RESPONSE TIME DISTRIBUTIONS FOR A MULTI-CLASS QUEUE WITH FEEDBACK*

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Abstract

A single server queue with feedback and multiple customer classes is analyzed. Arrival processes are independent Poisson processes. Each round of service is exponentially distributed. After receiving a round of service, a customer may depart or rejoin the end of the queue for more service with a probability that is dependent upon his class membership and number of rounds of service achieved. By properly defining customer classes, a wide range of non-exponential service time requirements are admissible in this model. Our main contribution is characterization of response time distributions for the customer classes. Our results generalize in some respects previous analyses of processor-sharing models. They also represent initial efforts to understand response time behavior along paths with loops in local balanced queueing networks.

1. Introduction

Many service facilities can be modeled as a feedback queue such as shown in Figure 1. Of interest in this paper is a single-server queue with infinite waiting room and R classes of customers. The arrival process of the r^{th} class is an independent Poisson process ($r = 1, 2, \ldots R$). Each new arrival joins the end of the queue. The customer at the head of the queue receives from the server a round of service which is an independent exponentially distributed random variable with mean $1/\mu$ seconds. After receiving a round of service, a customer may depart or rejoin the end of the queue for more service, depending upon his class membership and number of rounds of service achieved.

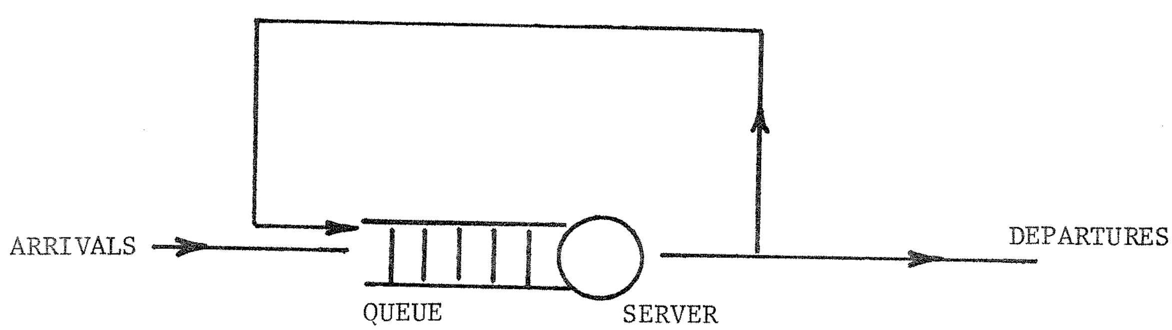

Figure 1. A feedback queue model.

The queue length distribution of the above model is readily available since the multi-class feedback queue described is an open queueing network satisfying local balance [1]. The contribution of this paper is to characterize response time distributions of the different classes of customers.

Relationship to prior work

Our feedback queue model is like a time-sharing model with exponentially distributed service "quantums." Time-sharing models were first studied by Kleinrock [2] who solved for the mean response time of a customer conditioning on his service requirement. He considered two cases: (a) constant quantum size Δ, and (b) the limiting case of $\Delta \to 0$ called processor-sharing. Customers were assumed to arrive according to a Poisson process. In case (a), the number of service quantums required by a customer is geometrically distributed. In case (b), the service requirements are characterized by an exponential distribution. (This is called the processor-sharing M/M/1 queue.) Kleinrock's conditional mean response time result was later shown to hold for a processor-sharing M/G/1 queue (i.e. service requirements characterized by a general distribution) as well by Sakata, Noguchi and Oizumi [3]. Higher order response time statistics are much harder to get. The response time distribution for the processor-sharing M/M/1 queue was obtained by Coffman, Muntz and Trotter [4].

Our feedback queue model is different from the time-sharing models in some respects. A round of service in our model, corresponding to a service quantum in time-sharing models, is exponentially distributed. Our model can be used, however, to approximate processor-sharing by making $1/\mu$ very small.

Distributions of service requirements that are admissible in our model are those with moment generating functions of the form

$$B_r^*(s) = \sum_{i=1}^{R} a_i^{(r)} \left(\frac{\mu}{s+\mu}\right)^i \qquad (1)$$

where $a_i^{(r)}$ can be interpreted as the probability of a class r customer requiring exactly i rounds of service. $\left\{a_i^{(r)}, 1 \le i \le R\right\}$ can be an arbitrary set of probabilities that sum to one and may be different for different customer classes.

