# A Sequent Calculus for Intensional Logic

Frank M. Brown

Department of Computer Sciences

The University of Texas at Austin

Austin, Texas

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## Abstract

We give an axiomatization of the modal quantificational logic which captures the notions of logical truth. This modal logic is stronger than S5. Next we describe a sequent calculus for this modal logic, and show that it is complete in the traditional sense.

We then show how two semantical systems can be represented in our modal logic, thus allowing our sequent calculus to prove theorems about the semantics of intensional logic. We end with a discussion of some potential applications of this sequent calculus to Artificial Intelligence.

Contents

## 1. Introduction

We first describe in section 2 a very strong modal logic which captures the notion of logical truth of the meaning of object language sentences. That is, for example, letting $\vdash$ be the modal symbol for logical truth, and M be our recursive meaning function [1] then $\vdash$ (MS) states that the meaning of the object language sentence S is logically true. Thus, amazing as it may seem, we can construct a definition of logical truth without the use of any set-theoretic concepts.

Next in section 3 we give a complete set of rewrite rules for this modal logic which may be used to form a sequent calculus for modal quantificational logic. In particular these rules are intended to be added to a classical sequent calculus such as: [2, 3, 4, 5, 6, 7].

Then in section 4 we describe how Kripke semantics [8] may be represented in this modal logic by the use of a few axioms.

In section 5 we describe how a new semantical system which we have recently developed [9] called B-Semantics can be represented in this modal logic by a single definition like axiom.

In section 6 we give some example proofs that could be obtained by this sequent calculus.

Finally in section 7 we describe some applications of this calculus to Artificial Intelligence.

## 2. Axiomatization of Modal Logic

After explaining in section 2.1 the logical notation that we use, we give in section 2.2 the logical axioms of our modal logic. In section 2.3 we discuss what we call the Possibility Problem of modal logic, and explain why any theory formulated in modal logic should also include certain specific non-logical axioms about possibility.

### 2.1. Notation

We now explain our notation.

The symbols of classical logic are listed below with their English Translations:

| | |
|---|---|
| $p \wedge q$ | p and q |
| $p \vee q$ | p or q |
| $p \rightarrow q$ | if p then q |
| $p \leftrightarrow q$ | p iff q |
| $\sim p$ | not p |
| $\blacksquare$ | true |
| $\square$ | false |
| $\forall x\ \phi x$ | for all objects x, $\phi x$ holds. |
| $\exists x\ \phi x$ | for some objects x, $\phi x$ holds. |
| $\forall p\ \phi p$ | for all propositions p, $\phi p$ holds. |
| $\exists p\ \ p$ | for some propositions p,  p holds. |

Letters such as x, y, z, range over objects, whereas letters such as p, q, r, w, u range over propositions.

The symbols of modal logic are:

| | |
|---|---|
| $\vdash p$ | p is logically true |
| $\vdash pq$ | p entails q |
| $\Diamond p$ | p is possible |
| (World p) | p is a world |

The last three modal symbols are defined in terms of the first one as follows:

$\vdash p\, q \quad =\text{df} \quad \vdash(p \rightarrow q)$

$\Diamond p \quad =\text{df} \quad \sim\vdash\sim p$

$(\text{World } p) \quad =\text{df} \quad (\Diamond p)\wedge\forall q(\vdash pq)\wedge(\vdash p(\sim q)))$

2.2. The Logical Axioms of Modal Logic

This modal logic consists of the symbols and laws of classical quantificational logic plus the unary necessity symbol $\vdash$ and the following axioms and inference rules:

R0: from p infer $\vdash p$

A1: $\vdash p \rightarrow p$

A2: $\vdash(p \rightarrow q) \rightarrow \vdash p \rightarrow \vdash q$

A3: $\vdash p \vee \vdash\sim\vdash p$

A4: $(\forall w \text{ World } w \rightarrow \vdash wp) \rightarrow \vdash p$

R0, A1, A2, and A3 are essentially the inference rule and axioms of S5 modal logic. Axiom A4 which we call Leibniz's postulate expresses his intuition that something is logically true if it is true in all possible worlds. A good introduction to modal logic is given in Hughes and Cresswell [10].

