ANALYSIS, DECOMPOSITION, SYNTHESIS, AND APPLICATIONS OF HIGHER RANK RELATIONS

by

Sung Yang Bang

August, 1974 TR-43

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*This paper was supported in part by National Science Foundation Grants GJ 36424 and GJ 31528.

ABSTRACT

Relations are an important concept and tool in information processing. For example, by using a set of relations, one can represent sentences in natural languages, patterns, and data bases. In general, the information about a finite set of elements can be expressed formally by a relational system $(S, \{\alpha_i\}_{i\varepsilon I})$ where S is the set of elements and $\{\alpha_i\}_{i\varepsilon I}$ is the set of relations of finite ranks on S. In this study, relations, especially relations of ranks higher than two are studied since relational systems generally contain higher rank relations.

The study of higher rank relations consists of three parts. First, higher rank relations themselves are analyzed since we know little about them. Secondly, since lower rank relations are generally simpler, more intuitive and easier to handle, the decomposition of a relation into lower rank relations is considered. Thirdly, the converse problem to decomposition, i.e. the synthesis of a set of relations into a higher rank relation, is considered. Syntheses are important from the point of compaction of relations too.

Bases on the results obtained from the study on relations, a relational data structure which is actually designed for the internal representation of relational systems is described. Noting that sets are unary relations and graphs are binary relations, it is pointed out that the relational data structure is more powerful and flexible than the existing data structures. Finally, the concept of relational systems with the operations of decomposition and synthesis are applied to pattern recognition and graph theory.

In short, this piece of research studies relations, especially higher rank relations and demonstrates the usefulness of the results.

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CHAPTER I

INTRODUCTION

1.1 Motivation

Relations are important in various fields of information processing. Relations are an important concept and tool especially in the areas of pattern recognition, semantic information processing, and data management systems. One of the most significant facts about relations is that most information can be conveniently represented by a set of relations defined on the set of primitives which concern us. This idea is exactly the model of most systems of pattern recognition, question answering and data management. Let us review some of the important uses of relations in these areas.

In pattern recognition, Miller and Shaw [26] point out the importance of syntactic approaches. Along this direction relational approaches in pattern recognition are described in Fu [11] and Rosenfeld [34]. Guzman [13] also thinks that any figure can be completely described by a set of relations. Winston [38] uses a labeled graph model for the representation of patterns inside computers. Evans [9,10] is also interested in syntactic methods in pattern recognition that use relations.

Example 1.1 The picture as shown in Fig. 1.1 can be described by the following relations [34].

$$\text{SQUARE} = \begin{Bmatrix} (a) \\ (c) \\ (d) \end{Bmatrix} \qquad \text{RECTANGLE} = \begin{Bmatrix} (b) \\ (e) \end{Bmatrix}$$

$\text{LEFT} = \left\{\begin{matrix}(a,b)\\(b,c)\end{matrix}\right\}$ $\text{ABOVE} = \{(b,d)\}$ $\text{INSIDE} = \{(e,d)\}$

$\text{CONGRUENT} = \{(a,c)\}$ $\text{LARGER} = \{(d,a)\}$ $\text{TALLER} = \{(b,d)\}$

$\text{BETWEEN} = \{(b,a,c)\}$ $\text{NEARER} = \left\{\begin{matrix}(b,c,a)\\(b,d,c)\end{matrix}\right\}$ ///

$\text{WIDER} = \{(e,b)\}$

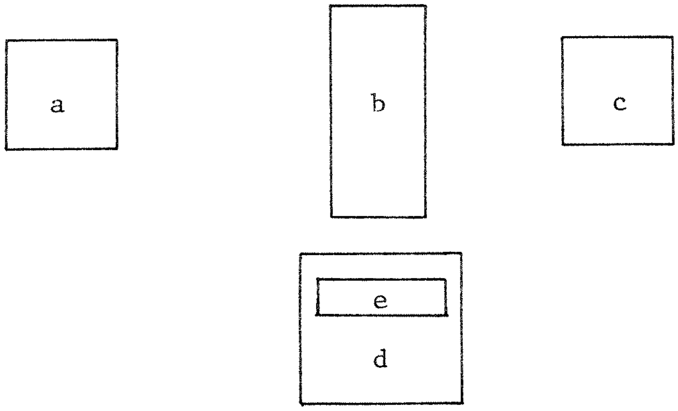


Fig. 1.1

While pattern recognition is concerned with this kind of abstraction and representation of patterns, natural language processing is interested in the abstraction and representation of sentences and concepts. Quillian [31] and Tesler, et al. [36] are primarily interested in the representation of memory structure. SIR by Raphael [31] is a question answering system in which well selected relations are used. The semantic network by Simmons is a model to represent English sentences [35].

<u>Example 1.2</u> In SIR [32] each sentence is represented by a relation. For example,

1. "The lamp is just to the right of the chair," is represented by a binary relation JRIGHT(LAMP, CHAIR).
2. "John gave a book to Jim," is represented by a ternary relation GIVE(JOHN, BOOK, JIM).

unique #	name	department
1	A	a
2	A	b
3	B	b
4	C	a

Fig. 1.3

supplier	part	project	quantity
1	2	5	17
1	3	5	23
2	3	7	9
2	7	5	4
4	1	1	12

Fig. 1.4

In the above records, unique # and the combination of supplier and part can be used as the indices of the records EMPLOYEE and SUPPLY, respectively, because they uniquely specify a row in the records. ///

1.2 Relational System

A relational system [39] is a pair $(S,\{\alpha_i\}_{i\varepsilon I})$ where S is a set of primitives and $\{\alpha_i\}_{i\varepsilon I}$ is a set of relations on S. This set of relations may contain unary relations, binary relations, ternary relations, and, in general, higher rank relations. The important point, as illustrated in the above examples, is that relational systems can be used to represent pattern descriptions, sentences in natural language, and data bases.

An efficient data management is critical if any system of visual and semantic information processing is going to be practical. It is even said that pattern recognition and semantic information processing are actually data management. On the other hand, pattern recognition and natural language

The internal representation of these sentences is given in Fig. 1.2 (1) and (2), respectively, where top nodes are atoms which have the bottom expressions in their property lists in LISP. ///

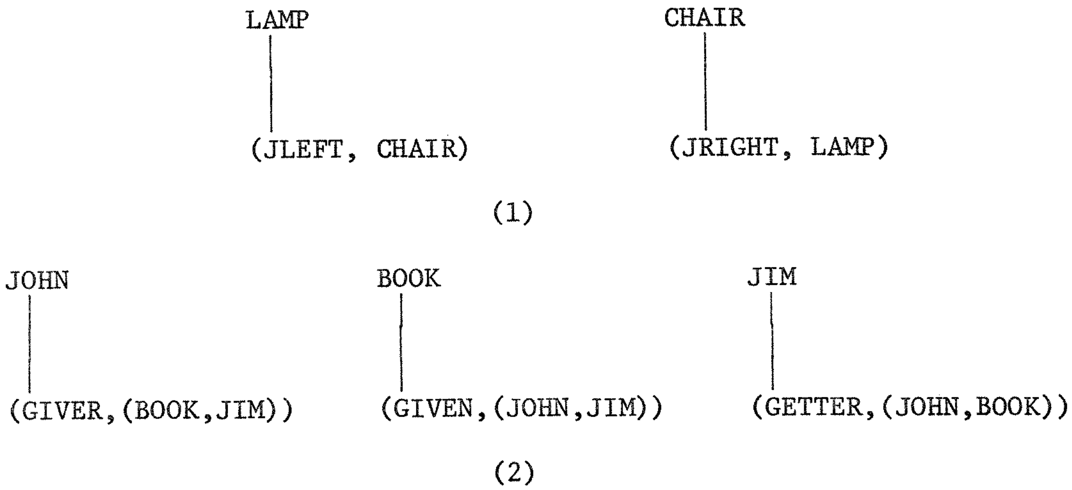


Fig. 1.2

The efficient processing and the representation of data are the concern of data management systems. Today there are many generalized data management systems available. In spite of the name "generalized" they are usually data-dependent [7]. For a more flexible and general model than hierarchical models, McGee [25] proposed a labeled graph model, and Childs [4,5] described a set theoretical model. Codd [7] regards even tables as relations and proposes a relational model.

Example 1.3 As will be stated later, tables can be regarded as relations which are subsets of cartesian products. Therefore each record of a data base is a relation [7]. For example the record of employees given in Fig. 1.3 is a ternary relation, and the record of supply shown in Fig. 1.4 is a 4-ary (quaternary) relation.

processing are often combined to have more general and more powerful information processing systems [8,20]. Therefore it seems quite natural to treat these different systems in a unified way. Also the abstraction by relational systems may describe the problems common to all these systems and solve them.

However, in this simple and unified form, there are many fundamental questions about relations. Some of the typical ones are: Is it reasonable to handle all these relations of different ranks in the same way, i.e., can higher rank relations be regarded as extensions of binary relations? What relationships exist between relations? Are there any convenient representations of a higher rank relation by lower rank ones and vice versa? These lead us to the study of relations. Another type of problem related to the adoption of a relational system is the internal representation of relational systems. None of the existing data structures seem suitable for it. Since a relational system generally has relations of ranks higher than two, the data structures using digraph model and set theory [4,25] are inadequate. For the same reason the associative structure of LEAP [28] is not enough. Codd's relational model [7] is too specific and limited to data bases only.

1.3 Structure of the Study

Our motivation is relational systems as stated above. However, we find that the theory of higher rank relations is lacking. Therefore, we initially attempt to study higher rank relations and subsequently apply the results of the study to relational systems.

We pursue our study as follows. In the next chapter relations are formally defined. Then higher rank relations are analyzed by analogy to binary relations. In other words, the properties of relations are presented and the extension of the concepts of compatibility relation and equivalence relation are considered. The decomposition and the synthesis of relations are the topics of Chapters 3 and 4, respectively. Since there has been much study of binary relations, it will be desirable to consider the relationships between higher rank relations and lower rank relations, especially the possibility of the representation of a high rank relation by binary relations and the construction of a high rank relation from binary relations. Various problems, including the characterizations related to these subjects, will be considered. Then Chapter 5 will describe the data structure for relational systems based on the results obtained in the preceding chapters. Finally some applications of decomposition and synthesis of relations are presented in Chapter 6.

Chapter 7 summarizes the study emphasizing the major results and describing them in the context of previous work.

Throughout this study, lemmas, theorems and corollaries are sequentially numbered from the beginning of each chapter. Figures and examples are numbered in the same way. Each new definition will be stated separately as far as possible but will not be numbered. It should be remembered here that $\subset$ is equivalent to $\subseteq$. If $\subsetneq$ is meant, that symbol is explicitly used.

CHAPTER II

ANALYSIS OF HIGHER RANK RELATIONS

2.1 Introduction

Although relations may be defined in various ways, we adopt a very standard and formal definition. It is true that a relation can be described independently of the objects on which the relation is defined: for example, "is to the right of". However, when we talk of a relation here, the set of objects on which the relation is defined is always specified, even if the set is infinite. The relations defined in this study are called abstract relations and distinguished from those defined by others.

In this chapter abstract relations are formally defined and analyzed by analogy to the analysis of binary relations. That is, the concepts for binary relations are extended so that binary relations are indeed simply a special case of general high rank relations. In this chapter, therefore, we are concerned with relations themselves and their properties while the relationships between relations will be our primary interests in the following chapters.

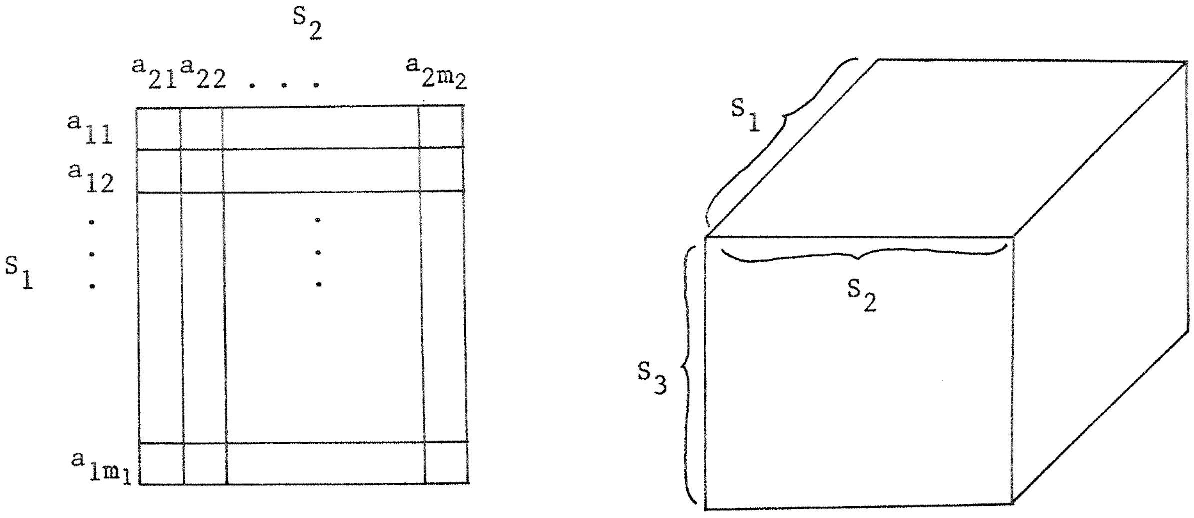
As it will be seen, the complexity of a relation seems to grow geometrically as the rank of the relation increases linearly. Another trouble in extending binary relations is the fact that the rank of binary relations is even while the ranks of general relations may be odd as well as even. It will also be mentioned later that the lack of a good method for the representation of higher rank relations like digraphs for binary relations makes it difficult to study higher rank relations intuitively.

In the next section preliminary definitions of abstract relations which will be used throughout this study are given. Then, in Section 2.3, we present properties of higher rank relations which are similar to reflexivity, symmetry, and transitivity of binary relations. Finally, we extend the equivalence and compatibility relations of binary relations to higher rank relations in Section 2.4.

2.2 Preliminaries

Definition: The cartesian product, $S \times S \times \ldots \times S_n$ (or $X_{i=1}^{n} S_i$) of the sets $S_i = \left\{a_1^i, a_2^i, \ldots, a_{m_i}^i\right\}$ is the set consisting of all ordered n-tuples $\left(a_{j_1}^1, a_{j_2}^2, \ldots, a_{j_n}^n\right)$ where $a_{j_i}^i \in S_i$.

If $n = 2$, a rectangular array having m_1 rows and m_2 columns, such that the rows and the columns are labeled $a_{11}, a_{12}, \ldots, a_{1m_1}$, and $a_{21}, a_{22}, \ldots, a_{2m_2}$, respectively, as shown in Fig. 2.1 (a), is a convenient way to represent a binary cartesian product since each position corresponds to a unique 2-tuple of the cartesian product. But when $n = 3$, a cubic representation of a ternary



(a) (b)

Fig. 2.1

cartesian product (as shown in Fig. 2.1 (b) similar to the binary case) is already less attractive. And it will not be practical at all when n is greater than 3. This difficulty of representation of higher order cartesian products is just the small tip of a huge iceberg of various difficulties which we will encounter in the study of general relations.

Definition: An n-ary relation α of an ordered set of sets S_i's is a subset of R_α of the cartesian product $X_{i=1}^{n} S_i$, i.e., $R_\alpha \subset X_{i=1}^{n} S_i$. We say that the rank (also "degree", "order", and "dimension" are used by others) of the relation α, denoted by $r(\alpha)$, is n.

We will usually use small Greek letters for relations. Also we will use R_α and α interchangeably when no confusion occurs.

Since we are going to handle two different kinds of sets, one of which is sets of objects over which relations are defined and the other of which is the relations themselves, we need to make a distinction between members of these different sets.

Definition: We call each member of a set S an element of S, and each member of R_α a member of the relation α.

The traditional notation $\alpha(a_1,a_2,\ldots,a_n)$ is equivalent to $(a_1,a_2,\ldots,a_n) \varepsilon R_\alpha$. We usually use small and capital letters for elements and members, respectively, like $R_\alpha = \{A_i\}_{i\varepsilon I} = \{(a_1^i,a_2^i,\ldots,a_n^i)\}_{i\varepsilon I}$. Although capital letters are used for both members of relations and element sets, earlier letters are used for members and later ones for sets among which S will be used almost exclusively.