Our model is also different from the feedback queue model of Takács [5]. In his model, each round of service can have a general distribution. However, he considered a single class of customers only and the number of rounds of service required by a customer is geometrically distributed; in other words, after each round of service, a customer always departs with probability (1-p) and rejoins the end of the queue with probability p (memoryless behavior).

The original motivation of this work stems from our efforts to characterize the response time in a network of queues. For a network of FCFS queues that satisfies local balance, J. Wong [6] found the response time distribution of customers traversing loop-free paths. Our results in this paper represent efforts to understand the response time behavior along paths with loops in the simplest form of queueing networks satisfying local balance.

Assumptions and definitions

We shall, without any loss of generality, consider the following model. There are R classes of customers. The arrival process of the rth class is Poisson at rate γ_r customers per second. A class r customer requires exactly r rounds of service. It should be obvious that if we can derive response time distributions for this model, response time distributions for any model with service time requirements characterized by Eq. (1) can be easily obtained.

Let t_r be the response time of attaining exactly r rounds of service; r = 1, 2, ... R and obviously $t_0 = 0$. We shall solve for its moment generating function

$$T_r^*(s) = E[e^{-st_r}]$$

where E[·] denotes the expectation of the function of random variable(s) inside the brackets.

We shall only consider steady-state results. For a single-server queue, stationarity is assured if the traffic intensity $\rho < 1$ where $\rho = \sum_{r=1}^{R} \gamma_r (r/\mu)$; see Cohen [7].

Let us follow the progress of a "tagged" customer and introduce some more notation. Upon his initial arrival, the tagged customer finds n_k customers in the queue (k = 1, 2, ..., R); each such customer has exactly k more rounds of service to go. The system state thus found at an arrival instant is denoted by $\underline{n} = (n_1, n_2, \ldots, n_R)$ and is described by the moment generating function

$$P^*(\underline{z}) = E[z_1^{n_1} z_2^{n_2} \ldots . z_R^{n_R}]$$

where $\underline{z}$ is the shorthand notation for $(z_1, z_2, \ldots, z_R)$.

At the end of the tagged customer's r^{th} round of service (given that he requires at least r rounds), let the system state at that instant be denoted by $\underline{m}^{(r)} = (m_1^{(r)}, m_2^{(r)}, \ldots, m_R^{(r)})$ where $m_k^{(r)}$ is the number of customers who have exactly k more rounds of service to go. Define $M^{(r)} = \sum_{k=1}^{R} m_k^{(r)}$.

In order to characterize $T_r^*(s)$, we shall need to first characterize the joint distribution of t_r and $\underline{m}^{(r)}$, which is described by

$$U_r^*(s, \underline{z}) = E[e^{-st_r} z_1^{m_1^{(r)}} z_2^{m_2^{(r)}} \ldots . z_R^{m_R^{(r)}}]$$

Summary of results

We derived a recursive equation relating $U_{r+1}^*(s,\underline{z})$ to $U_r^*(s, \underline{z})$ [Lemma 2]. An explicit solution of $U_r^*(s, \underline{z})$ is found, from which $T_r^*(s)$ is obtained [Theorem 1]. We then proved that the stationary distribution of $\underline{m}^{(r)}$, r=1, 2, ... R, is the same as that of $\underline{n}$ [Theorem 2]. With this result, we solve for the mean value of t_r [Theorem 3]; this last result is similar to the mean response time result of processor-sharing models. Finally, we provide an efficient recursive algorithm to calculate the second order statistics of t_r [Theorem 4].

2. The Analysis

Consider the system state $\underline{n} = (n_1, n_2, \ldots, n_R)$ at arrival instants. Recall that n_k is the number of customers with exactly k more rounds of service to go. Let us redefine the meaning of customer classes to correspond to $n_1, n_2, \ldots, n_R$. Hence the aggregate arrival rate of customers to the k^{th} class is

$$\lambda_k = \sum_{i=k}^{R} \gamma_i \qquad (2)$$

since any new arrival who requires at least k rounds of service must enter and leave the k^{th} class exactly once.

