

$$= \mu_{g_a \sigma^{-1}}(y,x).$$

Hence, for each $a \varepsilon \Sigma$, $\sigma^{-1} f_a \subseteq g_a \sigma^{-1}$. Similarly, we can prove that $\sigma g_a \subseteq f_a \sigma$. ||

Theorem 8. Let $M = [Q,\Sigma,f,F]$ and $M' = [Q',\Sigma,f',F']$ be two fuzzy automata over the same input alphabet Σ. Then $M \simeq_\sigma N$ implies $M_{q_o} \sim N_{q'_o}$ for any q_o and q'_o such that $\mu_\sigma(q_o,q'_o) = 1$.

Proof: We need to show that for each $x \varepsilon \Sigma^*$, $\vec{q}_o f_x \overleftarrow{F} = \vec{q}'_o f'_x \overleftarrow{F}'$.

If x is the empty sequence $\wedge$, then $\vec{q}_o f_\wedge \overleftarrow{F} = 1$ if $q_o \varepsilon F$ and is equal to 0, otherwise. Since $\mu_\sigma(q_o,q'_o) = 1$ and $\vec{F}\sigma \subseteq \vec{F}'$, we see that $\vec{q}_o f_\wedge \overleftarrow{F} \subseteq \vec{q}'_o f'_\wedge \overleftarrow{F}'$.

Let now $x = a_1 a_2 \ldots a_n \varepsilon \Sigma^*$, we have

$$\begin{aligned}
\vec{q}_o f_x \overleftarrow{F} &= \vec{q}_o f_{a_1} f_{a_2} \ldots f_{a_n} \overleftarrow{F} \\
&\subseteq \vec{q}'_o \sigma^{-1} f_{a_1} f_{a_2} \ldots f_{a_n} \overleftarrow{F} \\
&\subseteq \vec{q}'_o g_{a_1} \sigma^{-1} f_{a_2} \ldots f_{a_n} \overleftarrow{F} \text{, by lemma 11} \\
&\ \ \vdots \\
&\subseteq \vec{q}'_o g_{a_1} \ldots g_{a_n} \sigma^{-1} \overleftarrow{F} \text{, by repeated application of lemma 11} \\
&\subseteq \vec{q}'_o g_x \overleftarrow{F}' \text{, since } \vec{F}\sigma \subseteq \vec{F}' \iff \sigma^{-1} \overleftarrow{F} \subseteq \overleftarrow{F}'.
\end{aligned}$$

Similarly, we can prove that $\vec{q}'_o g_x \overleftarrow{F}' \subseteq \vec{q}_o f_x \overleftarrow{F}$, for each $x \varepsilon \Sigma^*$. Hence, $M \sim N$. ||

B. Application to Pattern Classification

Utilization of fuzzy relations to pattern classification and cluster construction has been discussed by several authors [1,3]. We will look at the clustering problem from the viewpoint of fuzzy graph.

Definition 6. A fuzzy graph G is a pair $[V,R]$, where V is a set of vertices, and R is a fuzzy relation on V. A vertex v is said to be ε-reachable from another vertex u, for some $0 < \varepsilon \leq 1$, iff there exists a positive integer k such that $\mu_{R^k}(u,v) \geq \varepsilon$. G is called strongly ε-connected iff every pair of vertices are mutually ε-reachable. G is said to be initial ε-connected iff there exists $v \in V$ such that every vertex u in G is ε-reachable from v.

Following the usual convention between binary relations and boolean matrices, we denote by M_G the corresponding fuzzy matrix of a fuzzy graph G. In other words, $(M_G)_{ij} = \mu_R(v_i,v_j)$.

The first part of the following result is due to Tamura, Higucchi and Tanaka [8]. The second part is quite straightforward and hence is given without proof.

Theorem 8. Let $G = [V,R]$ be a given finite fuzzy graph, consisting of n vertices.

(i) If R is reflexive, then there exists $k \leq n$ such that

$$M_G < M_G^2 < \ldots < M_G^k = M_G^{k+1}$$

(ii) If entries of the main diagonal of M_G are all zeros, then the sequence $\{M_G^i\}_{i \geq 1}$ is eventually periodic.

The previous result is now applied to clustering analysis. We assume that a data graph $G = [V,R]$ is given, where V is a set of data and $\mu_R(u,v)$ is a quantitive measure of the similarity of the two data items u and v. For $0 < \varepsilon \leq 1$, an ε-cluster in V is a maximal subset W of V such that each pair of elements in W is mutually ε-reachable. Therefore, the construction of ε-clusters of V is tantamount of finding all maximal strongly ε-connected subgraphs of G. An algorithm for the construction of ε-clusters is now given.

i) computer $R, R^2, \ldots, R^k$, where k is the smallest integer such that $R^k = R^{k+1}$;

ii) Let $S = \sum_{i=1}^{k} R^i$. Note that S is a similarity relation;

iii) construct F_ε of S.