One of the reasons why higher rank relations are not welcomed is that we do not have any decent way to represent a higher rank relation like a digraph for a binary relation [13]. Here we will use an extension of the graph representation of binary relations [30] although this is not satisfactory at all, especially when a relation has many members. More specifically, each i-th column of nodes belongs to the i-th set S_i of the corresponding cartesian product, and for each member $(a_1^i, a_2^i, \ldots, a_n^i) \epsilon R_\alpha$ we draw an arc from a_1^i in the first column to a_2^i in the second column, an arc from a_2^i in the second column to a_3^i in the third, and so on. In this case different from the binary case, however, we have to distinguish those arcs which represent one member from those which represent all other members of the relation.

Example 2.1 Let $S_1 = \{1,2,3\}$, $S_2 = \{a,b\}$, $S_3 = \{1,2\}$, and let $R_\alpha = \{(1,a,2),(1,a,1),(2,b,1),(3,a,1)\}$, then the diagram of the graph-like representation of α described above is shown in Fig. 2.2. ///

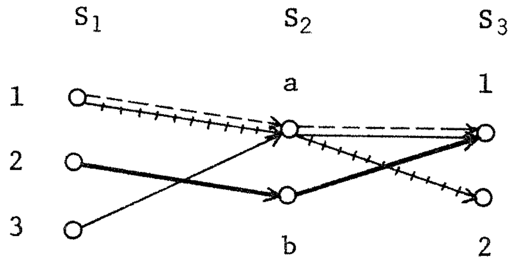


Fig. 2.2

Definition: Let $R_\alpha \subset (X_{i=1}^{\ell} S_i)x(X_{j=1}^{m} S'_j)$ and $R_\beta \subset (X_{j=1}^{m} S'_j)x(X_{k=1}^{n} S''_k)$, then the ordered pair (α,β) is m-composable if ℓ, n, m $\geq$ 1. The m-composite, denoted by $\alpha \underset{m}{\circ} \beta$, of α and β is an $(\ell+n)$-ary relation defined by

$$R_{\alpha \underset{m}{\circ} \beta} = \{(a_1,a_2,\ldots,a_\ell,a''_1,\ldots,a''_n) \mid \exists(a'_1,a'_2,\ldots,a') \epsilon X_{j=1}^{m} S'_j$$
$$\text{such that } (a_1,a_2,\ldots,a_\ell,a'_1,\ldots,a'_m) \epsilon R_\alpha$$
$$\text{and } (a'_1,a'_2,\ldots,a'_m,a''_1,\ldots,a'') \epsilon R_\beta\}.$$

Unlike binary relations, compositions of higher rank relations do not always possess an associative property. The following theorem gives a generalized associativity.

Theorem 2.1 Let (α,β) and (β,γ) be k- and m-composable, respectively, then if $k + m \leq r(\beta)$,

$$(\alpha \underset{k}{\circ} \beta) \underset{m}{\circ} \gamma = \alpha \underset{k}{\circ} (\beta \underset{m}{\circ} \gamma) = \alpha \underset{k}{\circ} \beta \underset{m}{\circ} \gamma.$$

The proof is trivial.

Definition: The inverse of an n-ary relation $\alpha \subset X_{i=1}^{n} S_i$ is an n-ary relation $\alpha^{-1} \subset X_{i=0}^{n-1} S_{n-i}$ specified by R_α^{-1} such that $(a_n, a_{n-1}, \ldots, a_1) \in R_\alpha^{-1}$ iff $(a_1, a_2, \ldots, a_n) \in R_\alpha$.

Definition: Let $\alpha, \beta \subset X_{i=1}^{n} S_i$, then α is said to be a subrelation of β if $\alpha \subset \beta$.

Definition: The complement of an n-ary relation $\alpha \subset X_{i=1}^{n} S_i$ is an n-ary relation $\bar{\alpha} \subset X_{i=1}^{n} S_i$ such that

$$\bar{\alpha} = \{(a_1, a_2, \ldots, a_n) \mid (a_1, a_2, \ldots, a_n) \notin \alpha\}.$$

Assuming further that the intersection ($\cap$) and the union ($\cup$) are defined on n-ary relations of the same cartesian product as the set theoretical intersection and union, respectively, the following can be easily proved.

Theorem 2.2

(1) $(\alpha \underset{k}{\circ} \beta)^{-1} = \beta^{-1} \underset{k}{\circ} \alpha^{-1}$.

(2) $(\alpha^{-1})^{-1} = \alpha$.

(3) $\alpha^{-1} = \beta^{-1}$ iff $\alpha = \beta$.

(4) $\alpha^{-1} \subset \beta^{-1}$ iff $\alpha \subset \beta$.

(5) $(\bar{\alpha})^{-1} = \overline{\alpha^{-1}}$

(6) $(\alpha \cap \beta)^{-1} = \alpha^{-1} \cap \beta^{-1}$.

(7) $(\alpha \cup \beta)^{-1} = \alpha^{-1} \cup \beta^{-1}$.

(8) $\gamma \underset{k}{\circ} (\alpha \cup \beta) = \gamma \underset{k}{\circ} \alpha \cup \gamma \underset{k}{\circ} \beta$.

(9) $\gamma \underset{k}{\circ} (\alpha \cap \beta) \subset \gamma \underset{k}{\circ} \alpha \cap \gamma \underset{k}{\circ} \beta$.

<u>Definition</u>: Let $\alpha \subset X_{i=1}^{n} S_i$ be an n-ary relation, then $\beta \subset X_{j=1}^{m} S_{i_j}$ where $m \leqslant n$ is said to be a <u>partial relation</u> (more specifically, an $(i_1, i_2, \ldots, i_m)$-(partial relation, denoted by $\alpha_{(i_1, i_2, \ldots, i_m)}$) of α if β is exactly the restriction of α to the cartesian product $X_{j=1}^{m} S_{i_j}$.

<u>Definition</u>: Let $R_\alpha \subset X_{i=1}^{n} S_i$, then we call $X_{i=1}^{n} S_i$ <u>the domain of R_α</u> and $X_{i=1}^{n} (R_\alpha)_i$ (the cartesian product of i-partial relations of R_α) the <u>range</u> of R_α. Further, we call S_i <u>i-th domain</u> (or domain i) of R_α and $(R_\alpha)_i$ <u>i-th range</u> (or range i) of R_α.

Next we present the concepts of onto and one-to-one for general relations which are slightly different from the counterparts of functions and which seem natural when we consider the fact that there is no difference between domains and codomains of relations.

<u>Definition</u>: An n-ary relation α is said to be an <u>onto relation</u> if $\alpha_1 \times \alpha_2 \times \ldots \times \alpha_n = X_{i=1}^{n} S_i$ where α_i is the i-partial (or (i)-partial) relation of α, i.e., if the range of α is the same as the domain of α.

<u>Definition</u>: An n-ary relation α is said to be a <u>one-to-one relation</u> if $A = (a_1, a_2, \ldots, a_n)$, $B = (b_1, b_2, \ldots, b_n) \in \alpha$ and $A \neq B$ implies $a_i \neq b_i \; \forall \; i$.

These are equivalent to saying that a relation α is onto or one-to-one if $\forall$ j the projection function $\phi_j : R_{\alpha_i} \rightarrow S_j$ is onto or one-to-one, respectively. We have the following characterization of an onto and one-to-one relation.

Lemma 2.1 If $\alpha \subset X_{i=1}^{n} S_i$ is one-to-one and onto, then $|S_i| = |S_j|$ $\forall$ i,j and hence $|\alpha| = |S_i|$.

Proof: Suppose that $|S_i| \neq |S_j|$, say $|S_i| > |S_j|$ without loss of generality. Let $|S_i| = p$ and $|S_j| = q$, then $p > q$. Since α is onto, $\exists$ a member corresponding to each $x_k \in S_i$, which means that there are p members in α. But the number of different elements in S_j is $q < p$, hence a contradiction. Therefore, $|S_i| = |S_j| = |\alpha|$ $\forall$ i,j.

Q.E.D.

Theorem 2.3 An n-ary relation $\alpha \subset X_{i=1}^{n} S_i$ where $|S_i| = |S_j|$ $\forall$ i,j is onto and one-to-one iff $\alpha_{i,i+1}$, the restriction of α to $S_i \times S_{i+1}$, is an onto and one-to-one function from S_i to S_{i+1} $\forall$ $i = 1,2,\ldots,n-1$.

Proof:

($\Longrightarrow$) Assume that α is an onto and one-to-one relation, then clearly $\alpha_{i,i+1} = S_i \times S_{i+1}$. And $A_{i,i+1} \neq A'_{i,i+1}$ $\forall$ $A,A' \in \alpha$ since $A \neq A'$ implies $a_j \neq a'_j$ $\forall$ j. Therefore $\alpha_{i,i+1}$ is an onto and one-to-one function from S_i to S_{i+1}.

($\Longleftarrow$) Conversely, assume that $\alpha_{i,i+1}$ of α is an onto and one-to-one function from S_i to S_{i+1} $\forall$ $i = 1,2,\ldots,n-1$. Then the onto-ness of $\alpha_{i,i+1}$ guarantees that the relation α is onto and the one-to-one-ness of them implies that α is an one-to-one relation.

Q.E.D.

Example 2.2 An example of a non-trivial onto and one-to-one relation is given in Fig. 2.3. ///

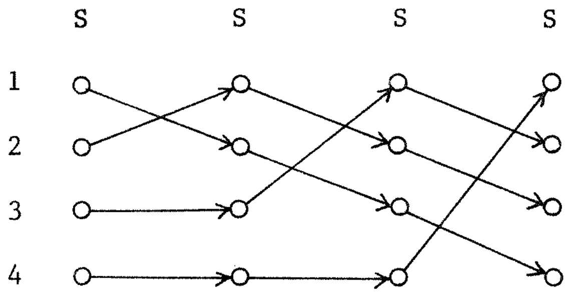


Fig. 2.3

Before we move over to the next section, we mention the isomorphism of relations.

The way we write and today's overwhelming sequential processing of information enforce relations to be represented by sequences of elements on which a relation holds. Indeed we used n-tuples to define relations earlier in this section. Obviously this is not the best way to express some relations at all, especially those in which the order of elements is immaterial. For example, let α be a ternary relation such that three elements are in the relation α if each pair is connected. Let the set $S = \{a,b,c,d\}$ as shown in Fig. 2.4, then α may be represented by $\{\{a,b,c\}\}$ if $\{a,b,c\}$ is understood to denote not a triple but a set of three elements since the order of the three elements a, b, and c is insignificant. But since the representation has to be sequential, α should be expressed as

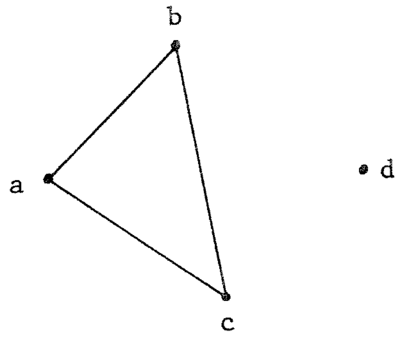


Fig. 2.4

$\alpha = \{(a,b,c),(a,c,b),(b,a,c),(b,c,a),(c,a,b),(c,b,a)\}$, which seems to be very redundant.

This sequencing gives us another problem. Obviously $\alpha = \{(a,b),(b,c)\}$ and $\beta = \{(a,b),(b,c)\}$ are the same relation according to our definition of relations. But how about this case: $\alpha = \{(a,b),(b,c)\}$ and $\beta = \{(b,a),(c,b)\}$? Apparently they are different. Hence α and β may have been completely different in the original meaning of the relations. However, after another glance at them it can be thought that they provide us exactly the same information. In other words, if we know the relationship between α and β in terms of the order of domains, we can obtain one relation from the other automatically. It is a matter of simple conversion. That is, if $(x,y) \in \alpha$, then $(y,x) \in \beta$ and vice versa. This is what we mean by saying that two relations provide us the same information about S. Let us call two such relations equivalent. In general, i.e., in the case of n-ary relations, it is assumed that if α is obtained from β by simply changing the order of the sequence of domains, then α and β provide the same information. In this sense two equivalent relations are thought to be isomorphic.

Let us consider the enumeration of relations. Suppose $|S| = n$, then the number of different (but maybe isomorphic) m-ary relations is 2^{n^m} since n^m is the number of different m-tuples of n elements. The number 2^{n^m} is quite large even when n and m are fairly moderate. Therefore we realize that an m-ary relation is very powerful to represent patterns although the complexity of the relation grows geometrically as the number of elements of S grows linearly. This indicates that the number of primitives can be very small theoretically to classify a large number of patterns if the choices of primitives and relations are right. The number of non-isomorphic m-ary relations on S is left as an open problem.

2.3 Properties

Hereafter we will consider only n-ary relations in S^n for some S unless it is specified otherwise. In other words, we will be concerned with only those relations defined on a certain set of elements. Of course all definitions defined in the previous section are valid and will be used. In this section we will try to extend the familiar properties of binary relations, i.e., reflexivity, symmetry, and transitivity, to higher rank relations. One can extend those properties in many diverse ways, but in this chapter only some of the extensions which seem very natural will be considered.

Reflexivity

First we consider reflexivities of higher rank relations.

Definition: (Strong Reflexivity) An n-ary relation α is strongly reflexive if $(a,a,\ldots,a) \in \alpha \ \forall \ a \in S$.

Definition: (Weak Reflexivity) An n-ary relation α is weakly reflexive if $(a_1,a_2,\ldots,a_n) \in \alpha \Rightarrow (a_1,a_1,\ldots,a_1),(a_1,a_2,\ldots,a_2),\ldots,$ $(a_1,a_2,\ldots,a_{n-1},a_{n-1}) \in \alpha$.

The strong reflexivity is a direct extension of the binary reflexivity and the weak reflexivity is an extension of the weak reflexivity by Yeh [40]. We find that the following anti-reflexivity is sometimes useful.

Definition: (Anti-reflexivity) An n-ary relation α is anti-reflexive if $(a_1,a_2,\ldots,a_n) \in \alpha \Rightarrow a_i \neq a_j \ \forall \ i \neq j$.

A binary relation with the anti-reflexivity property corresponds to a digraph without loops. Such cases occur frequently in applications, so it is not a new concept at all.

Example 2.2 Let $\alpha \subset (Z^+)^n$ and $\alpha = \{(a_1,a_2,\ldots,a_n) | a_1|a_2, a_2|a_3,\ldots,$ and $a_{n-1}|a_n\}$, then obviously α is weakly reflexive. If we further require that $a_1 < a_2 < \ldots < a_n \;\forall\; (a_1,a_2,\ldots,a_n) \in \alpha$, then α is now anti-reflexive. A strong reflexive relation can be found easily. For example, let $\beta \subset (Z^+)^n$ be $\beta = \{(b_1,b_2,\ldots,b_n) | b_1 < b_2 < \ldots < b_n\}$, then β is strongly reflexive since $(i,i,\ldots,i) \in \beta \;\forall\; i \in Z^+$. ///

Symmetry

Next we consider symmetric properties of higher rank relations. The following three symmetries seem to be fundamental.

Definition: (Strong Symmetry) An n-ary relation α is strongly symmetric if $(a_1,a_2,\ldots,a_n) \in \alpha \Rightarrow (a_{p(1)},a_{p(2)},\ldots,a_{p(n)}) \in \alpha$ for any permutation p on $(1,2,\ldots,n)$.

Definition: (Weak Symmetry) An n-ary relation α is weakly symmetric if $(a_1,a_2,\ldots,a_n) \in \alpha \Rightarrow (a_n,a_{n-1},\ldots,a_1) \in \alpha$.

Definition: (Circularity) An n-ary relation α is circular if $(a_1,a_2,\ldots,a_n) \in \alpha \Rightarrow (a_n,a_1,\ldots,a_{n-1}),(a_{n-1},a_n,\ldots,a_{n-2}),\ldots,(a_2,a_3,\ldots,a_1) \in \alpha$, or equivalently, $(a_1,a_2,\ldots,a_n) \in \alpha \Rightarrow (a_n,a_1,\ldots,a_{n-1}) \in \alpha$.