Lemma 1. The moment generating function of $\underline{n}$ is

$$P^*(\underline{z}) = \frac{1 - \rho}{1 - \sum_{k=1}^{R} \rho_k z_k} \qquad (3)$$

where $\rho_k = \lambda_k/\mu$ and $\rho = \sum_{k=1}^{R} \rho_k$.

Proof. Given Poisson arrival processes and that each round of service is exponentially distributed with the same mean $(1/\mu)$, we have an open queueing network that satisfies local balance [1]. Eq. (3) has been obtained by Reiser and Kobayashi [8]. (Q. E. D.)

Since each round of service is exponentially distributed, it has the moment generating function

$$B^*(s) = \frac{\mu}{s + \mu} \qquad (4)$$

A recursive solution of $U_r^*(s, \underline{z})$ is next given.

Lemma 2

$$U_0^*(s, \underline{z}) = P^*(\underline{z}) \qquad (5)$$

$$U_{r+1}^*(s, \underline{z}) = y_1(s, \underline{z})\, U_r^*(s, \underline{y}(s, \underline{z})) \qquad r \geq 0 \qquad (6)$$

where

$$\underline{y}(s, \underline{z}) = (y_1(s, \underline{z}), y_2(s, \underline{z}), \ldots, y_R(s, \underline{z})),$$

$$y_1(s, \underline{z}) = B^*(s + \sum_{i=1}^{R} \gamma_i(1 - z_i)),$$

and

$$y_k(s, \underline{z}) = z_{k-1}\, y_1(s, \underline{z}) \qquad \text{for } 2 \leq k \leq R$$

Proof. For $r = 0$, $t_0 = 0$ and $\underline{m}^{(0)} = \underline{n}$. This and the definition of $U_r^*(s, \underline{z})$ yield (5) at once.

To show (6), consider the time period between t_r and t_{r+1} during which the server served $M^{(r)} + 1$ customers, where $M^{(r)} = \sum_{k=1}^{R} m_k^{(r)}$ and the extra one is for the tagged customer's $(r + 1)^{st}$ round. During the same time period, each class k customer became a class (k - 1) customer where k = 2, 3, ... R. Furthermore, let $A_k(t)$ be the number of external new arrivals to class k during time $t (= t_{r+1} - t_r)$ according to a Poisson process of rate γ_k customers per second. We note that class R is an exception in that its $m_R^{(r+1)}$ customers are all new arrivals. Thus, conditioning on t_r and $\underline{m}^{(r)}$, we have

$$U_{r+1}^*(s, \underline{z}/t_r, \underline{m}^{(r)}) = E[e^{-s(t+t_r)} z_1^{m_2^{(r)}+A_1(t)} z_2^{m_3^{(r)}+A_2(t)} \cdots z_R^{A_R(t)} / t_r, \underline{m}^{(r)}]$$

$$= e^{-st_r} (\prod_{k=2}^{R} z_{k-1}^{m_k^{(r)}}) E[e^{-st} z_1^{A_1(t)} z_2^{A_2(t)} \cdots z_R^{A_R(t)} / M^{(r)}].$$

The last quantity on the right hand side is $(y_1(s, \underline{z}))^{M^{(r)}+1}$ because t is the sum of $M^{(r)}+1$ independent identically distributed random variables with the moment generating function $B^*(s)$. The above equation can be rewritten as

$$U^*_{r+1}(s,\ \underline{z}/t_r,\ \underline{m}^{(r)}) = y_1(s,\ \underline{z})\ \{e^{-st_r}\ y_1(s,\ \underline{z})^{m_1^{(r)}} \prod_{k=2}^{R} (z_{k-1}\ y_1(s,\ \underline{z}))^{m_k^{(r)}}\ \}$$

Unconditioning on t_r and $\underline{m}^{(r)}$, (6) follows. (Q.E.D.)

Explicit solutions for $U_r^*(s,\ \underline{z})$ and $T_r^*(s)$ can now be shown.