2.3. The Possibility Problem

The possibility problem of modal logic is that from the logical axioms of modal logic we cannot prove certain elementary facts about the possibility of conjunctions of distinct possible negated atomic expressions consisting of non-logical symbols. For example, if we have a theory formulated in our modal logic which contains the non-logical atomic expression (ON A B) then since $\sim$(ON A B) is not logically true, it follows that (ON A B) must be possible. Yet $\Diamond$(ON A B) is not a theorem of our modal logic.

Thus, for any theory expressed in modal logic, a certain number of non-logical axioms dealing with possibility should also be added. Fro example, in the case of the propositional logic, or in the case of the quantificational logic over a finite domain since it reduces to propositional logic, one sufficient but inefficient axiomatization would be to assert the possibility of all consistent disjunctions of conjunctions of literals as additional non-logical axioms:

$$\Diamond(\vee(\wedge \text{Literals}))$$

A more computationally efficient axiomatization which is obtained by noting that the possibility of a disjunction of sentences is implied by the possibility of any one of those sentences:

$$\Diamond p \rightarrow \Diamond(p \vee q)$$

is to assert only the possibility of all consistent conjunctions of literals:

$$\Diamond(\wedge \text{literals})$$

Using our meaning function [1] this may be done in a finite manner by adding the single axiom:

$$(\text{Conj } S) \wedge (\text{Consist } S) \rightarrow \Diamond(M\ S)$$

where Conj and Consist are recursive functions defined as follows:

(Conj S) = df (Lit S) $\vee$ $\exists$T $\exists$R (S=[T$\wedge$R] $\wedge$ (Lit T) $\wedge$ (Conj R))

(Lit S) = df ($\exists$T S=[ʹ$\sim$ T] $\wedge$ (Atomicsent T)) $\vee$ (Atomicsent S)

(Consist [ ]) = df ■

(Consist [S, L]) = df (Consist 2 S L) $\wedge$ (Consist L)

(Consist 2 S [ ]) = df ■

(Consist 2 S[T, L] = df $\sim$(Opp S T) $\wedge$ (Consist 2 S L)

(Opp S T) = df ($\exists$R S=[ʹ$\sim$R] $\wedge$ T=R) $\vee$ ($\exists$R T=[ʹ$\sim$R] $\wedge$ S=R)

The methods for representing object language expressions in our logic and for obtaining their meanings are defined in [1,7] [1977d]. For example [T ʹ$\wedge$ R] is essentially a structural descriptive name of

(M T) $\wedge$ (M R).

## 3. A Sequent Calculus for Modal Logic

We give in section 3.1 some theorems of our modal logic which when used as rewrite rules will form the basis of a method for translating every expression of our modal logic into the standard form described in section 3.2. In section 3.3 we show that these rewrite rules form the basis of a complete proof procedure for modal logic. In section 3.4 we list some definitions which are also to be used as rewrite rules.

### 3.1. Rules of the Sequent Calculus

We list below thirteen theorems of our modal logic of the form $p \leftrightarrow q$ or $r \rightarrow (p \leftrightarrow q)$ which may be used to rewrite rules replacing p by q in any context in which r is a hypothesis. The symbols World* and $\vdash$* have the same meaning respectively as world and $\vdash$ , but are never to be replaced by their definitions in a proof procedure using these rules. Furthermore any initial theorem given to such a proof procedure must not itself contain the world* or $\vdash$* symbols, although it could of course contain the World and $\vdash$ symbols.

$\vdash$ : $(\vdash p) \leftrightarrow \forall w(\text{World*} w) \rightarrow \vdash{*}w\ p)$

$\vdash \wedge$: $(\text{World*}\ w) \rightarrow ((\vdash{*}w(p \wedge q)) \leftrightarrow (\vdash{*}w\ p \wedge \vdash{*}w\ q))$

$\vdash \vee$: $(\text{World*}\ w) \rightarrow ((\vdash{*}w(p \vee q)) \leftrightarrow (\vdash{*}w\ p \vee \vdash{*}w\ q))$

$\vdash \rightarrow$: $(\text{World*}\ w) \rightarrow ((\vdash{*}w(p \rightarrow q)) \leftrightarrow (\vdash{*}w\ p \rightarrow \vdash{*}w\ q))$