These three are all direct extensions of binary symmetry. We give one more symmetry which is more general and will be used later.

Definition: (Complete Symmetry) An n-ary relation α is completely symmetric if $(a_1,a_2,\ldots,a_n) \in \alpha \Rightarrow T^n \subset \alpha$ where T = {all distinct a_i's in $(a_1,a_2,\ldots,a_n)$}.

We give the following two kinds of anti-symmetry.

Definition: (Strong Anti-symmetry) An n-ary relation α is strongly anti-symmetric if $(a_1,a_2,\ldots,a_n) \in \alpha$ and any of its permutation $(a_{p(1)},a_{p(2)},\ldots,a_{p(n)}) \in \alpha \Rightarrow a_i = a_{p(i)} \;\forall\; i$.

Definition: *(Weak Anti-symmetry)* An n-ary relation α is *weakly anti-symmetric* if $(a_1,a_2,\ldots,a_n)$, $(a_n,a_{n-1},\ldots,a_1) \varepsilon \alpha \Rightarrow a_i = a_j \ \forall \ i,j \ni i + j = n+1$.

It should be noted that a completely symmetric relation is also both weakly symmetric and circular, and that the strong anti-symmetry implies the weak anti-symmetry.

Example 2.3 Let $G = (X,\Gamma)$ be a graph where X is a set of nodes and $\Gamma \subset X^2$ is a set of edges, and let us define α, β, $\gamma \subset X^4$ as follows:

$\alpha = \{(a_1,a_2,a_3,a_4) | \{a_1,a_2,a_3,a_4\}$ is a complete subgraph of $G\}$,

$\beta = \{(a_1,a_2,a_3,a_4) | (a_1,a_2),(a_2,a_3),(a_3,a_4) \ \varepsilon \ \Gamma\}$,

$\gamma = \{(a_1,a_2,a_3,a_4) | (a_1,a_2),(a_2,a_3),(a_3,a_4),(a_4,a_1) \ \varepsilon \ \Gamma\}$.

Then it is easy to see that α is strongly symmetric, β is weakly symmetric, and γ is circular. ///

Transitivity

Finally we look at transitivities of higher rank relations.

Definition: The k-star composition of $\alpha \subset S^m x S^k$ and $\beta \subset S^k x S^n$, denoted by $\alpha \underset{k}{*} \beta$, is an (m+n+1)-ary relation such that $\alpha \underset{k}{*} \beta = \{(a_1,a_2,\ldots,a_m,b_i,c_1,c_2,\ldots,c_n) \mid (a_1,a_2,\ldots,a_m,b_1,b_2,\ldots,b_k) \ \varepsilon \ \alpha$ and $(b_1,b_2,\ldots,b_k,c_1, c_2,\ldots,c_n) \ \varepsilon \ \beta$, and $b_i \ \varepsilon \{b_1,b_2,\ldots,b_k\}\}$.

Definition: *(Compositional Transitivity)* An n-ary relation α is *compositionally transitive* if $\alpha \underset{k}{\circ} \alpha \subset \alpha$ when $n = 2k$ and $\alpha \underset{k}{*} \alpha \subset \alpha$ when $n = 2k+1$ where k is a positive integer.

Definition: *(Chain Transitivity)* An n-ary relation α is *chain-transitive* if $(a_1^1 a_2^1 \ldots a_n^1),(a_1^2 a_2^2 \ldots a_n^2),\ldots,(a_1^n a_2^n \ldots a_n^n) \ \varepsilon \ \alpha$ and $a_2^1 = a_2^2, a_3^2 = a_3^3, \ldots, a_n^{n-1} = a_n^n$, $a_1^1 = a_1^n \Rightarrow (a_1^1 a_2^2 \ldots a_n^n) \ \varepsilon \ \alpha$.

The necessity of the strange looking k-star composition comes from the fact that the rank of a general relation can be odd as well as even. Chain transitivity is also found to be natural as will be seen below.

Example 2.4 Let R be an integral domain (commutative ring with no zero divisor) and $\alpha \subset R^4$ such that $\alpha = \{(a_1,a_2,a_3,a_4) | a_1/a_2 = a_3/a_4\}$, then clearly α is compositionally transitive. ///

Example 2.5 Let $G = (X,\Gamma)$ be a graph and $\alpha \subset X^4$ be a set of cycles of length 4, i.e. $\alpha = \{(a_1,a_2,a_3,a_4) | (a_1,a_2),(a_2,a_3),(a_3,a_4),(a_4,a_1) \in \Gamma$ and $a_i \neq a_j \ \forall\ i \neq j\}$, then α has the chain-transitivity property as illustrated in Fig. 2.3, where $X = \{1,2,\ldots,12\}$. Namely, $(12,3,2,1)$, $(4,3,6,5),(8,7,6,9),(12,11,10,9) \in \alpha$ implies $(12,3,6,9) \in \alpha$. ///

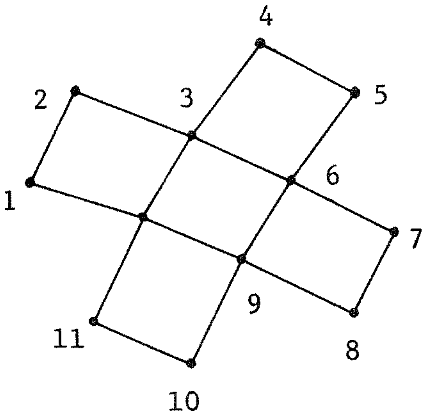


Fig. 2.3

We define another transitivity which will be used later.

Definition: (Terminal Transitivity) An n-ary relation α is terminal-transitive if $(x,a_1,a_2,\ldots,a_{n-1}),(a_1,a_2,\ldots,a_{n-1},y) \in \alpha \Rightarrow (x,a_1,\ldots,a_{i-1},a_{i+1},\ldots a_{n-1},y) \in \alpha \ \forall i = 1,2,\ldots,n-1$.

This property is not uncommon.

Example 2.6 Let S be a set of real numbers, then the n-ary relation $\alpha \subset S^n$ defined by $\alpha = \{(a_1,a_2,\ldots,a_n) | a_1 < a_2 < \ldots < a_n\}$ obviously has the property of terminal transitivity. Namely if $(x_1,x_2,\ldots,x_n),(x_2,x_3,\ldots,x_{n+1}) \in \alpha$, then $(x_1,x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_{n+1}) \in \alpha \ \forall\ i$. ///

Unlike the reflexivities and symmetries, there is no inclusion relation among the transitivities defined here.

2.4 Two Ternary Relations

It may be appropriate to give some practical higher rank relations at this point by means of the properties defined in the previous section so that they can be used whenever some examples are needed and those are relevant.

There seem to be two large sources for higher rank relations: linear relations and complete relations. By a linear relation we mean an n-ary relation in which only those binary relations between two consecutive elements, i.e., a_i and a_{i+1} for $i = 1,2,\ldots,n-1$, are critical while a complete relation is one in which all binary relations between two elements, i.e., a_i and a_j $\forall$ $i \neq j$, are essential factors to decide whether each $(a_1,a_2,\ldots,a_n) \varepsilon \alpha$ or not. Intuitively, linear relations are regarded as sequences and complete relations as groups. For example, a sequence of four numbers (a_1,a_2,a_3,a_4) is the former type and a group of four persons (b_1,b_2,b_3,b_4) is the latter type.

In spite of all these higher rank relations, what we need here are those which seem to be higher rank relations in nature and which will lose their meaning significantly when they are expressed otherwise. It turns out that such pure high rank relations are surprisingly rare [11,13] even though it is not seldom that for various reasons people use higher rank relations which can be also treated by their binary-expressed relations. Here we will give two of the simplest yet useful higher rank relations both of which are of rank 3. One is the "between" relation [13] which seems to be a genuine ternary relation. Another is the "triangle" relation [11] which most people prefer to handle as a ternary relation.

Definition: The between relation is a ternary relation $\beta \subset S^3$ which satisfies the following properties:

(i) $(a,b,c) \in \beta \Rightarrow a,b,c$ are all distinct (anti-reflexivity),

(ii) $(a,b,c) \in \beta \Rightarrow (c,b,a) \in \beta$ (weak symmetry),

(iii) $(a,b,c),(b,c,d) \in \beta \Rightarrow (a,b,d),(a,c,d) \in \beta$ (compositional transitivity),

(iv) $(a,b,d),(a,c,d) \in \beta \Rightarrow (b,c,d) \in \beta$ or $(c,b,d) \in \beta$ but not both,

(v) $(a,b,c),(a,b,d) \in \beta \Rightarrow (b,c,d) \in \beta$ or $(b,d,c) \in \beta$ but not both.

Example 2.7 Let S be a set of points on a plane and let $\alpha \subset S^3$ be defined as $\alpha = \{ (a,b,c) \mid$ three distinct points a, b and c constitute a straight line and b is between a and c$\}$. It is fairly easy to see that α satisfies all five conditions above and, hence, actually is a between relation. The properties (i) and (ii) are direct from the definition itself. The other properties come from the fact that if $(a,b,c) \in \alpha$, then a, b and c are on a line. It may be of interest to notice that S can be divided so that a pair of classes can have at most one point in common, and that each class can be identified by at most two points, as seen in an example of Fig. 2.4. These observations throw light on an extension of the equivalence relation of binary relations which will be a topic of the next section. ///

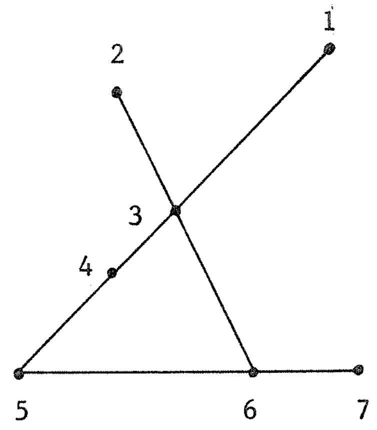


Fig. 2.4

Definition: The *triangle relation* is a ternary relation $\tau \subset S^3$ which satisfies the following properties:

(i) $(a,b,c) \in \tau \Rightarrow a,b,c$ are all distinct (anti-reflexivity),

(ii) $(a,b,c) \in \tau \Rightarrow$ all of its permutations $\in \tau$ (strong symmetry),

(iii) $(a,b,x),(a,y,c),(z,b,c) \in \tau \Rightarrow (a,b,c) \in \tau$ (chain-transitivity).

Example 2.8 Let G be a graph $G = (X,\Gamma)$ without loops and let $\alpha \subset X^3$ be defined as $\alpha = \{ (a,b,c) \mid (a,b),(b,c),(a,c) \in \Gamma\}$, then α is a triangle relation. For since G is a graph without loops, (i) is satisfied. The properties (ii) and (iii) are obvious from the fact that G is a graph, not a digraph. A simple example is given in Fig. 2.5, where $X = \{1,2,3,4,5\}$ and α is defined by the set of $(1,2,5),(2,3,4),(2,3,5),(2,4,5),(3,4,5)$ and all their permutations. ///

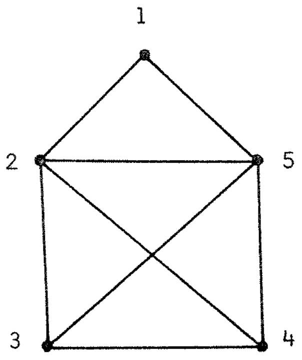


Fig. 2.5

2.5 k-compatibility and k-equivalence Relations

The properties of reflexivity, symmetry, and transitivity constitute an interesting set of attributes for the categorization of binary relations on a set. They are fully exhibited in [29]. An it is pointed out that among them the compatibility and equivalence relations are important as well as interesting. It is doubtful that it is fruitful to categorize higher rank relations by the extended properties described previously even if we can do it since the categorization will become less neat than that of binary relations because of the large number of properties. However we will try to extend the relations of compatibility and equivalance which seem to be interesting.

Definition: A k-compatibility relation $\alpha \subset S^k$ $(k \geq 2)$ is a k-ary relation on S which has the strong reflexivity and the complete symmetry properties.

Note that the degenerate case of k-compatibility relation, i.e., $k = 2$, is exactly the same as the compatibility relation of binary relations since the strong reflexivity and the complete symmetry become simply the reflexivity and the symmetry of binary relations, respectively. In that case, therefore, we may call it a compatibility relation instead of a 2-compatibility relation.

Definition: Given a k-compatibility relation α on S, a k-compatibility class or k-compatible, induced by α, is a subset C of S, such that $(a_1, a_2, \ldots, a_k) \in \alpha \; \forall \; a_1, a_2, \ldots, a_k \in C$.

Definition: Given a k-compatibility relation α on S, a maximal k-compatibility class or a maximal k-compatible is a k-compatibility class which is not properly contained in any other k-compatibility class.

Definition: The k-complete cover of S with respect to α, denoted by $C_\alpha(S)$, is the collection of all and only the maximal k-compatibility classes induced by α on S.

Lemma 2.1 If α is a k-compatibility relation on a countable set S and C is a k-compatibility class, then there is some maximal k-compatibility class C' such that $C \subset C'$.

Proof Since S is countable, the elements of S can be numbered; that is, $S = \{a_1, a_2, \ldots\}$. We define a sequence $C_0 \subset C_1 \subset C_2 \subset \ldots$ of k-compatibles by the following rule: $C_0 = C$. Given C_i, C_{i+1} is defined as $C_i \cup \{a_j\}$ where j is the smallest integer for which $a_j \notin C_i$ and C_{i+1} is a k-compatible. If there is no such j, then C_i is the last k-compatible of the sequence and $C_i = C'$. Thus C' is k-compatible and maximal. Since $C \subset C'$, the lemma is proved.

Q.E.D.

Theorem 2.4 There is a one-to-one correspondence between k-compatibility relations on S and k-complete covers $C_\alpha(S)$.

Proof Obviously a k-compatibility relation α induces a k-complete cover $C_\alpha(S)$ and it is unique. Next we show the converse.

Suppose that there are two distinct k-compatibility relations α and α' such that $C_\alpha(S) = C_{\alpha'}(S)$. Since $\alpha \neq \alpha'$, there is at least one member $A = (a_1,a_2,\ldots,a_k)$ such that $A \in \alpha$ but $A \notin \alpha'$. Then by Lemma 2.1 there is some member, i.e., a maximal k-compatible, C of $C_\alpha(S)$ which contains all distinct elements $a_{i_1},a_{i_2},\ldots,a_{i_j}$ of A. Since $C_\alpha(S) = C_{\alpha'}(S)$, there is a member C' of $C_{\alpha'}(S)$ which is identical to C; hence it contains $a_{i_1},a_{i_2},\ldots,a_{i_j}$. Then since C' is a k-compatible, any sequence of k elements of $a_{i_1},a_{i_2},\ldots,$ a_{i_j} is in α' by the property of the complete symmetry of α' if $j \geq 2$ and of the strong reflexivity of α' if $j = 1$, which is a contradiction. Therefore if $C_\alpha(S) = C_{\alpha'}(S)$, $\alpha = \alpha'$.

Q.E.D.

Example 2.9 Given a set $S = \{a,b,c,d\}$ and the 3-complete cover $C_\alpha(S) = \{\{a,b,c\},\{a,b,d\},\{c,d\}\}$, we can readily derive the corresponding 3-compatibility relation α as shown in Fig. 2.6, where we show those members of α derived from each maximal 3-compatible separately for clarity, and further, multiple arcs are simplified to simple arcs.