Theorem 1. (i) $$U_r^*(s,\ \underline{z}) = \frac{1 - \rho}{P_r(s) - \sum_{k=1}^{R} Q_{k,r}(s) z_k} \qquad r \geq 0 \tag{7}$$

where $P_r(s)$ and $Q_{k,r}(s)$ are polynomials in s, and given by

$$\begin{bmatrix} P_r(s) \\ Q_{1,r}(s) \\ Q_{2,r}(s) \\ \cdot \\ \cdot \\ \cdot \\ Q_{R-1,r}(s) \\ Q_{R,r}(s) \end{bmatrix} = \begin{bmatrix} (1 + \frac{s}{\mu} + \rho_1) & -1 & 0 & 0 & \cdots & 0 \\ \gamma_1/\mu & 0 & 1 & 0 & & 0 \\ \gamma_2/\mu & 0 & 0 & 1 & & 0 \\ \cdot & \cdot & & \ddots & & \cdot \\ \cdot & \cdot & & & \ddots & \cdot \\ \cdot & \cdot & & & 1 & 0 \\ \gamma_{R-1}/\mu & 0 & 0 & \cdots & 0 & 1 \\ \gamma_R/\mu & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}^r \begin{bmatrix} 1 \\ \rho_1 \\ \rho_2 \\ \cdot \\ \cdot \\ \cdot \\ \rho_{R-1} \\ \rho_R \end{bmatrix} \tag{8}$$

(ii) $$T_r^*(s) = \frac{1 - \rho}{P_r(s) - \sum_{k=1}^{R} Q_{k,r}(s)} \tag{9}$$

Proof. (i) Because of (3) and (5), (7) holds for r = 0 with $P_0(s) = 1$ and $Q_{k,0}(s) = \rho_k$ for $1 \leq k \leq R$. Assuming that (7) holds for r, we use (6) and (4) to express $U^*_{r+1}(s,\ \underline{z})$ as follows.

$$U^*_{r+1}(s, \underline{z}) = \frac{1}{1 + \frac{s}{\mu} + \sum_{i=1}^{R} \frac{\gamma_i}{\mu}(1-z_i)} \cdot \frac{1 - \rho}{P_r(s) - \dfrac{Q_{1,r}(s) - \sum_{k=1}^{R-1} Q_{k+1,r}(s)\, z_k}{1 + \frac{s}{\mu} + \sum_{i=1}^{R} \frac{\gamma_i}{\mu}(1 - z_i)}}$$

$$= \frac{1 - \rho}{\{(1 + \frac{s}{\mu} + \sum_{i=1}^{R} \frac{\gamma_i}{\mu})P_r(s) - Q_{1,r}(s)\} - \sum_{k=1}^{R-1} [\frac{\gamma_k}{\mu}P_r(s) + Q_{k+1,r}(s)]z_k - \frac{\gamma_R}{\mu} z_R P_r(s)}$$

Thus, the form of (7) is maintained, and it is evident from the above that

$$\begin{bmatrix} P_{r+1}(s) \\ Q_{1,r+1}(s) \\ \cdot \\ \cdot \\ \cdot \\ Q_{R,r+1}(s) \end{bmatrix} = \begin{bmatrix} (1 + \frac{s}{\mu} + \rho_1) & -1 & 0 & \cdots & 0 \\ \gamma_1/\mu & 0 & 1 & & \vdots \\ \cdot & & & \ddots & \\ \cdot & & & \cdot & 0 \\ \cdot & & & 0 & 1 \\ \gamma_R/\mu & \cdots & & 0 & 0 \end{bmatrix} \begin{bmatrix} P_r(s) \\ Q_{1,r}(s) \\ \cdot \\ \cdot \\ \cdot \\ Q_{R,r}(s) \end{bmatrix} \qquad (10)$$

The recursion in (10) started at $r = 0$ clearly yields (8).

(ii) (9) follows from (7) and $T^*_r(s) = U^*_r(s, \underline{1})$. (Q.E.D.)