Then, each element in F_ε is an ε-cluster.

We may also define an ε-cluster in V as a maximal subset W of V such that every element of W is ε-reachable from a special element v in W. In this case, the construction of ε-clusters is equivalent to finding all maximal initial ε-connected subgraphs of G. Note, however, that the relation induced by initial ε-connected subgraphs is not in general a similarity relation.

C. Application to Information Network.

A model of information network was proposed by Nance, Karfhage and Bhat [5] utilizing the concepts of directed graph. The most significant result of their work is the establishment of a measure of flexibility of a network. More specifically, let N be a network with m edges and n nodes, then the measure of flexibility of N, denoted by $Z(N)$, is defined as follows:

$$Z(N) = \frac{m - n}{n(n-2)} \tag{12}$$

While equation (12) is quite useful in classifying certain graph structures related to information network, it also has some drawbacks in that it is insensitive to certain classes of graphs. Also, it seems that the use of fuzzy graphs is a more desirable model for information network. The weights in each arc could be used as parameters such as number of channels between stations, costs for sending messages, etc. Thus, we propose here the use of a fuzzy graph to model an information network. Let N have n nodes; we define two measures (due to Y. Bang) of N: flexibility and balancedness, denoted by Z(N) and B(N) respectively, in the following:

$$Z(N) = \frac{\sum_{i=1}^{n} \sum_{j\neq i} \mu_R(v_i,v_j)}{n(n-1)} \qquad (13)$$

$$B(N) = \frac{\sum_{i} \left| \sum_{j} {}_{R}(v_i,v_j) - \sum_{k} \mu_R(v_k,v_i) \right|}{n(n-1)} \qquad (14)$$

It is readily seen that the proposed two measures given in (13) and (14) are much more sensitive to the structure of graphs than the one given in (12).

VI. Concluding Remarks

We have investigated certain algebraic properties of fuzzy relations and fuzzy relational systems in the previous sections. One of the main advantages resulted in the development of such an algebra is that it greatly simplifies the manipulation of fuzzy relation. Such a formal tool, it is hoped, will readily lend itself to aid in exploring the wide spectrum of potential applications of fuzzy relational systems some of which have already been touched upon in the previous section.

Let us remark here that the main motivation for the investigation of fuzzy logic and fuzzy relational systems came from the emergence of problems involving decisions upon ill-defined classes of events in the general area of artificial intelligence. Natural applications of fuzzy relational systems to problems of pattern classification and system approximation have been mentioned in the previous section as well as many other authors [2,6,9,15]. To conclude this section, we would like to propose here a machine model for further investigations, based on fuzzy relational system, which can perform computation on soft-information, i.e. information pertinent to the classes of events which cannot be defined precisely. In particular, the machine should have the capability of recognizing quantitive semantics. The machine model is developed based on the following rationales: (i) the machine can perform computation when provided with soft data. However, the machine must make "hard" decisions depending upon the global confidence value of the input words which is determined in terms of some internal interpretative scheme of the machine; (ii) the machine must have the capability of trying out different alternatives. With the above premises in mind, a description of the proposed machine model, which is a generalization of the usual Turing machine with a reference table [1] is given in the following: The machines consist of an input tape, a finite control and a reference table. The input alphabet belongs to some finite soft algebra [7]. (It should be noted that, in general, some restriction should be placed upon the algebra. For example, the natural restriction for the soft algebra of fuzzy functions is that the grade of membership should only range over the rationals). Upon receiving an input symbol, the machine will evaluate the global confidence, initially set to zero, of that part of the input word received so far based on the input symbol, current state, and the value of the current reference table entry. A decision is made by the machine depending upon the final confidence value of the input word relative to some pre-determined threshold value, and the terminal state of the machine. However, during the course of computation, the machine will reset itself to some initial configuration whenever the global confidence value of the input word at that particular time fell below a given value.

We observe that the manner in which the proposed machine operates is similar to the sequential decoding algorithm (with backtracking) of recurrent codes as are encountered in the communication engineering [10]. Therefore, we believe that the machine being proposed can execute nondeterministic algorithms discussed by Floyd [2], and perhaps fuzzy algorithms discussed by Zadeh [15].

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