///

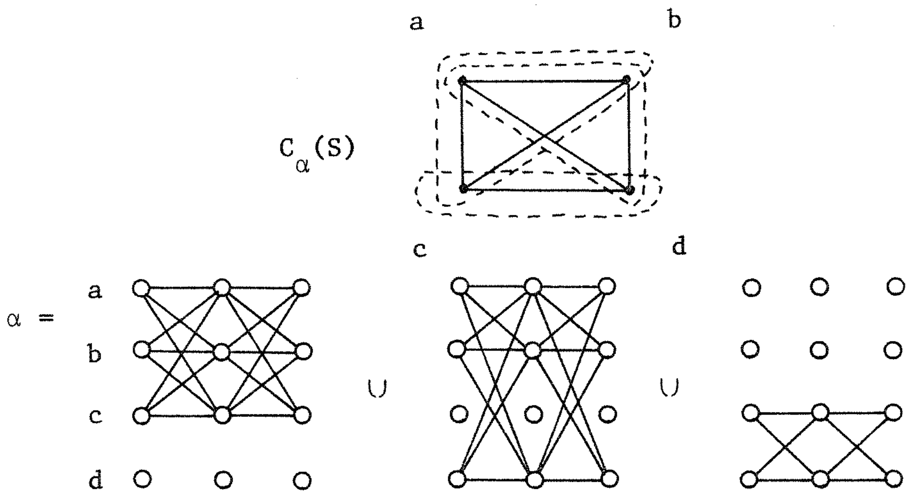


Fig. 2.6

Next we characterize k-compatibility relations.

Theorem 2.5 A k-ary relation $\alpha \subset S^k$ is a k-compatibility relation iff there exist a set T and an onto binary relation $\rho \subset S \times T$ such that $\alpha = \{(a_1,a_2,\ldots,a_k) \in S^k \mid a_i \in \rho^{-1}\rho(a_j)\ \forall\ i,j\}$.

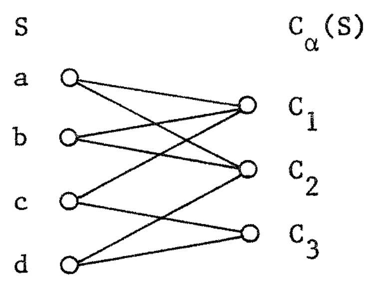
Proof

($\Rightarrow$) Suppose that α is a k-compatibility relation, then there is a k-complete cover $C_\alpha(S)$ induced by α. Let T be $C_\alpha(S)$ and ρ be defined as follows: $\rho(a) = \{C \in C_\alpha(S) \mid a \in C\}$. Now assume that $(a_1,a_2,\ldots,a_k) \in \alpha$, then there is a maximal k-compatible C such that $a_i \in C\ \forall\ i$. Since $a_i \in \rho^{-1}(C)\ \forall\ i$, $a_i \in \rho^{-1}\rho(a_j)\ \forall\ i,j$.

($\Leftarrow$) Conversely suppose that α is a k-ary relation such that $\alpha = \{(a_1,a_2,\ldots,a_k) \in S^k \mid a_i \in \rho^{-1}\rho(a_j)\ \forall\ i,j\}$ where ρ is an onto binary relation such that $\rho \subset S \times T$. First since ρ is onto, $a \in \rho^{-1}\rho(a)\ \forall\ a \in S$. Hence $(a,a,\ldots,a) \in \alpha\ \forall\ a \in S$, i.e., α is strongly reflexive. Next assume that $(a_1,a_2,\ldots,a_k) \in \alpha$, then $(a_{p(1)},a_{p(2)},\ldots,a_{p(K)}) \in \alpha$ for any permutation p of (1,2,...,k) directly form the hypothesis, i.e., α is strongly symmetric. Therefore α is a k-compatibility relation.

Q.E.D.

Example 2.10 The onto binary relation ρ above for Example 2.9 is shown in Fig. 2.7. ///



By adding one more property to k-compatibility relations we define k-equivalence relations in the following.

Definition: A k-equivalence relation $\alpha \subset S^k$ $(k \geq 2)$ is a k-ary relation on a set S which is a k-compatibility relation and has the property of terminal transitivity.

Again note that when k = 2, a 2-equivalence relation is simply an equivalence relation of binary relations.

Definition: A maximal k-compatible of a k-equivalence relation is termed a k-equivalence class.

Lemma 2.2 For a k-equivalence relation α, any two distinct members E_1 and E_2 of $C_\alpha(S)$ have at most k-2 elements of S in common.

Proof Suppose that E_1 and E_2 have more than k-2 elements in common. Let the arbitrary k-1 elements of them be $e_1,e_2,\ldots,e_{k-1}$. If $E_1 = E_1 \cap E_2$, then $E_1 \subset E_2$. Otherwise let $a \in E_1$ and $b \in E_2$, then since $e_1,e_2,\ldots,e_{k-1} \in E_1 \cap E_2$, $(a,e_1,e_2,\ldots,e_{k-1}),(e_1,e_2,\ldots,e_{k-1},b) \in \alpha$. Then by the terminal transitivity $(a,e_1,e_2,\ldots,e_{k-2},b) \in \alpha$, which means $a \in E_2$.

Hence $E_1 \subset E_2$. Similarly we can show that $E_2 \subset E_1$, thus obtaining $E_1 = E_2$.

Q.E.D.

Now we are ready to extend the partition concept.

Definition: A n-partition of a set S is a collection of subsets of S, such that each element of S belongs to at least one member of the collection and two different members of the collection have at most n elements in common.

By this definition an 0-partition is the same as an ordinary partition. Actually this extension is so natural that the reader can find the same definition in Jardine and Sibson [18] in which it is called a k-partition which allows any two members of the partition to have a maximum of k-1 elements in common.

Theorem 2.6 There is a one-to-one correspondence between k-equivalence relations on S and (k-2)-partitions of S.

Proof It is trivial that the complete cover of a k-equivalence relation is a (k-2)-partition, and it is unique.

Conversely we show that a k-partition uniquely specifies a (k+2)-equivalence relation whose complete cover $C_\alpha(S)$ coincides with the k-partition. Given a k-partition P, define a (k+2)-ary relation α as follows: $(a_1, a_2, \ldots, a_{k+2}) \varepsilon \alpha$ iff $a_1, a_2, \ldots, a_{k+2}$ belong to the same subset of P. Then α is strongly reflexive because each element belongs to some subset of P. α is also completely symmetric since if $a_1, a_2, \ldots, a_{k+2}$ belong to the same subset of P, then any subset of them is also contained in the subset of P. Finally if $(x, a_1, \ldots, a_{k+1}), (a_1 \ldots, a_{k+1}, y) \varepsilon \alpha$, then $x, a_1, \ldots, a_{k+1}$, and y are in the same subset of P since P is a k-partition.

Next we show that each class C of the original k-partition is a maximal (k+2)-compatible. First C is by hypothesis a (k+2)-compatible. Secondly if $(x,a_1,\ldots,a_{k+1}) \varepsilon \alpha \forall a_1,a_1,\ldots,a_{k+1} \varepsilon C$, then $x \varepsilon C$, a contradiction. Hence C is maximal.

Q.E.D.

Example 2.11 Let S = {a,b,c,d,e,f,g} and the 1-partition of S be {{a,b,c},{c,d,e,f} ,{f,g}} as shown in Fig. 2.8, then we have the 3-equivalence relation corresponding to the 1-partition, and its diagram is given in Fig. 2.9. As seen in the diagram, there exists a 3-equivalence relation among any three elements which belong to the same 1-partition class but not among three in different classes.

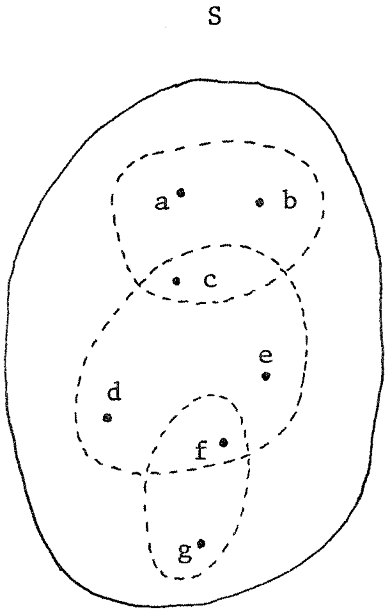
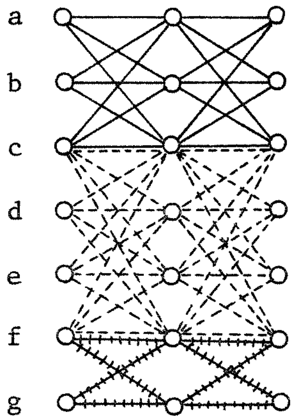


Fig. 2.8



Here, for clarity, the arcs of those members of S belonging to the same class are drawn by the same type of line and multiple lines are replaced by a single line.

Fig. 2.9

Finally we characterize k-equivalence relations.

Theorem 2.7 A k-ary relation $\alpha \subset S^k$ is a k-equivalence relation iff there exists an onto binary relation $\rho \subset S \times T$ such that $|\rho^{-1}(x) \cap \rho^{-1}(y)| \leq k-2 \ \forall\ x,y \in T$ and $\alpha = \{(a_1,a_2,\ldots,a_k) | a_j \in \rho^{-1}\rho(a_i) \ \forall\ i,j\}$.

Proof

($\Rightarrow$) Since $C_\alpha(S)$ is a (k-2)-partition, $|\rho^{-1}(x) \cap \rho^{-1}(y)| \leq k-2$ $\forall\ x,y \in T$. And since α is also a k-compatibility relation, the second half is satisfied by Theorem 2.5.

($\Leftarrow$) Conversely assume that $\alpha = \{(a_1,a_2,\ldots,a_k) | a_j \in \rho^{-1}\rho(a_i)$ $\forall\ i,j\}$ where ρ is an onto binary relation from S to T such that $|\rho^{-1}(x) \cap \rho^{-1}(y)| \leq k-2 \ \forall\ x,y \in T$. Then by Theorem 2.5 α is a k-compatibility relation. We have to show that α is also terminal-transitive. Let $(s,a_1,a_2,\ldots,a_{k-1}),(a_1,a_2,\ldots,a_{k-1},t) \in \alpha$, then $\exists\ x \in T \ni x \in \{\bigcap_{i=1}^{k-1}\rho(a_i)\} \cap \rho(s)$ and $\exists\ y \in T \ni y \in \{\bigcap_{i=1}^{k-1}\rho(a_i)\} \cap \rho(t)$ by the hypothesis. If $x \neq y$, $|\rho^{-1}(x) \cap \rho^{-1}(y)| \geq |\{a_1,a_2,\ldots,a_{k-1}\}| \geq k-2$, which contradicts the assumption that $|\rho^{-1}(x) \cap \rho^{-1}(y)| \leq k-2 \ \forall\ x,y \in T$. Therefore $x = y$. Hence, $(s,a_2,a_3,\ldots,a_{k-1},t),(s,a_1,a_3,\ldots,a_{k-1},t),\ldots,(s,a_1,a_2,\ldots,a_{k-2},t) \in \alpha$, namely α is terminal-transitive, proving the theorem.

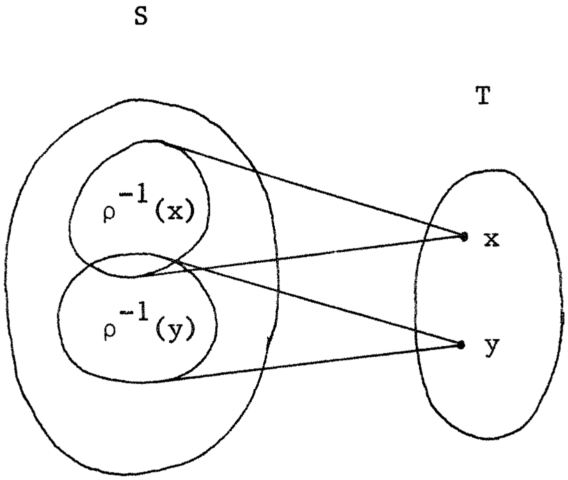


Fig. 2.10

Q.E.D.

CHAPTER III

DECOMPOSITION OF RELATIONS

3.1 Introduction

Higher rank relations are generally less intuitive and more complicated than binary relations, as seen in the previous chapter. Furthermore, since binary relations have a practical significance and at the same time can be handled by digraphs, they have been studied and utilized extensively. A question one may naturally ask at this point is whether or not higher rank relations can be decomposed to binary ones so that we can make use of the many results about binary relations which we already have or so that higher rank relations can be analyzed by means of simple binary relations. An intuitive concensus about the relationship between higher rank relations and binary relations has been that any higher rank relation can be decomposed into lower rank relations and hence, eventually, into binary relations. This seems to be one of the reasons why the study of higher rank relations has been consequently neglected.

It seems very natural, of course, to assume that a set of elements is in a certain high rank relation if and only if each pair of elements of the set is in a certain binary relation. For example, Fu [11] mentions two methods for decomposition of relations. He says, " . . . a relation $r(x_1,x_2,\dots,x_n)$ can be transformed into a decomposition of binary relations, such as $r_1(x_1,r_2(x_2,\dots,r_{n-1}(x_{n-1},x_n))\dots)$, or into a conjunction of binary relations $r_1(x_{11},x_{12}) \wedge r_2(x_{21},x_{22}) \wedge \dots \wedge r_k(x_{k1},x_{k2})$, or into a combination of these." Note that "relation" here is used for "member of a relation" in our definitions. It would be reasonable to assume that he is describing the decomposition not of a relation $r \subset S^n$ but a member of r, $r(x_1,x_2,\ldots,x_n)$. However, since these decompositions look very natural and we are concerned with the decomposition of relations, we will next look at them as methods for the decomposition of relations.

Although the first form looks neat, it is nothing but a rewriting of the original form. For r_1 is a relation on $S \times S^{n-1} = S^n$, r_2 on $S \times S^{n-2} = S^{n-1}$, and so on. That is, the rank of the relation r_1 has not been decreased at all. Naturally this kind of decomposition is not to reduce the ranks but to provide structures to relations. Therefore the first form is meaningless here but useful for some applications, especially for structuring data structures such as in Lisp [22].

In the second form above, k is not specified. It is convenient, however, to fix $k = n(n-1)/2$ for a uniform treatment. And, as a matter of fact, it is always possible to fix k that way. If a pair of elements is in more than one binary relation, we assign the pair to a new binary relation which is composed of them. And, on the other hand, if a pair of elements is in no binary relation, then we assume that they have a "don't-care" relation, so that every pair is in one and only one binary relation. One more thing is worthy of comment. Even though we always deal with ordered pairs for binary relations, we will use only one of the two related ordered pairs, i.e., an ordered pair and its inverse, since if one pair is specified by a binary relation, the other is automatically specified by the inverse of the binary relation.

Now we are ready to ask, "Can any higher rank relation be represented by the second form?" In other words, given an n-ary relation $\alpha \subset S^n$, do there exist $n(n-1)/2$ binary relations α_{ij} such that $\alpha = \{(a_1,a_2,\ldots,a_n) \mid \bigwedge_{i<j} (a_i,a_j) \in \alpha_{ij}]\}$? The answer is not trivial although obviously it is

always possible to express a member of a higher rank relation by a conjunction of the corresponding binaries as claimed by Fu. The question above can be interpreted in terms of digraphs, since a binary relation corresponds to a digraph. That is, is a higher rank relation really representable by a set of digraphs? Here we are talking of the decomposition of a higher rank relation into binary relations which, in turn, reproduce exactly the same higher rank relation. Let us take a look at the following simple example.