For $r = 1$, 2 and 3, we show $U^*_r(s, \underline{z})$ below.

$$U_1^*(s, \underline{z}) = \frac{1 - \rho}{1 + \frac{s}{\mu} - \sum_{k=1}^{R} \rho_k z_k}$$

$$U_2^*(s, \underline{z}) = \frac{1 - \rho}{(1 + \frac{s}{\mu})^2 + \frac{s}{\mu}\rho_1 - \sum_{k=1}^{R} (\rho_k + \frac{s}{\mu^2}\gamma_k) z_k}$$

$$U_3^*(s,\underline{z}) = (1-\rho)/\left\{(1+\tfrac{s}{\mu})^3 + 2(\tfrac{s}{\mu})^2 \rho_1 + \tfrac{s}{\mu}(\rho_2+2\rho_1+\rho_1^2) - \sum_{i=1}^{R-1}\left\{\tfrac{\gamma_i}{\mu}[(1+\tfrac{s}{\mu})^2 + \tfrac{s}{\mu}\rho_1]\right.\right.$$

$$\left.\left. + [\rho_{i+1} + \frac{\gamma_{i+1}}{\mu}\frac{s}{\mu}]\right\} z_i - \frac{\gamma_R}{\mu}[(1+\tfrac{s}{\mu})^2 + \tfrac{s}{\mu}\rho_1]\, z_R\right\}$$

From the above, we obtain $T_r^*(s)$ for r = 1, 2 and 3 by letting $\underline{z} = \underline{1}$ in $U_r^*(s, \underline{z})$.

$$T_1^*(s) = \frac{1-\rho}{(1+\frac{s}{\mu}) - \rho}$$

$$T_2^*(s) = \frac{1-\rho}{(1+\frac{s}{\mu})^2 - \rho}$$

$$T_3^*(s) = \frac{1-\rho}{(1+\frac{s}{\mu})^3 + \rho_1(\frac{s}{\mu})^2 - \rho}$$

We note that the solutions for $U_r^*(s, \underline{z})$ and $T_r^*(s)$ become quite complex if one tries to solve for $P_r(s)$ and $Q_{k,r}(s)$ explicitly using the matrix equation (8) when $r \geq 4$. In what follows, we turn our attention to finding the moments of t_r. To do so, we need the following result concerning the distribution of $\underline{m}^{(r)}$.

Theorem 2. For any $r \geq 0$, $\underline{m}^{(r)}$ and $\underline{n}$ have the same stationary distribution. That is

$$U_r^*(0, \underline{z}) = E[z_1^{m_1^{(r)}} z_2^{m_2^{(r)}} \cdots z_R^{m_R^{(r)}}] = P^*(\underline{z}) \qquad (11)$$

Proof. By (5), (11) holds true for r = 0. Assume that (11) holds true for some r so that $U_r^*(0, \underline{z}) = P^*(\underline{z})$. By (6) and the induction hypothesis,

$$U_{r+1}^*(0, \underline{z}) = y_1(0, \underline{z}) . \frac{1-\rho}{1 - \sum_{k=1}^{R} \rho_k\, y_k(0, \underline{z})}$$

$$= \frac{1-\rho}{\frac{1}{y_1(0, \underline{z})} - (\rho_1 + \sum_{k=1}^{R-1} \rho_{k+1} z_k)}$$

$$= \frac{1 - \rho}{1 + \sum_{i=1}^{R} \frac{\gamma_i}{\mu} (1-z_i) - \rho_1 - \sum_{k=1}^{R-1} \rho_{k+1} z_k}$$

$$= \frac{1 - \rho}{1 - \sum_{k=1}^{R} \rho_k z_k}$$

which is $P^*(\underline{z})$. The last equality is obtained using the following relationships:

$$\rho_1 = \frac{\lambda_1}{\mu} = \sum_{i=1}^{R} \frac{\gamma_i}{\mu}$$

and

$$\rho_k = \frac{\lambda_k}{\mu} + \rho_{k+1} \qquad \text{for} \qquad 1 \leq k \leq R - 1. \qquad \text{(Q.E.D.)}$$

The moments of t_r can be obtained from the moment generating function of t_r and $\underline{m}^{(r)}$ as follows.

$$E[t_r^{\ n}] = (-1)^n \frac{\partial^n}{\partial s^n} U_r^*(s, \underline{z}) \Big|_{s = 0,\ \underline{z} = \underline{1}}$$

$$= (-1)^n \frac{\partial^n}{\partial s^n} U_r^*(s,z,z, \ldots, z) \Big|_{s = 0,\ z = 1} \qquad (12)$$