$\vdash \leftrightarrow$: $(\text{World*}\ w) \rightarrow ((\vdash{*}w(p \leftrightarrow q)) \leftrightarrow (\vdash{*}w\ p \leftrightarrow \vdash{*}w\ q))$

$\vdash \sim$: $(\text{World*}\ w) \rightarrow ((\vdash{*}w(\sim p)) \leftrightarrow \sim \vdash{*}w\ p)$

$\vdash \Box$: $(\text{World*}\ w) \rightarrow ((\vdash{*}w\ \Box\ ) \leftrightarrow \Box)$

$\vdash \Box$: $(\text{World*}\ w) \rightarrow ((\vdash{*}w\ \Box) \leftrightarrow \Box)$

$\vdash \forall$: $(\text{World*}\ w) \rightarrow ((\vdash{*}w(\forall x\ \phi x)) \leftrightarrow (\forall x \vdash{*}w\ \phi\ x))$

$\vdash \exists$: $(\text{World*}\ w) \rightarrow ((\vdash{*}w(\exists x\ \phi x)) \leftrightarrow (\exists x \vdash{*}w\ \phi\ w))$

$\vdash$ a: $(\text{World*}\ w) \rightarrow ((\vdash{*}w(\forall p\ \phi p)) \leftrightarrow (\forall p \vdash{*}w\ \phi\ p)$

$\vdash$ e: $(\text{World*}\ w) \rightarrow ((\vdash{*}w(\exists p\ \phi p)) \leftrightarrow (\exists p \vdash{*}w\ \phi\ p))$

$\vdash \vdash$: $(\text{World*}\ w) \rightarrow ((\vdash{*}w(\vdash p)) \leftrightarrow \vdash p$

The $\vdash\forall$ and $\vdash\exists$ theorems pertain to quantifiers of object language variables whereas the $\vdash$a, $\vdash$e theorems pertain to quantifiers for propositional variables. The $\vdash\forall$ and $\vdash\exists$ theorems are equivalent to the fact that something is an object iff it is logically true that it is an object. All these theorems hold regardless of whether propositions are objects or not.

In order to try to prove a theorem $\psi$ with a proof procedure using these rules, sometimes it must actually try to prove $\vdash \psi$ instead. There is a deep and beautiful reason for this which is basically that this initial $\vdash$ inserted before $\psi$ is a symbol of the metalanguage of this logic as are all the $\vdash$* and World* symbols. Essentially $\vdash \psi$ is the statement in the metatheory that $\psi$ is logically true, and it is this rather than $\psi$ itself which we are trying to prove.

Unlike sequent calculi for weaker modal logics [11] our sequent calculus leads to a very efficient proof procedure, as can be seen from the fact that the $\vdash$ law is an explicit definition of $\vdash$ in terms of $\vdash$*w, and that the reamining twelve laws are essentially a contextual definition which eliminates all occurrences of the $\vdash$*w symbol which do not occur immediately before an atomic sentence. Thus in a proof procedure based on this sequent calculus it makes no essential difference as to which of the thirteen laws is first applied. Furthermore it makes no essential difference to which subformula of the theorem being proven a law is first applied. All possible strategies of applying these laws, so long as they are applied as many times as possible will lead to the standard form described in section 3.2.

## 3.2. The Standard Form

If the rewrite rules given in section 3.1 are supplemented by enough laws of classical logic which when used as rewrite rules suffice to put the sentences of classical logic into skolemized prenex conjunctive normal form, then every sentence of our modal logic will be rewritten to an equivalent sentence in the following standard form:

(1) First a sequence of universal quantifiers (ie. skolem functions) consisting of:

(a) object variable quantifiers: $\forall x$

(b) proposition variable quantifiers: $\forall p$ (except those introduced by the $\vdash$ and $\vdash\vdash$ rules)

(c) Propositional variable quantifiers introduced by the $\vdash$ rule. These quantifiers are essentially treated as world quantifers as the (World*w) hypothesis is always kept next to the quantifiers: $(\forall w(\text{World*}w) \rightarrow \ldots)$.

(2) Second a sequence of existential quantifiers (ie. unification variables) consisting of:

(a) object variable quantifiers: $\exists x$

(b) proposition variable quantifiers: $\exists p$ (except those introduced by the $\models$ and $\vdash\vdash$ rules)

(c) Propositional variable quantifiers introduced by the $\models$ rule. These quantifiers are essentially treated as World quantifiers as the (World*w) hypothesis is always kept next to the quantifiers: $(\exists w(\text{World*}w) \ldots)$.