Example 3.1 Let α be a ternary relation $\alpha = \{(1,1,2),(2,1,1),(2,2,2)\}$ on a set of elements $S = \{1,2\}$, then each member can be expressed by a conjunction of its binary components:

$$\alpha(1,1,2) = \alpha_{12}(1,1) \wedge \alpha_{13}(1,2) \wedge \alpha_{23}(1,2),$$

$$\alpha(2,1,1) = \alpha_{12}(2,1) \wedge \alpha_{13}(2,1) \wedge \alpha_{23}(1,1),$$

$$\alpha(2,2,2) = \alpha_{12}(2,2) \wedge \alpha_{13}(2,2) \wedge \alpha_{23}(2,2).$$

Therefore the corresponding binary relations are:

$$\alpha_{12} = \left\{\begin{matrix}(1,1)\\(2,1)\\(2,2)\end{matrix}\right\}, \quad \alpha_{13} = \left\{\begin{matrix}(1,2)\\(2,1)\\(2,2)\end{matrix}\right\}, \quad \alpha_{23} = \left\{\begin{matrix}(1,1)\\(1,2)\\(2,2)\end{matrix}\right\}.$$

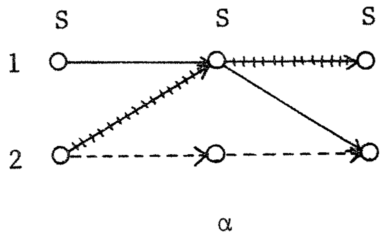


Fig. 3.1

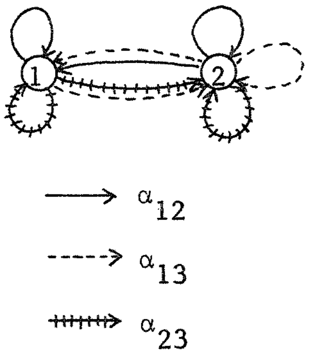


Fig. 3.2

The diagram of the ternary relation α is given in Fig. 3.1. The reader may recognize the binary components given above also directly in the diagram. Now let us represent the decomposition of this ternary relation by a labeled diagraph instead of three different digraphs as shown in Fig. 3.2. Here we used different lines for different labels just for clarity. Do not be confused by the use of the same lines as those of members of α in Fig. 3.1. The question is whether this exactly represents the ternary relation α above. Let $\alpha_2^* = \{(a,b,c) \mid (a,b) \in \alpha_{12} \wedge (a,c) \in \alpha_{13} \wedge (b,c) \in \alpha_{23}\}$, then we readily find that $\alpha_2^* \neq \alpha$, actually, that $\alpha_2^* = \{(2,1,2)\} \cup \alpha$, as also seen from the figure. In other words $\alpha_2^* = \{(i,j,k) \mid \textcircled{i} \longrightarrow \textcircled{j} \longrightarrow \textcircled{k}$ in the graph, $i,j,k = 1,2\}$. ///

It may be helpful to give another example which, this time, is exactly the same as a conjunction of its binary partial relations.

Example 3.2 Let $\beta = \{(1,1,1),(1,1,2),(2,2,1),(2,2,2)\} \subset S^3$ where $S = \{1,2\}$, then

$$\beta_{12} = \left\{\begin{matrix}(1,1)\\(2,2)\end{matrix}\right\}, \quad \beta_{13} = \left\{\begin{matrix}(1,1)\\(1,2)\\(2,1)\\(2,2)\end{matrix}\right\}, \text{ and } \beta_{23} = \left\{\begin{matrix}(1,1)\\(1,2)\\(2,1)\\(2,2)\end{matrix}\right\}.$$

Therefore $\beta = \beta_2^* = \{(a,b,c) \mid (a,b) \in \beta_{12} \wedge (a,c) \in \beta_{13} \wedge (b,c) \in \beta_{23}\}$. ///

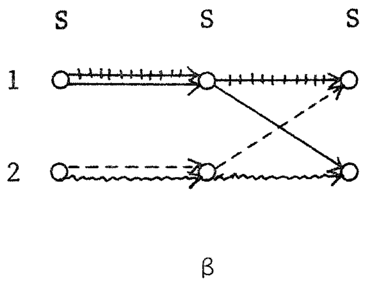


Fig. 3.3

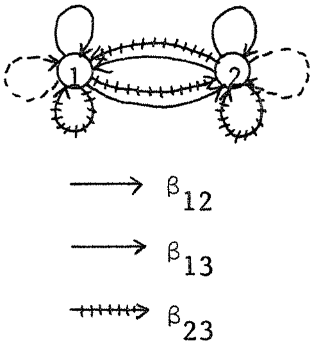


Fig. 3.4

From the discussion and the examples above we notice that a higher rank relation may be represented by the second form, i.e., by a conjunction of its binary partial relations although this is not always possible. If the rank of the relation is greater than three, we can think of the decomposition into relations of some reduced ranks greater than two as well as into binary relations.

A similar decomposition of relations first appeared in Codd [7] and then was studied by Rissanen and Delobel [33]. The motivation of their studies was data base systems. Codd proposed a relational model for data base systems which he claimed was superior to traditional hierarchical models which are still overwhelmingly dominant in generalized data base systems. At this moment RDMS [19] of General Motors seems to be the only data base system which adopted the relational model outside IBM. In the sequel the decomposition of relations is briefly and informally described.

A file is a set of records. Each record is a list of data, i.e., a list of attribute-value pairs. In other words a file is a table in which each row is a record, i.e., an n-tuple of data. Since the order of attribute-value pairs in records is predetermined for operational reasons, and, hence, is important, a file can be regarded as a relation, more specifically, a linear relation. Then it is apparently desirable to decompose files into simpler files, i.e., to decompose tables into simpler tables, in such a way that all information that could be obtained from the original files can be also retrieved and at the same time any information which did not exist in the original files should not come from the new simpler files.

Suppose that a file F is a table consisting of n-tuples $(a_1, a_2, \ldots, a_n)$, i.e., $F = \{(a_1, a_2, \ldots, a_n)\}$, where a_i is the value of the record corresponding to the attribute A_i. Let $\alpha = (A_{k_1}, A_{k_2}, \ldots, A_{k_i})$ and

$\beta = (A_{m_1}, A_{m_2}, \ldots, A_{m_j})$ be the sequences of two subsets of the attributes $\{A_1, A_2, \ldots, A_n\}$ such that $\{\alpha\} \cap \{\beta\} \neq \phi$ and $\{\alpha\} \cup \{\beta\} = \{A_1, A_2, \ldots, A_n\}$. The α- and β-projections of F, P_α and P_β, are defined as the relations of F restricted to α- and β-domain of $X_{i=1}^{n} A_i$, respectively. The natural join of P_α and P_β, denoted by $P_\alpha * P_\beta$, is specified by $P_\alpha * P_\beta = \{(a_1, a_2, \ldots, a_n) | (a_{k_1}, a_{k_2}, \ldots, a_{k_i}) \in P_\alpha \wedge (a_{m_1}, a_{m_2}, \ldots, a_{m_j}) \in P_\beta\}$ where $(k_1, k_2, \ldots, k_i)$ and $(m_1, m_2, \ldots, m_j)$ are the subsequence of $(1, 2, \ldots, n)$ corresponding to α and β, respectively. Now F is decomposable if $F = P_\alpha * P_\beta$. This decomposition is illustrated in Example 3.3.

First note that again it is not always possible to decompose a relation even though in this case a relation is decomposed into a conjunction of only two reduced relations, which is slightly different from our case in which an n-ary relation is decomposed into $\binom{n}{k}$k-ary relations. This difficulty (that a file is not always decomposable) introduces the so-called normalization of files in relational data base systems. Since, furthermore, the object of Codd's study of the decomposition is data base systems, he is concerned with only those decompositions in which the set of attributes $\alpha \cap \beta$ is the key of the file. One of the theoretical results appearing in the paper by Rissanen and Delobel is Delobel and Heath's corollary that if there exists a functional dependence $P_{\alpha \cap \beta} \longrightarrow P_\alpha$ or P_β, then $P_\alpha * P_\beta = F$. This is significant because if $\alpha \cap \beta$ is a key, then there is always a functional dependence from $P_{\alpha \cap \beta}$ to at least one of P_α and P_β.

Example 3.3 Suppose that files F and F' are defined by the following tables where A, B, C, D and E are the attributes of the files.

$$F = \begin{array}{c} A\ B\ C\ D\ E \\ \left\{\begin{array}{c}(1,1,1,1,1)\\(1,2,2,1,2)\\(1,2,2,2,2)\\(2,1,2,1,1)\end{array}\right\}. \end{array} \qquad F' = \begin{array}{c} A\ B\ C\ D\ E \\ \left\{\begin{array}{c}(1,1,1,1,1)\\(2,2,2,1,2)\\(1,2,2,2,2)\\(2,1,2,1,1)\end{array}\right\}. \end{array}$$

And let $\alpha = (A,B,C)$ and $\beta = (B,C,D,E)$ so that $\alpha \cap \beta = (B,C)$. Then the projections P_α, P_β, P'_α and P'_β are obtained as follows:

$$P_\alpha = \begin{array}{c} A\ B\ C \\ \left\{\begin{array}{c}(1,1,1)\\(1,2,2)\\(2,1,2)\end{array}\right\}, \end{array} \qquad P_\beta = \begin{array}{c} B\ C\ D\ E \\ \left\{\begin{array}{c}(1,1,1,1)\\(2,2,1,2)\\(2,2,2,2)\\(1,2,1,1)\end{array}\right\}, \end{array} \qquad P_{\alpha \cap \beta} = \begin{array}{c} B\ C \\ \left\{\begin{array}{c}(1,1)\\(2,2)\\(1,2)\end{array}\right\}, \end{array}$$

$$P'_\alpha = \begin{array}{c} A\ B\ C \\ \left\{\begin{array}{c}(1,1,1)\\(2,2,2)\\(1,2,2)\\(2,1,2)\end{array}\right\}, \end{array} \qquad P'_\beta = \begin{array}{c} B\ C\ D\ E \\ \left\{\begin{array}{c}(1,1,1,1)\\(2,2,1,2)\\(2,2,2,2)\\(1,2,1,1)\end{array}\right\}, \end{array} \qquad P'_{\alpha \cap \beta} = \begin{array}{c} B\ C \\ \left\{\begin{array}{c}(1,1)\\(2,2)\\(1,2)\end{array}\right\}. \end{array}$$

And the natural joins of P_α and P_β, and P'_α and P'_β are given as follows:

$$\begin{aligned} P_\alpha * P_\beta &= \{\ (a,b,c,d,e)\ |(a,b,c)\ \varepsilon\ P_\alpha \wedge\ (b,c,d,e)\ \varepsilon\ P_\beta\} \\ &= \left\{\begin{array}{c}(1,1,1,1,1)\\(1,2,2,1,2)\\(1,2,2,2,2)\\(2,1,2,1,1)\end{array}\right\}, \end{aligned}$$

$$\begin{aligned} P'_\alpha * P'_\beta &= \{\ (a,b,c,d,e)|\ (a,b,c)\ \varepsilon\ P'_\alpha\ \wedge\ (b,c,d,e)\ \varepsilon\ P'_\beta\} \\ &= \left\{\begin{array}{c}(1,1,1,1,1)\\(2,2,2,1,2)\\(2,2,2,2,2)\\(1,2,2,1,2)\\(1,2,2,2,2)\\(2,1,2,1,1)\end{array}\right\}. \end{aligned}$$

Therefore F is decomposable while F' is not by α and β since $F = P_\alpha * P_\beta$ and $F' \neq P'_\alpha * P'_\beta$.

Note that there is a functional dependence from $P_{\alpha \cap \beta}$ to P_α, i.e., $\forall$ (a,b,c,d,e),(a',b',c',d',d') ε F (a,b,c) $\neq$ (a',b',c') implies (b,c) $\neq$ (b',c'), While there is not from $P'_{\alpha \cap \beta}$ to P'_α nor to P'_β.

The structure of the rest of this chapter is as follows:

First in Section 3.2 we define decompositions mentioned above and describe a hierarchical structure of decompositions and an approach to general cases. Then the decomposition is characterized in various ways in Section 3.3. And finally the decomposability of the two ternary relations described in Chapter 2 is discussed in Section 3.4.

3.2 Decomposition of Relations

We will consider the decomposition of a relation into its partial relations mentioned in the previous section.

Definition: Let $L = (\ell_1, \ell_2, \ldots, \ell_k)$ be a sequence such that $1 \leqslant \ell_1 < \ell_2 < \ldots < \ell_k \leqslant n$, then $(\ell_1, \ell_2, \ldots, \ell_k)$ is called the L-factor (or factor L) of the sequence (1,2,...,n). Each such L-factor is a k-ary factor of the sequence.

Definition: Let $\alpha \subset S^n$, then we refer to i-th S of S^n as the i-th domain (or domain i) of α. More generally, if $L = (\ell_1, \ell_2, \ldots, \ell_k)$, we refer to the sequence of ℓ_1-, ℓ_2-,..., ℓ_k-th S's of S^n as the L-domain (or domain L) of α. Each such L-domain is called a k-ary domain of the relation.

In terms of these, the L-partial relation α_L of a relation $\alpha \subset S^n$ is a k-ary relation of α restricted to the L-domain. And each such L-partial relation is called a k-ary partial relation of α.

Definition: The L-component of a member $A = (a_1, a_2, \ldots, a_n)$ of a relation is defined as $(a_{\ell_1}, a_{\ell_2}, \ldots, a_{\ell_k})$ where $L = (\ell_1, \ell_2, \ldots, \ell_k)$. Each such L-component is called a k-ary component of A.

Definition: Let α be a set of n-tuples, i.e., $\alpha = \{(a_1, a_2, \ldots, a_n)\}$, then an operator, projection is defined as follows. Given a factor L of the sequence $(1,2,\ldots,n)$, the L-projection of α, denoted by $P_L(\alpha)$, is $P_L(\alpha) = \{(a_{\ell_1}, a_{\ell_2}, \ldots, a_{\ell_k}) \mid (a_1, a_2, \ldots, a_n) \in \alpha\}$. Each such L-projection is called a k-ary projection of α.

In terms of projections, the L-partial relation α_L of α is obtained by $P_L(\alpha)$ and the L-component of a member A is obtained by $P_L(\{A\})$, or simply $P_L(A)$.

Next we give a natural way of combination of partial relations.

Definition: The natural join, $\alpha_{L_1} * \alpha_{L_2}$, of two partial relations α_{L_1} and α_{L_2} of a relation $\alpha \subset S^n$ is an n-ary relation specified by $\{A \in S^n \mid P_{L_1}(A) \in \alpha_{L_1} \wedge P_{L_2}(A) \in \alpha_{L_2}\}$.

Definition: The k-ary closure, α_k^*, of a relation $\alpha \subset S^n$ is an n-ary relation specified by $\alpha_k^* = \{A \in S^n \mid P_L(A) \in \alpha_L \ \forall$ k-ary factor $L\}$.

Now we are ready to give the definition of decompositions.

Definition: (k-ary Regular Decomposition) A relation $\alpha \subset S_n^n$ is k-ary (regular) decomposable iff $\alpha = \alpha_k^*$, $(k < n)$.

When $k = n$, it is trivial. If $\alpha = S^n$ or α consists of one member, α is always k-ary decomposable for any k. A simple but non-trivial example is given below.

Example 3.4 Let $\alpha \subset S^4$ and $\alpha = \{(1,4,2,3),(2,3,4,3),(4,2,1,3),(4,3,2,1)\}$ where $S = \{1,2,3,4\}$ as shown in Fig. 3.5, then its binary partial relations are:

$$\alpha_{12} = \left\{\begin{matrix}(1,4)\\(2,3)\\(4,2)\\(4,3)\end{matrix}\right\},$$

$$\alpha_{13} = \left\{\begin{matrix}(1,2)\\(2,4)\\(4,1)\\(4,2)\end{matrix}\right\},$$

$$\alpha_{14} = \left\{\begin{matrix}(1,3)\\(2,3)\\(4,1)\\(4,3)\end{matrix}\right\},$$

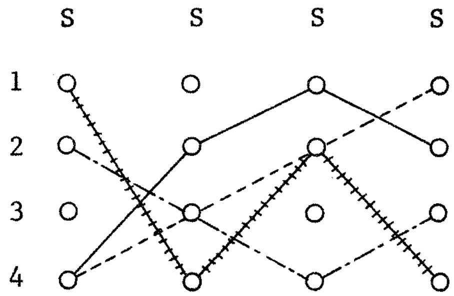


Fig. 3.5

$$\alpha_{23} = \left\{\begin{matrix}(2,1)\\(3,2)\\(4,2)\\(3,4)\end{matrix}\right\}, \quad \alpha_{24} = \left\{\begin{matrix}(2,3)\\(3,1)\\(3,3)\\(4,3)\end{matrix}\right\}, \quad \alpha_{34} = \left\{\begin{matrix}(1,3)\\(2,3)\\(2,1)\\(4,3)\end{matrix}\right\}.$$

Therefore the binary closure of α is

$$\alpha_2^* = \{ (1,4,2,3),(2,3,4,3),(4,2,1,3),(4,3,2,1),(4,3,2,3)\}.$$

And its ternary partial relations are:

$$\alpha_{123} = \left\{\begin{matrix}(1,4,2)\\(2,3,4)\\(4,2,1)\\(4,3,2)\end{matrix}\right\}, \qquad \alpha_{124} = \left\{\begin{matrix}(1,4,3)\\(2,3,3)\\(4,2,3)\\(4,3,1)\end{matrix}\right\},$$

$$\alpha_{134} = \left\{\begin{matrix}(1,2,3)\\(2,4,3)\\(4,1,3)\\(4,2,1)\end{matrix}\right\}, \qquad \alpha_{234} = \left\{\begin{matrix}(2,1,3)\\(3,2,1)\\(3,4,3)\\(4,2,3)\end{matrix}\right\}.$$

Therefore the ternary closure of α is

$$\alpha_3^* = \{ (1,4,2,3),(2,3,4,3),(4,2,1,3),(4,3,2,1)\}.$$

Since $\alpha_2^* = \{(4,3,2,3)\} \cup \alpha$ and $\alpha_3^* = \alpha$, α is ternary decomposable but not binary decomposable. ///

Next we describe an interesting hierarchical structure among closures of a relation which seems natural.