Theorem 3. The conditional mean response time is

$$E[t_r] = \frac{r/\mu}{1-\rho} \tag{13}$$

Proof. Using (6) and (12), we have

$$E[t_{r+1}] = -\frac{\partial}{\partial s}\{B^*(s+\lambda_1(1-z))\ U_r^*(s, \underline{y}(z,z, \ldots, z))\}\Big|_{s=0,\ z=1}$$

$$= \frac{1}{\mu} - E[\frac{\partial}{\partial s}\{e^{-st_r}(B^*(s))^{M^{(r)}}\}]_{s=0}$$

$$= \frac{1}{\mu} - \{-E[t_r] - E[M^{(r)}].\frac{1}{\mu}\}$$

By (11),

$$E[M^{(r)}] = \frac{\partial}{\partial z} P^*(z,z, \ldots, z)\Big|_{z=1} = \frac{\rho}{1-\rho}$$

Substituting this into the above expression for $E[t_{r+1}]$, we have

$$E[t_{r+1}] = \frac{1/\mu}{1-\rho} + E[t_r]$$

which yields (13) by induction starting with $E[t_0] = 0$. (Q.E.D.)

Theorem 4. The second order statistics of the conditional response time can be found recursively using

$$\mathrm{Var}(t_{r+1}) = \mathrm{Var}(t_r) + \frac{1-2\rho r}{\mu^2(1-\rho)^2} + \frac{2}{\mu} E[t_r M^{(r)}] \tag{14}$$

$$E[t_{r+1}M^{(r+1)}] = \sum_{i=1}^{R} E[t_{r+1}m_i^{(r+1)}] \tag{15}$$

and

$$E[t_{r+1}m_i^{(r+1)}] = \frac{2\rho_i}{\mu\ (1-\rho)^2} + \frac{r\gamma_i}{\mu^2(1-\rho)} + \frac{\gamma_i}{\mu} E[t_r M^{(r)}] + E[t_r\ m_{i+1}^{(r)}] \quad 1 \le i \le R \tag{16}$$

where $\mathrm{Var}(t_r)$ is the variance of t_r and $E[t_r\ m_{R+1}^{(r)}]$ is zero, with the initial condition

$$\mathrm{Var}(t_0) = 0$$

$$E[t_0\ m_i^{(0)}] = 0 \qquad \text{for} \quad 1 \le i \le R$$

Proof.

$$E[t_{r+1}^2] = \frac{\partial^2}{\partial s^2} [y_1(s, \underline{z})\ U_r^*(s, \underline{z})]\Big|_{s = 0,\ \underline{z} = \underline{1}}$$

$$= \frac{\partial}{\partial s}\left\{\frac{\partial}{\partial s} \{y_1(s, \underline{1})\ E[e^{-st_r}(y_1(s, \underline{1}))^{M^{(r)}}]\}\right\}_{s = 0}$$

(where $y_1(s, \underline{1}) = \frac{1}{1 + (s/\mu)}$)

$$= \frac{\partial}{\partial s}\left\{- \frac{1}{\mu} (y_1(s,\underline{1}))^2\ E\ [e^{-st_r}(y_1(s,\underline{1}))^{M^{(r)}}] + y_1(s,\underline{1})E\Big[-t_r e^{-st_r}(y_1(s,\underline{1}))^{M^{(r)}}\right.$$

$$\left. + M^{(r)} e^{-st_r} y_1(s,\underline{1})^{M^{(r)}+1}(- \frac{1}{\mu})\Big]\right\}_{s=0}$$

$$= (- \frac{1}{\mu})\left\{-E[t_r] + (-\frac{1}{\mu})(E[M^{(r)}] + 2)\right\} - E\Big[t_r\{-t_r+(-\frac{1}{\mu})(M^{(r)}+1)\}\Big]$$

$$+ E\Big[M^{(r)}(-\frac{1}{\mu})\{-t_r+(-\frac{1}{\mu})(M^{(r)}+2)\}\Big]$$

$$= \{\frac{E[t_r]}{\mu} + \frac{E[M^{(r)}]}{\mu^2} + \frac{2}{\mu^2}\} + \{E[t_r^2] + \frac{E[t_r M^{(r)}]}{\mu} + \frac{E[t_r]}{\mu}\}$$

$$+ \{\frac{E[t_r M^{(r)}]}{\mu} + \frac{E[(M^{(r)})^2]}{\mu^2} + \frac{2E[M^{(r)}]}{\mu^2}\}$$

$$= \{\frac{2}{\mu^2} + \frac{2E[t_r]}{\mu} + \frac{3E[M^{(r)}]}{\mu^2} + \frac{E[(M^{(r)})^2]}{\mu^2}\} + \frac{2}{\mu} E[t_r\ M^{(r)}] + E[t_r^2]$$