Lemma 3.1 For any relations α, $\alpha^*_{k+1} \subset \alpha^*_k$.

Proof Let $A \varepsilon \alpha^*_{k+1}$, then by the definition $P_L(A) \varepsilon \alpha_L$ $\forall$ (k+1)-ary factor L, in other words, $\exists A' \varepsilon \alpha \ni P_L(A) = P_L(A') \varepsilon \alpha_L$ $\forall$ (k+1)-ary factor L. Now let L' be a k-ary factor, then always $\exists$ (k+1)-ary factor L $\ni$ L' is a k-ary factor of L. Therefore $P_{L'}(A) = P_{L'}(P_L(A)) = P_{L'}(P_L(A')) \varepsilon \alpha_{L'}$. Hence $P_{L'}(A) \varepsilon \alpha_{L'}$, $\forall$ k-ary factor L', i.e., $A \varepsilon \alpha^*_k$.

Q.E.D.

Since $\alpha = \alpha^*_n \subset \alpha^*_{n-1}$ for any n-ary relation α, we have the following hierarchy of closures immediately from Lemma 3.1.

Theorem 3.1 For any n-ary relation $\alpha \subset S^n$,

$$\alpha \subset \alpha^*_{n-1} \subset \alpha^*_{n-2} \subset \ldots \subset \alpha^*_2 \subset \alpha^*_1 \subset S^n.$$

Example 3.5 Let $\alpha \subset S^4$ and $\alpha = \{(2,1,3,1),(2,3,1,2),(4,3,1,1),(4,3,3,2),(4,4,1,2)\}$ where $S = \{1,2,3,4\}$ as shown in Fig. 3.6, then its binary partial relations are:

$$\alpha_{12} = \left\{\begin{matrix}(2,1)\\(2,3)\\(4,3)\\(4,4)\end{matrix}\right\}, \quad \alpha_{13} = \left\{\begin{matrix}(2,1)\\(2,3)\\(4,1)\\(4,3)\end{matrix}\right\}, \quad \alpha_{14} = \left\{\begin{matrix}(2,1)\\(2,2)\\(4,1)\\(4,2)\end{matrix}\right\}, \quad \alpha_{23} = \left\{\begin{matrix}(1,3)\\(3,1)\\(3,3)\\(4,1)\end{matrix}\right\},$$

$$\alpha_{24} = \left\{\begin{matrix}(1,1)\\(3,1)\\(3,2)\\(4,2)\end{matrix}\right\}, \quad \alpha_{34} = \left\{\begin{matrix}(1,1)\\(1,2)\\(3,1)\\(3,2)\end{matrix}\right\}.$$

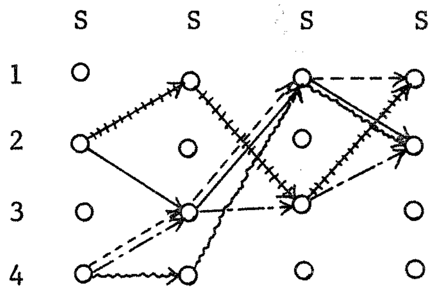


Fig. 3.6

Hence $\alpha^*_2 = \{(2,3,1,1),(2,3,3,2),(4,3,1,2)\} \cup \alpha$.

And its ternary partial relations are:

$$\alpha_{123} = \begin{Bmatrix} (2,1,3) \\ (2,3,1) \\ (4,3,1) \\ (4,3,3) \\ (4,4,1) \end{Bmatrix}, \quad \alpha_{124} = \begin{Bmatrix} (2,1,1) \\ (2,3,2) \\ (4,3,1) \\ (4,3,2) \\ (4,4,2) \end{Bmatrix}, \quad \alpha_{134} = \begin{Bmatrix} (2,1,2) \\ (2,3,1) \\ (4,1,1) \\ (4,1,2) \\ (4,3,2) \end{Bmatrix}, \quad \alpha_{234} = \begin{Bmatrix} (1,3,1) \\ (3,1,1) \\ (3,1,2) \\ (3,3,2) \\ (4,1,2) \end{Bmatrix}.$$

Hence $\alpha_3^* = \{ (4,3,1,2)\} \cup \alpha$.

Therefore we obtain $\alpha \subsetneqq \alpha_3^* \subsetneqq \alpha_2^*$. ///

The following corollary is also an immediate conclusion of Theorem 3.1.

Corollary 3.1 If α is not a k-ary decomposable, it is not j-ary decomposable for any $j < k$.

Since we are concerned with the decomposition of relations into relations of lower ranks, it is appropriate to look at the decomposition into relations of the lowest possible ranks. Let $\alpha \subset S^n$ and suppose that α is (n-1)-ary decomposable. Then we obtain n (n-1)-ary partial relations which constitute the (n-1)-ary closure of α. Next suppose that some of them are (n-2)-ary decomposable. Then for each of them we obtain n-1 (n-2)-ary partial relations which constitute the (n-2)-ary closure of it. In this way we perform the successive regular decompositions to α and its subsequent partial relations until no further decomposition is possible to any of the subsequent partial relations.

Definition: The unique decomposition thus obtained is called the maximal decomposition of α, denoted by α^*. Each partial relation which results from the maximal decomposition and which is not further decomposable is termed an elementary relation of α. Equivalently, $\alpha^* = \{ A \in S^n | P_L(A) \in \alpha_L \; \forall \; L \ni \alpha_L$ is an elementary relation of $\alpha\}$.

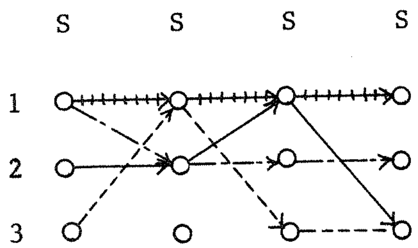


Fig. 3.7

Example 3.6 Let $\alpha \subset S^4$ be defined by

$$\alpha = \begin{Bmatrix} (1,1,1,1) \\ (1,2,2,2) \\ (2,2,1,3) \\ (3,1,3,3) \end{Bmatrix},$$

and its diagram be shown in Fig. 3.7. Then Fig. 3.8 illustrates the successive regular decompositions of α described above. Each terminal node represents an elementary relation of α. ///

Suppose that we obtained the maximal decompositions of a set of relations, instead of one, on S. Then it is quite possible that some of the elementary relations are identical. In other words, it will be necessary to keep only the irredundant set of elementary relations to reproduce the entire set of relations. This idea is equivalent to what underlies the irredundancy of a relational data base system by Codd [7].

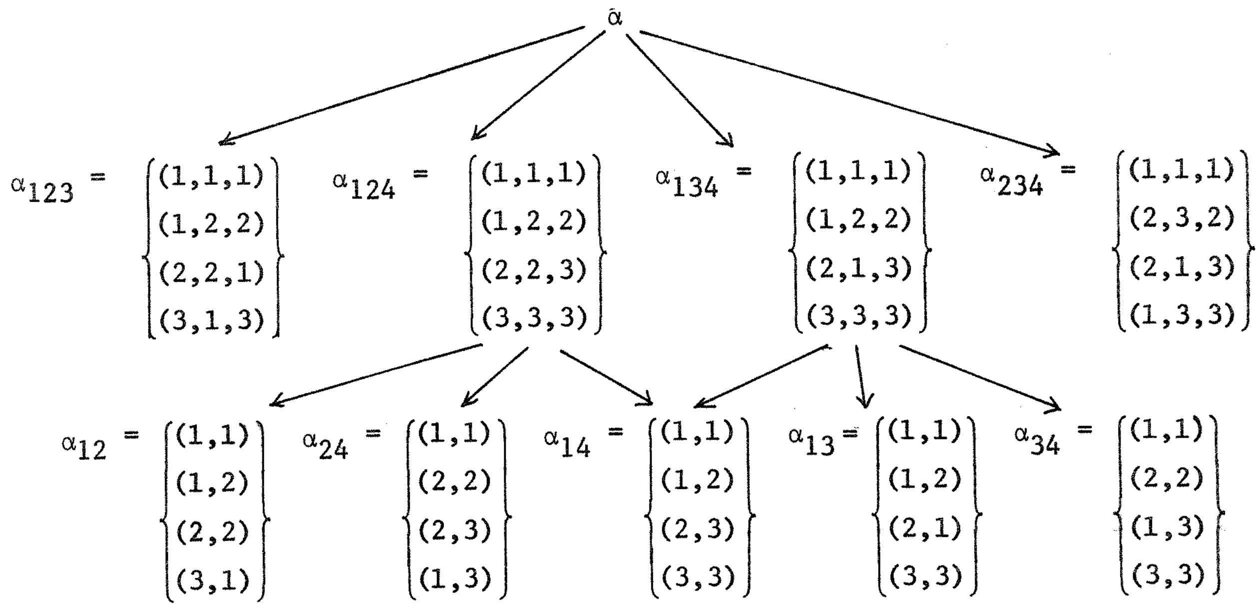


Fig. 3.8

Apparently it is not always possible to decompose a relation $\alpha \subset S^n$ into even (n-1)-ary partial relations. Therefore it may be appropriate to consider non-decomposable cases so that we can always have the representation of a relation by its k-ary relations for arbitrary k. One way to decompose a relation which is not decomposable is to divide the relation into a set of subrelations. That is, let $\alpha = \bigcup_i \alpha^i \ni \alpha^i = (\alpha^i)^*_k$, then $\alpha = \bigcup_i (\alpha^i)^*_k$.

This approach to the decomposition of non-decomposable relations may be justified by the following arguments. We have already mentioned that a binary relations can be a combination of binary relations. For example, $\alpha(a,b)$ may mean that a is bigger than b but not to the left of b or that a is not bigger than b but to the left of b. In other words, $\alpha = (\beta \wedge \neg\gamma) \vee (\neg\beta \wedge \gamma)$ where β is "bigger than" and γ is "to the left of". Now suppose that a set of relation $\{\alpha^i\}_{i \varepsilon I}$ on S are all binary decomposable. Then it may be possible to generate new relations from them by using only ordinary set theoretical operations, i.e., intersection, union, complement and subtraction. For example, $\alpha^1 \cup \alpha^2$ may be interpreted as a relation which is α^1 <u>or</u> α^2 on S. In this case it is very conceivable that most of these new relations are not regular decomposable even though $\{\alpha^i\}_{i \varepsilon I}$ are. Therefore, when we have a non-decomposable relation, it is very natural to replace it by an equivalent set theoretical expression of other relations which are decomposable. Among many possible expressions, a disjunctive form $\alpha = \bigcup_{i \varepsilon J} \alpha^i$ is better simply because at least $\alpha^i \subset \alpha \; \forall \; i \; \varepsilon \; J$. But note that even this disjunctive form is not unique.

Before going further, we need a new concept. Let $A \varepsilon (\alpha^*_k - \alpha)$, then each k-ary component of A has to be identical to the corresponding k-ary component of some member of α. Therefore it may be convenient to define the following term.

Definition: (k-ary covering) Let $A \varepsilon \alpha \subset S^n$, then for an m-ary factor M the M-component of A is said to be k-ary covered by a set of relations $\{A_i\} \subset S^n$ $(k \leq m \leq n)$ if $P_L(A) \varepsilon P_L(\{A_i\})$ $\forall$ k-ary factor L of M. When $k = m$, it may be said instead that it is covered by $\{A_i\}$. Also note that when $m = n$, the m-ary component of A is A itself.

The approach to the decomposition of a general case suggested above may be done as follows. For each $A \varepsilon (\alpha_k^* - \alpha)$, those members which k-ary cover A are divided into different subrelations α^i so that A is not contained in $(\alpha^i)_k^*$ anymore. We choose a disjunctive form $\bigcup_{i \varepsilon I} \alpha^i$ which satisfies the division requirement of covering relations for each $A \varepsilon (\alpha_k^* - \alpha)$. Then $\alpha = \bigcup_{i \varepsilon I} (\alpha^i)_k^*$. The following example illustrates the process.

Example 3.7 In Example 3.5 let us name the members as

$$\begin{aligned} A_1 &= (2,1,3,1) & \text{and} \quad B_1 &= (2,3,1,1) \\ A_2 &= (2,3,1,2) & B_2 &= (2,3,3,2) \\ A_3 &= (4,3,1,1) & B_3 &= (4,3,1,2), \\ A_4 &= (4,3,3,2) & & \\ A_5 &= (4,4,1,2) & & \end{aligned}$$

then $\alpha_2^* = \{B_1, B_2, B_3\} \cup \alpha$ where $\alpha = \{A_1, A_2, A_3, A_4, A_5\}$.

Let us express the binary covering of each B_i by a conjunction of members of α, such that each term of the conjunction is the disjunction of A_j's which cover the corresponding binary component of B_i:

(1) $B_1 = A_2 \wedge A_2 \wedge A_1 \wedge (A_2 \vee A_3) \wedge A_3 \wedge A_3 = A_1 \wedge A_2 \wedge A_3$,

(2) $B_2 = A_2 \wedge A_1 \wedge A_1 \wedge A_2 \wedge A_4 \wedge (A_2 \vee A_4) \wedge A_4 = A_1 \wedge A_2 \wedge A_4$,

(3) $B_3 = (A_3 \vee A_4) \wedge (A_3 \vee A_5) \wedge (A_4 \vee A_5) \wedge (A_2 \vee A_3) \wedge (A_2 \vee A_4) \wedge (A_2 \vee A_5)$.

(1) indicates that A_1, A_2 and A_3 cannot be in the same subrelation. (2) indicates that A_1, A_2 and A_4 cannot be in the same subrelation. And (3) indicates that A_2, A_3, A_4 and A_5 cannot be in the same subrelation.

Therefore, for example, let $\alpha^1 = \{A_1, A_3, A_5\}$ and $\alpha^2 = \{A_2, A_4\}$, then $\alpha^1 = (\alpha^1)^*_2$ and $\alpha^2 = (\alpha^2)^*_2$, which gives us a binary decomposition of α, $\alpha = (\alpha^1)^*_2 \cup (\alpha^2)^*_2$. The diagrams of α and its two subrelations α^1 and α^2 are shown in Fig. 3.9. ///

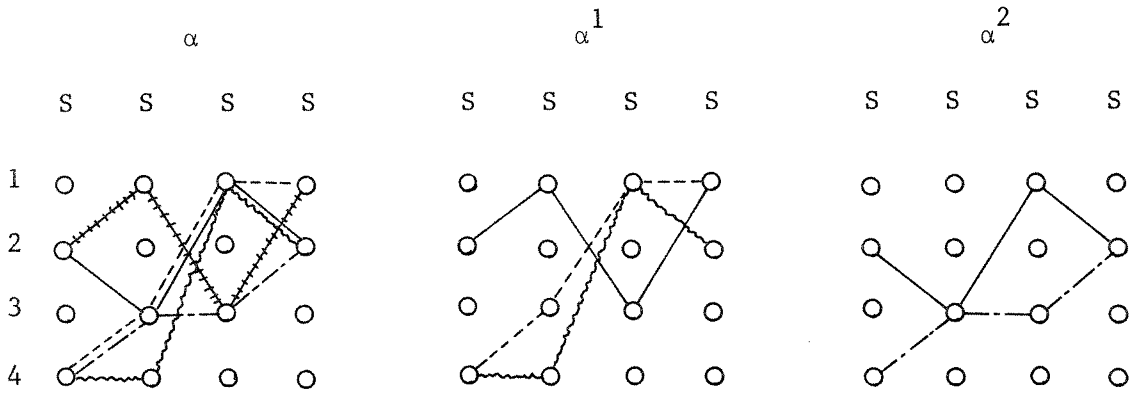


Fig. 3.9

3.3 Characterization of Decompositions

In this section we will be concerned with decomposability, that is, whether a given relation is k-ary regular decomposable for a certain k, when it is decomposable, and the like. Before we go into general cases, it may be beneficial to have the results of the following simple cases which will provide a rough guidance to this section.

Definition: A binary relation α is said to be a function if $a_2 \neq b_2$ implies $a_1 = b_1 \; \forall \; (a_1, a_2), (b_1, b_2) \in \alpha$.

Lemma 3.2 For a ternary relation $\alpha \subset S^3$, if there is an $A \in \alpha^*_2$ such that $A \notin \alpha$, then $|\alpha| > 2$ and all binary partial relations and their inverses are not functions.

Proof Let $A = (a_1,a_2,a_3) \varepsilon \alpha_2^*$ and $A \notin \alpha$, then $P_{ij}(A) \varepsilon P_{ij}(\alpha) \forall$ i,j. This means that there are $A_1,A_2,A_3 \varepsilon \alpha$ where $A_i = (a_1^i,a_2^i,a_3^i)$, i = 1,2,3 such that

$$(a_1^1,a_2^1) = (a_1,a_2),$$
$$(a_1^2,a_3^2) = (a_1,a_3),$$
$$(a_2^3,a_3^3) = (a_2,a_3).$$

Suppose that $A_1 = A_2$, then $A_1 = A_2 = A$. Hence $A \varepsilon \alpha$, which is a contradiction. Similarly, if any two of A_1, A_2 and A_3 are the same, $A \varepsilon \alpha$, which contradicts the assumption that $A \notin \alpha$. Therefore $|\alpha| > 2$.

Since $A_i \varepsilon \alpha \forall$ i, $\{(a_1^1,a_2^1),(a_1^2,a_2^2),(a_1^3,a_2^3)\} \varepsilon \alpha_{12}$ where $a_2^1 = a_2^3 = a_2$ and $a_1^1 = a_1^2 = a_1$. Suppose $a_2^2 = a_2$, then we have $(a_1^2,a_2^2,a_3^2) = (a_1,a_2,a_3)$, hence $A \varepsilon \alpha$ since $(a_1^2,a_2^2) = (a_1,a_2)$. Therefore, since $a_2^1 \neq a_2^2$ while $a_1^1 = a_1^2$, α_{12} is not a function. Similarly α_{12}^{-1} cannot be a function. By symmetry, the two other partial relations and their inverses are not functions.

Q.E.D.

Definition: It is said that there is a functional dependence from α_L to α, denoted by $\alpha_L \Rightarrow \alpha$, if $\forall A,A' \varepsilon \alpha$ $A \neq A'$ implies $P_L(A) = P_L(A')$. If there is not a functional dependence from α_L to α, we will use the notation $\alpha_L \not\Rightarrow \alpha$.

Lemma 3.3 For a 4-ary relation $\alpha \subset S^4$, if there is an $A \varepsilon \alpha_3^*$ such that $A \notin \alpha$, then $|\alpha| > 3$ and for each ternary factor L of (1,2,3,4) $\alpha_{L'} \not\Rightarrow \alpha_L$ for any binary factor L' of L.

Proof Let $A = (a_1,a_2,a_3,a_4) \varepsilon \alpha_3^*$ and $A \notin \alpha$, then $P_{ijk}(A) \varepsilon P_{ijk}(\alpha)$ $\forall$ i,j,k. This means that there are $A_1,A_2,A_3,A_4 \varepsilon \alpha$ where $A_i = (a_1^i,a_2^i,a_3^i,a_4^i)$, i = 1,2,3,4 such that

$$(a_1^1, a_2^1, a_3^1) = (a_1, a_2, a_3),$$
$$(a_1^2, a_2^2, a_4^2) = (a_1, a_2, a_4),$$
$$(a_1^3, a_3^3, a_4^3) = (a_1, a_3, a_4),$$
$$(a_2^4, a_3^4, a_4^4) = (a_2, a_3, a_4).$$

Suppose that $A_1 = A_2$, then $A_1 = A_2 = A$, therefore, $A \varepsilon \alpha$, which is a contradiction. Similarly, if any two of A_i's are the same, $A \varepsilon \alpha$ which contradicts the assumption that $A \notin \alpha$. Therefore $|\alpha| > 3$.

$$\text{Since}\quad (a_1^1, a_2^1) = (a_1^2, a_2^2) = (a_1, a_2),$$
$$(a_1^1, a_3^1) = (a_1^3, a_3^3) = (a_1, a_3),$$
$$(a_2^1, a_3^1) = (a_2^4, a_3^4) = (a_2, a_3),$$

$a_3^1 \neq a_3^2$. Otherwise $A = A_1 = A_2 \varepsilon \alpha$. Therefore $\alpha_{12} \nrightarrow \alpha_{123}$. Similarly it can be proved that α_{13}, $\alpha_{23} \nrightarrow \alpha_{123}$. And by symmetry the rest of the cases can also be proved.

Q.E.D.

In the previous section the concept of covering was introduced and it was pointed out that even if a member A of α_k^* is not a member of α, there exist a set of $\{A_i\} \subset \alpha$ which k-ary cover A. We have the following condition on such a set $\{A_i\}$.

Lemma 3.4 If $A \varepsilon (\alpha_k^* - \alpha)$ and A is k-ary covered by $\{A_i\} \subset \alpha$, then

$$|\{A_i\}| \geq k+1.$$

Proof Since $A \varepsilon (\alpha_k^* - \alpha)$, $A \neq A_i \; \forall \; A_i \varepsilon \alpha$. Obviously $A \neq A_i$ iff $\exists \; j \ni a_j \neq a_j^i$ where $A = (a_1, a_2, \ldots, a_n)$ and $A_i = (a_1^i, a_2^i, \ldots, a_n^i)$. Therefore there must exist at least one $a_{j(i)}$ which does not appear in A_i for each A_i. Let us consider the best case, that is, each A_i is different from A just at one domain j(i). Suppose without loss of generality that A and A_1 differ only at domain 1, i.e., $a_1 \neq a_1^1$ and $a_j = a_j^1 \; \forall \; j \neq 1$. Then the number of k-ary components of A which are not covered by A_i is $\binom{n-1}{k-1}$. Therefore these $\binom{n-1}{k-1}$ k-ary components whose first element is a_1 have to be covered by other A_i's. Let A_2 be a member with $a_1^2 = a_1$. A_2 always exists since at least (a_1,a_2) has to be covered by some A_i. Since $A_2 \neq A$, let $a_2^2 \neq a_2$ and $a_j^2 = a_j \;\forall\; j \neq 2$, again without loss of generality. The number of k-ary components of A which have not been covered by A_2 among $\binom{n-1}{k-2}$ k-ary components above is $\binom{n-2}{k-2}$. Repeating this process, we eventually come to the point where the number of k-ary components which are not covered by A_k either is $\binom{n-k}{0}$. Therefore we need one more member A_{k+1} to completely cover A. Since $A_1,A_2,\ldots,A_{k+1}$ obtained above is a best possible case, $|\{A_i\}| \geq k+1$ in general.

Q.E.D.

The above process will be illustrated by the following example.

Example 3.8 Let $\alpha \subset S^6$ and A = $(1,1,1,1,1,1)$ ε $(\alpha_2^* - \alpha)$, then $\{A_i\} = \{A_1 = (2,1,1,1,1,1),\ A_2 = (1,2,1,1,1,1),\ A_3 = (1,1,2,1,1)\}$ is a best case obtained by the procedure described above, and $\{A_i\}$ is shown in Fig. 3.10. That is each A_i differs from A at only one domain. Since A_1 differs from A at domain 1, A is covered by A_1 except for (1,i)-components, i = 2,3,4,5,6. Among them A_2 covers (1,3),(1,4),(1,5) and (1,6)-components of A. Finally the remaining (1,2)-component is covered by A_3. Note that $|\{A_i\}| = 3$. ///

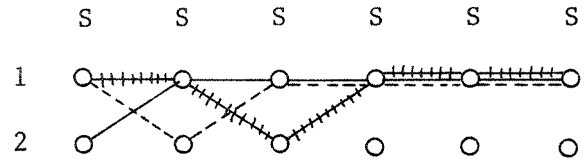


Fig. 3.10

The following is an immediate consequence of Lemma 3.4 on the minimum number of members necessary for a relation to be not k-ary decomposable.

Corollary 3.2 If $\alpha \subsetneq \alpha_k^*$, then $|\alpha| > k$.

Now we go on to the characterization of regular decompositions.

Theorem 3.2 An n-ary relation $\alpha \subset S^n$ is k-ary regular decomposable iff for each $A \in S^n \ni P_L(A) \in \alpha_L$ $\forall$ k-ary factor L of $(1,2,\ldots,n)$, $A \in \alpha$.

Proof

($\Rightarrow$) If $\alpha = \alpha_k^* = \{A \in S^n \mid \bigwedge_{\text{k-ary factor } L} P_L(A) \in \alpha_L\}$, clearly the second half is true.

($\Leftarrow$) If for each $A \in S^n \ni P_L(A) \in \alpha_L$ $\forall$ k-ary factor L, $A \in \alpha$, then $\alpha_k^* \subset \alpha$. On the other hand, $\alpha \subset \alpha_k^*$ by Theorem 3.1. Hence $\alpha = \alpha_k^*$.

Q.E.D.

The reader may notice that the characterization of the decomposition above is almost useless in practice because finding those A's which satisfy the above condition is equivalent to checking whether or not $\alpha = \alpha_k^*$. Therefore a different and more practical method to decide the decomposability of relations is needed. The following theorem is a generalization of Lemmas 3.2 and 3.3.

Theorem 3.3 If $\exists$ $A \in (\alpha_{n-1}^* - \alpha)$ for $\alpha \subset S^n$, then for any (n-1)-ary factor L, $\alpha_L \not\Rightarrow \alpha_L$ $\forall$ (n-2)-ary factor L' of L.

Proof Let A be covered by $\{A_i\}$ including $\{A_1, A_2, \ldots, A_n\}$, since $|\{A_i\}| \geq n$ is guaranteed by Lemma 3.4, such that

$$(a_1, a_2, \ldots, a_{n-1}) = (a_1^n, a_2^n, \ldots, a_{n-1}^n) \text{ and } a_n \neq a_n^n,$$

$$(a_1, a_2, \ldots, a_{n-2}, a_n) = (a_1^{n-1}, a_2^{n-1}, \ldots, a_{n-2}^{n-1}, a_n^{n-1}) \text{ and } a_{n-1} \neq a_{n-1}^{n-1},$$

$$\vdots$$

$$(a_2, a_3, \ldots, a_n) = (a_2^1, a_3^1, \ldots, a_n^1) \text{ and } a_1 \neq a_1^1.$$

Then $(a_1,a_2,\ldots,a_{n-2}) = (a_1^{n-1},a_2^{n-1},\ldots,a_{n-2}^{n-1})$ but $a_{n-1}^{n} \neq a_{n-1}^{n-1}$ because otherwise $A \in \alpha$. Hence $\alpha_{12\ldots(n-2)} \not\Rightarrow \alpha_{12\ldots(n-1)}$. Similarly we can show that $\alpha_{L'} \not\Rightarrow \alpha_{12\ldots(n-1)}$ for any other (n-2)-ary factor L' of (1,2,...,n-1). Furthermore, by symmetry, for every other (n-1)-ary factor L $\alpha_{L'} \not\Rightarrow \alpha_L$ $\forall$ (n-2)-ary L' of L. Q.E.D.

By simply negating the statement of Theorem 3.3 we have the following corrollary. However we give an alternate proof using Delobel and Heath's corollary which was stated in Section 3.1.

Corollary 3.3 If there exists an (n-1)-ary partial relation α_L of α such that $\alpha_{L'} \Rightarrow \alpha_L$ for some (n-2)-ary factor L' of L, then the n-ary relation α is (n-1)-ary regular decomposable.

Proof Suppose that L = (1,2,...,n-1) and L' = (2,3,...,n-1), without loss of generality. And let M = (2,3,...,n), then $L \cap M = L'$. Hence $\alpha_L * \alpha_M = \alpha$ by Delobel and Heath's corollary since $\alpha_{L'} \Rightarrow \alpha_L$ by the hypothesis. Then since $\alpha = \alpha_L * \alpha_M \supset \overset{*}{\alpha}_{n-1} \supset \alpha$ because L and M are (n-1)-ary factor, $\alpha = \overset{*}{\alpha}_{n-1}$.

Q.E.D.

3.4 Examples

Next we will decide the binary regular decomposability of two ternary relations described earlier in Chapter 2, which we restate below for convenience.

A "between" relation β is a ternary relation $\beta \subset S^3$ which satisfies the following properties:

(1) $(a,b,c) \in \beta \Rightarrow a,b,c$ are all distinct (anti-reflexive),

(2) $(a,b,c) \in \beta \Rightarrow (c,b,a) \in \beta$ (weak symmetric),

(3) $(a,b,c),(b,c,d) \in \beta \Rightarrow (a,c,d),(a,b,d) \in \beta$ (compositional transitive),

(4) $(a,b,c),(a,c,d) \in \beta \Rightarrow (b,c,d) \in \beta$ or $(b,d,c) \in \beta$ but not both,

(5) $(a,b,c),(a,b,d) \in \beta \Rightarrow (b,c,d) \in \beta$ or $(b,d,c) \in \beta$ but not both.

A "triangle" relation τ is a ternary relation $\tau \subset S^3$ which satisfies the following properties:

(1) $(a,b,c) \in \tau \Rightarrow a,b,c$ are all distinct (anti-reflexive),

(2) $(a,b,c) \in \tau \Rightarrow$ all its permutations are in τ (strong symmetric),

(3) $(a,b,x),(a,y,c),(z,b,c) \in \tau \Rightarrow (a,b,c) \in \tau$ (chain transitive).

Example 3.9 First, a "between" relation is not always binary decomposable. We give an example. Let $\beta = \{(1,2,6),(6,2,1),(3,2,4),(4,2,3),(4,5,6),(6,5,4)\}$, then apparently β is a "between" relation on $S = \{1,2,3,4,5,6\}$ and the diagram of the corresponding set of points on a plane is shown in Fig. 3.11. Its binary partial relations are:

$$\beta_{12} = \left\{\begin{matrix}(1,2)\\(3,6)\\(4,2)\\(4,5)\\(6,2)\\(6,5)\end{matrix}\right\}, \quad \beta_{13} = \left\{\begin{matrix}(1,2)\\(3,4)\\(4,3)\\(4,6)\\(6,1)\\(6,4)\end{matrix}\right\}, \quad \beta_{23} = \left\{\begin{matrix}(2,1)\\(2,3)\\(2,4)\\(2,6)\\(5,4)\\(5,6)\end{matrix}\right\}.$$

Therefore $\beta_2^* = \{(4,2,6),(6,2,4)\} \cup \beta \neq \beta$.

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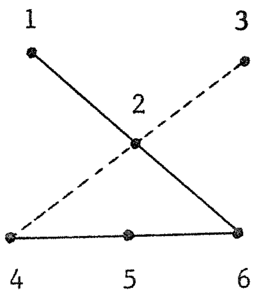


Fig. 3.11

Theorem 3.4 A "triangle" relation τ is binary decomposable.

Proof Suppose that $\exists\ A = (a,b,c) \in (\tau_2^* - \tau)$, then, from Lemma 3.2, there must exist $A_1,A_2,A_3 \in \tau$ such that

$$A_1 = (a,b,x) \text{ and } x \neq c,$$

$$A_2 = (a,y,c) \text{ and } y \neq b,$$

$$A_3 = (z,b,c) \text{ and } z \neq a.$$

But then $(a,b,c) \in \tau$ by the property (3) of "triangle" relations. Hence

$A \varepsilon \tau$ which is a contradiction. Therefore $\tau_2^* \subset \tau$, that is $\tau_2^* = \tau$.

Q.E.D.