(where the terms bracketted by {} can be evaluated using (11) and (13))

$$= \frac{2(1+r(1-\rho))}{\mu^2(1-\rho)^2} + \frac{2}{\mu} E[t_r\ M^{(r)}] + E[t_r^2]$$

Substituting $E[t_r^2] = Var(t_r) + (E[t_r])^2$ into the above yields (14).

The validity of (15) is obvious since $M^{(r+1)} = \sum_{i=1}^{R} m_i^{(r+1)}$. To solve for (16) below for $1 \leq i \leq R$, we shall interpret $m_{R+1}^{(r)}$ as zero.

$$E[t_{r+1}\, m_i^{(r+1)}] = -\frac{\partial^2}{\partial s\, \partial z_i}\{y_1(s,\underline{z})\; U_r^*(s,\underline{y}(s,\underline{z}))\}\Big|_{s=0,\ \underline{z}=\underline{1}}$$

$$= \Big[-\frac{\partial}{\partial z_i}\Big\{[\frac{\partial}{\partial s} y_1(s,\underline{z})].U_r^*(s,\underline{y}(s,\underline{z})) + y_1(s,\underline{z}).E\Big[z_1^{m_2^{(r)}} \ldots\, z_{R-1}^{m_R^{(r)}}$$

$$.\frac{\partial}{\partial s}\{e^{-st_r}(y_1(s,\underline{z}))^{M^{(r)}}\}\Big]\Big\}_{s=0,\ z_j=1 \text{ for } j\neq i}\Big]_{z_i=1}$$

$$= \frac{\partial}{\partial z_i}\Big\{\frac{1}{\mu}(y_1(z_i))^2 E[z_i^{m_{i+1}^{(r)}}(y_1(z_i))^{M^{(r)}}] + y_1(z_i)E[z_i^{m_{i+1}^{(r)}}\{t_r(y_1(z_i))^{M^{(r)}}$$

$$+ M^{(r)}(y_1(z_i))^{M^{(r)}-1}.\frac{1}{\mu}.(y_1(z_i))^2\}]\Big\}_{z_i=1}$$

$$\text{(where } y_1(z_i) = y_1(0,\underline{z})\Big|_{z_j=1 \text{ for } j\neq i} = \frac{1}{1+\frac{\gamma_i}{\mu}(1-z_i)}$$

$$= \frac{2\gamma_i}{\mu^2} + \frac{1}{\mu}\{E[m_{i+1}^{(r)}] + E[M^{(r)}]\frac{\gamma_i}{\mu}\} + \frac{\gamma_i}{\mu}\{E[t_r] + \frac{1}{\mu}E[M^{(r)}]\}$$

$$+ \{\frac{\gamma_i}{\mu} E[t_r\, M^{(r)}] + \frac{\gamma_i}{\mu^2}E[M^{(r)}(M^{(r)}+1)]\} + \{E[t_r\, m_{i+1}^{(r)}] + \frac{1}{\mu}E[M^{(r)} m_{i+1}^{(r)}]\}$$

$$= \Big\{\frac{2\gamma_i}{\mu^2} + \frac{1}{\mu}E[m_{i+1}^{(r)}] + \frac{3\gamma_i}{\mu^2}E[M^{(r)}] + \frac{\gamma_i}{\mu}E[t_r]$$

$$+ \frac{\gamma_i}{\mu^2}E[(M^{(r)})^2]\Big\} + \frac{\gamma_i}{\mu}E[t_r\, M^{(r)}] + E[t_r\, m_{i+1}^{(r)}]$$

where the terms bracketted by {} can be evaluated using (11) and (13) to yield (16).

(Q.E.D.)

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