CHAPTER IV

SYNTHESIS OF RELATIONS

4.1 Introduction

In the previous chapter we considered the decomposition of relations. And we have seen that some relations are decomposable while some others are not, and finally we have determined when they are. If a relation is decomposable, of course we want to decompose it into lower rank relations so that we can have simpler relations. However a question arises: how the replacement of the relation by its decomposition affects other matters. For example, one may want to make sure that the original relational system $(S, \{\alpha_i\}_{i\varepsilon I})$ is the same as the new relational system $(S, \{\alpha'_j\}_{j\varepsilon J})$ by some necessary criteria. We will not approach directly these problems induced by decompositions. Instead we will study them in a more general form, synthesis. Synthesis is the converse problem of the decomposition described in Chapter 3. That is, synthesis is a way to construct a higher rank relation from a given set of relations.

The structure of this chapter is as follows. First, synthesis is defined in Section 4.2 and various types of syntheses are classified by placing restrictions on the mapping. Next, Section 4.3 is concerned with the relation between decompositions and syntheses. In Section 4.4, after defining a relation space as the set of all relations that can be constructed from a set of relations, we conclude that decompositions do not reduce the relation space of the original set of relations. Another important problem related to syntheses, the compaction of relations, is discussed in Section 4.5. Given a set of relations, what is the lowest ranked relation which represents them? Since each original relation has to be retrieved from the new relation, the synthesis should not lose information. In some cases this problem is important since, from the data structure point of view, up to $\binom{n}{2}$ different binary relations can be compacted into one n-ary relation which may provide a great deal of reduction in the storage space of the information. Finally the properties defined in chapter 2 which are preserved through decompositions and syntheses are summarized in Section 4.6.

4.2 Syntheses of Relations

Although there are many ways to construct a relation from a set of relations, we want to have something that is considered to be the converse of decomposition. For example, we want to construct a higher rank relation from a set of given relations in such a way that the original relations are obtained directly as the partial relations of the constructed relation. In this section, however, we define the most general form of these constructions, called syntheses, and later we will place restrictions on them to obtain more useful results. First we need more terms.

Definition: Let $\alpha = \{(a_1,a_2,\ldots,a_n)\}$ be an n-ary relation and π be a permutation of the n-tuple $(1,2,\ldots,a\ )$ i.e. $(\pi(1),\pi(2),\ldots,\pi(n))$, then we call $\pi(\alpha) = \{(a_{\pi(1)},a_{\pi(2)},\ldots,a_{\pi(n)}) \mid (a_1,a_2,\ldots,a_n) \in \alpha\}$ a permutation of α (by π).

Definition: Let α, β be two n-ary relations on S, then we say that α and β are equivalent, denoted by $\alpha \simeq \beta$, if α is a permutation of β by some π, i.e. $\exists\ \pi \ni \alpha = \pi\ \ (\beta)$. If α and β are equivalent, α is an equivalent of β and vice versa.

Definition: A relation α is derivable from a relation β if α is an equivalent of some partial relation of β.

Now we define a synthesis as follows.

Definition: (m-ary synthesis or simply m-synthesis) Let $\Omega = \{\alpha^1, \alpha^2, \ldots, \alpha^n\}$ be a set of relations of not necessarily the same rank on S and Σ be a set of non-void factors of $(1,2,\ldots,m)$, i.e. $\Sigma = \{(i_1, i_2, \ldots, i_j) | 1 \leqslant i_1 < i_2 < \ldots < i_j \leqslant m\}$, we generate a new set of relation $\hat{\Omega} = \{\beta | \beta = \pi(\alpha)$ for some π and $\alpha \in \Omega\} \cup \{S, S^2, \ldots, S^m\}$. Now let ψ be a mapping $\psi : \Sigma \longrightarrow \hat{\Omega}$ such that if $L \in \Sigma$ is a k-tuple, then $\psi(L)$ is a relation of rank k in $\hat{\Omega}$. Then the (type 0) m-synthesis of Ω by ψ, denoted by $\psi(\Omega)$, is an m-ary relation on S defined by $\psi(\Omega) = \{A \in S^m \mid P_L(A) \in \psi(L) \ \forall \ L \in \Sigma\} = \{(a_1, a_2, \ldots, a_m) \mid \forall (i_1, i_2, \ldots, i_j) \in \Sigma \left[(a_{i_1}, a_{i_2}, \ldots, a_{i_j}) \in \psi((i_1, i_2, \ldots, i_j))\right]\}$.

Definition: In a synthesis $\psi(\Omega)$, each $L \in \Sigma$ is called essential if there exist $\alpha^i \in \Omega$ such that $\psi(L)$ is a permutation of α^i, and each $\alpha^i \in \Omega$ is also essential if there exists $K \in \Sigma$ such that $\psi(K)$ is a permutation of α^j. Otherwise they are non-essential.

Note the generality of the above definition. For example $\psi(\Omega)$ may be ϕ or S^m itself, even if Ω is not empty. One of the reasons for this generality comes from the fact that all α^i's in Ω may not be essential in the synthesis $\psi(\Omega)$. Various syntheses related to decomposition and compaction will be studied in the other sections. In this section we confine our attention to the mapping itself. First we look at an exmaple of a synthesis.

Example 4.1 Let $\Omega = \{\alpha, \beta\}$ where $\alpha = \{(1)\}$ and $\beta = \{(1,1),(1,2)\}$ on $S = \{1,2\}$, then $\hat{\Omega} = \{\alpha, \beta, \beta^{-1}\} \cup \{S, S^2, S^3\}$ and $\Sigma = \{(1),(2),(3),(1,2),(1,3),(2,3),(1,2,3)\}$ for $m = 3$. And let the mapping ψ be defined by

$$\begin{aligned} \psi: \quad (1,2,3) &\longrightarrow S^3 \\ (1,2) &\longrightarrow \beta \\ (1,3) &\longrightarrow S^2 \\ (2,3) &\longrightarrow \beta^{-1} \\ (1) &\longrightarrow \alpha \\ (2) &\longrightarrow S \\ (3) &\longrightarrow S, \end{aligned}$$

then the 3-synthesis of Ω by ψ is easily worked out, and

$$\psi(\Omega) = \{ (a,b,c) \mid (a,b,c) \varepsilon S^3 \wedge (a,b) \varepsilon \beta \wedge (a,c) \varepsilon S^2 \wedge (b,c) \varepsilon \beta^{-1} \wedge (a) \varepsilon \alpha$$

$$(b) \varepsilon S \wedge (c) \varepsilon S\} \qquad \text{(i)}$$

$$= \{ (a,b,c) \mid (a,b) \varepsilon \beta \wedge (b,c) \varepsilon \beta^{-1} \wedge (a) \varepsilon \alpha\} \qquad \text{(ii)}$$

$$= \{ (1,1,1), (1,2,1)\}. \qquad ///$$

In the above definition of an m-synthesis a set of dummy relations $\{S, S^2, \ldots, S^m\}$ is added to the co-domain of ψ so that $\psi(x)$ is defined for all $x \varepsilon \Sigma$ without affecting what the essential factors do in the synthesis. This point can be seen clearly in the above example. That is, the specification (i) of $\Psi(\Omega)$ is exactly the same as (ii) since $\psi((2))$, $\psi((3))$, $\psi((1,3))$ and $\psi((1,2,3))$ do not contribute to specify $\psi(\Omega)$ at all. Let us change $\psi((2,3))$ from β^{-1} to S^2 and leave the others unchanged in the new mapping ψ', then the new synthesis $\psi'(\Omega)$ is $\{(1,1,1),(1,1,2),(1,2,1),(1,2,2)\}$ which is equivalent to $\{(a,b,c) \mid (a,b) \varepsilon \beta \wedge c \ \varepsilon S\}$. This case may be of interest because $\psi(\Omega)$ could be constructed by a 2-synthesis instead of a 3-synthesis and by concatenating S with it. Similarly, if the essential factors do not overlap, the synthesis could be obtained from the natural join of syntheses of lower ranks. Suppose, in the above example, that the essential factors are only (1) and (2,3) such that $\psi((1)) = \alpha$ and $\psi((2,3)) = \beta$, then $\psi(\Omega)$ could be obtained by the natural join of a 1-synthesis and a 2-synthesis.

Definition: An m-synthesis $\psi(\Omega)$ ($m \geq 2$) is type 1 if every binary factor of the form (i,i+1), i = 1,2,...,m-1, is covered by essential factors.

Definition: An m-synthesis $\psi(\Omega)$ ($m \geq 2$) is type 2 if every binary factor is covered by some essential factor.

Definition: A k-ary factor is called quasi-essential in an m-synthesis if each (k-1)-ary factor of it is either essential or quasi-essential.

Definition: An m-synthesis $\psi(\Omega)$ ($m \geq 2$) is type 3 if each (m-1)-ary factor is quasi-essential.

The following example illustrates the differences among the syntheses type 0, type 1, type 2, and type 3 defined above.

Example 4.2 Let $\hat{\Omega} = \{\alpha, \beta\}$ where α and β are ternary and binary relations, respectively, on $S = \{a,b\}$ such that

$$\alpha = \begin{Bmatrix} (a,a,b) \\ (a,b,a) \\ (b,a,a) \\ (b,a,b) \end{Bmatrix} \quad \text{and} \quad \beta = \begin{Bmatrix} (a,a) \\ (a,b) \\ (b,a) \end{Bmatrix},$$

then $\Omega = \{$all permutations of α and $\beta\} \cup \{S, S^2, S^3, S^4\}$ and $\Sigma = \{(1),(2),\ldots,(1,2,3,4)\}$ for $m = 4$.

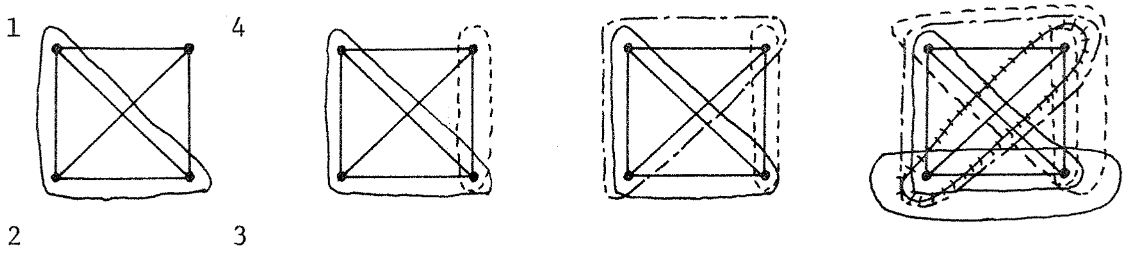
Let us define mappings ψ_0, ψ_1, ψ_2, and ψ_3 as follows (in each mapping the factors except for those indicated below are all non-essential and omitted for clarity):

ψ_0: $(1,2,3) \longrightarrow \alpha$

ψ_1: $(1,2,3) \longrightarrow \alpha$
$(3,4) \longrightarrow \beta$

$$\psi_2: \quad \begin{array}{r} (1,2,3) \longrightarrow \alpha \\ (1,2,4) \longrightarrow \alpha \\ (3,4) \longrightarrow \beta \end{array} \qquad\qquad \psi_3: \quad \begin{array}{r} (1,2,3) \longrightarrow \alpha \\ (1,2,4) \longrightarrow \alpha \\ (1,3,4) \longrightarrow \alpha \\ (2,3) \longrightarrow \beta \\ (2,4) \longrightarrow \beta \\ (3,4) \longrightarrow \beta \end{array}$$

The diagrams of the coverings of binary factors by essential factors are shown in Fig. 4.1.



ψ_0 ψ_1 ψ_2 ψ_3

Fig. 4.1

Obviously the syntheses $\psi_0(\Omega)$, $\psi_1(\Omega)$, $\psi_2(\Omega)$, and $\psi_3(\Omega)$ are type 0, 1, 2, and 3, respectively. It is easy to verify that the four syntheses yield the following 4-ary relations:

$$\psi_0(\Omega) = \left\{ \begin{array}{c} (a,a,b,a) \\ (a,a,b,b) \\ (a,b,a,a) \\ (a,b,a,b) \\ (b,a,a,a) \\ (b,a,a,b) \\ (b,a,b,a) \\ (b,a,b,b) \end{array} \right\} \qquad \psi_1(\Omega) = \left\{ \begin{array}{c} (a,a,b,a) \\ (a,b,a,a) \\ (a,b,a,b) \\ (b,a,a,b) \\ (b,a,b,a) \end{array} \right\}$$

$$\psi_2(\Omega) = \begin{Bmatrix}(a,b,a,a)\\(b,a,a,b)\\(b,a,b,a)\end{Bmatrix} \qquad \psi_3(\Omega) = \{(b,a,a,b)\}.$$

///

The following special kind of syntheses may be also of interest.

Definition: An m-synthesis $\psi(\Omega)$ is called k-regular if every k-ary factor is essential and all other factors are non-essential.

Note that a k-regular m-synthesis ($k \geq 2$) is always of type 3 and, hence, is a special case of type 3.

4.3 Loss-free Syntheses

In this section we consider the relation between decompositions and syntheses. Since a decomposition is the representation of a relation by a set of lower rank relations, there always exists a synthesis which yields the original relation from the set of relations. Is the converse also true? In other words, suppose that $\alpha = \psi(\Omega)$, then does the decomposition of α give exactly the same set of relations as Ω? Obviously not. First of all, if some relation of Ω is not essential in ψ, then it cannot be expected that the relation is derivable from α.

Definition: A synthesis $\psi(\Omega)$ is said to be full if each relation of Ω is essential.

Therefore a full synthesis $\psi(\Omega)$ is one in which every relation of the given set of relations participates in constructing the relation $\psi(\Omega)$.

Since, in an arbitrary set of relations, some relation may be an equivalent of another and some may be derivable from another, the following definition of a normal form will be useful. Let E be the set of all elementary relations in a given set of relations, Ω, i.e., $E = \{\alpha^* \mid \alpha \in \Omega\}$, then we can partition E by the equivalence relation induced by the relation "equivalent ($\simeq$)", i.e., $\alpha \simeq \beta$ if $\exists$ a permutation $\pi \ni \alpha = \pi(\beta)$. Let us denote the set of the equivalence classes by [E] and each class of [E] by $[\beta]$ where β is a representative of the class $[\beta]$.

Definition: The skeleton of a set of relations Ω, denoted by Ω^*, is the collection of class representatives, i.e., $\Omega^* = \{\beta \mid [\beta] \in [E]\}$. If $\Omega = \Omega^*$, we say that Ω is in a skeleton form.

Now let us assume further that $\alpha = \psi(\Omega)$ is full and in a skeleton form. In general, it is still the case that not all relations in Ω can be obtained by decomposing α. Therefore we need to distinguish the following special syntheses from others.

Definition: A synthesis $\psi(\Omega)$ is loss-free (lossless) if each essential relation is equivalent to the projection of $\psi(\Omega)$ corresponding to its essential factor, i.e., $P_L(\psi(\Omega)) = \psi(L)\ \forall$ essential factor L.

Example 4.3 Let $\Omega = \{\alpha,\beta,\gamma\}$ where α, β, and γ are specified as:

$$\alpha = \begin{Bmatrix}(1,1,2)\\(2,1,2)\\(2,2,1)\end{Bmatrix}, \qquad \beta = \begin{Bmatrix}(2,1)\\(2,2)\end{Bmatrix}, \qquad \gamma = \{(2,2)\}.$$

Let a 4-synthesis be defined by ψ as follows:

$$\begin{aligned}\psi:\quad (1,2,3) &\longrightarrow \alpha\\ (3,4) &\longrightarrow \beta\\ (1,4) &\longrightarrow \gamma\end{aligned}$$

and let all other factors be non-essential.

Then $\psi(\Omega) = \{(2,1,2,2)\}$. Therefore $\psi(\Omega)$ is not lossless since α, β are not derivable from $\psi(\Omega)$. Only $\gamma = \psi(\Omega)_{1,4}$.

Next let a 5-synthesis be defined by ψ' as follows:

$$\begin{aligned}\psi':\quad (1,2) &\longrightarrow \gamma\\ (2,3) &\longrightarrow \beta\\ (3,4,5) &\longrightarrow \alpha\end{aligned}$$

and let all other factors be non-essential,