(3) And finally a matrix consisting of

(a) a conjunction: $\wedge$

(b) of a disjunction: v (ie. of sequents)

(c) of negated: $\sim$ or unnegated

(d) atoms consisting of an entailment symbol $\vdash^{*}w$ whose first argument is a variable quantified by a quantifier of type (1c) or (2c), and whose second arguement is either (i) a variable quantified by a quantifier of type (1b) or (2b), or (ii) a nonlogical atomic sentence containing no variables of type (1c) or (2c).

Schematically this standard form can be represented as:

$$\forall X \forall p \forall w(\text{World*}w) \rightarrow \exists x \exists p \exists w(\text{World*}w) \wedge \text{Matrix}$$

where the matrix is of the form:

$$\wedge\ v\{\sim\} \vdash^{*}w \ \{^{p}_{(\phi p x)}$$

where $\phi$ is a non-logical symbol.

The fact that sorted quantifiers can be pulled out of the matrix and skolemized can be justified by the following theorems of classical logic:

$\rightarrow \forall w$: $(\forall w(\text{World*}w) \rightarrow (s \wedge (t v \phi w))) \leftrightarrow (s \wedge (t v(\forall w(\text{World*}w) \rightarrow \phi w)))$

$\rightarrow \exists w$: $(\exists w(\text{World*}w) \wedge (s \wedge (t v \phi w))) \leftrightarrow s \wedge (t v(\exists w W \text{World*}w) \wedge \phi w)))$

Skolem: $(\exists x \Pi x \wedge (\forall a \quad a \rightarrow \phi x a)) \leftrightarrow (\forall a(\forall x \Pi x \rightarrow \quad (a x)) \rightarrow (\exists x \Pi x \wedge \phi x(a x)))$

The $\rightarrow \forall w$ and $\rightarrow \exists w$ theorems depend on the truth of the theorem: $\exists w(\text{World } w)$. The "a" in "(ax)" of the Skolem theorme represents a skolem function.

## 3.3. Completeness and Expressibility

We consider the fragment of our modal logic which does not contain any propositional variables and where propositions are not objects. Given the standard form described in section 3.2 we can prove the completeness of this fragment relative to classical state logic. Classical state logic is a classical quantificational logic containing two distinct sorts, namely objects and worlds such that each non-logical symbol contains exactly one argument position which contains a world variable. This completeness proof is carried out merely by forming an isomorphism between expressions of our modal logic and expressions of state logic by translating each atom $\vdash *w(\phi^n X_1 \ldots X_n)$ of our modal logic containing an n-ary non-logical symbol $\phi$ into an (n+1)-ary atom of state logic: $(\phi^{n+1} X_1 \ldots X_n w)$.

We now state the Completeness Theorem:

Completeness Theorem

For every sentence (a) of this fragment of our modal logic, there is an equivalent sentence (b) of our modal logic such that there exists a sentence (s) of classical state logic which is isomorphic to (b). Therefore (s) is provable iff (b) and (a) are provable. Using this isomorphism we may also obtain an Expressibility Theorem for our modal logic:

Expressibility Theorem

For every sentence (t) of state logic there is an equivalent sentence (s) of state logic in skolemized prenex conjunctive normal form such that there exists a sentence (b) of our modal logic which is isomorphic to (s).

The expressibility theorem shows that everything expressible in state logic is expressible in our modal logic.

### 3.4. Rules defining modal concepts

The laws are all of the form $p \leftrightarrow q$ are to be used only to replace an expression of the form $p\theta$ by an expression of the form $q\theta$.

| | | | |
|---|---|---|---|
| $D\vdash$ : $\vdash pq$ | $\leftrightarrow$ | $\vdash(p \to q)$ | "p entails q" |
| $D\equiv$ : $p \equiv q$ | $\leftrightarrow$ | $\vdash(p \leftrightarrow q)$ | "p is synonomous to q" |
| $D\Diamond$ : $\Diamond p$ | $\leftrightarrow$ | $\sim\vdash\sim p$ | "p is possible" |
| Ddet : (Det pq) | $\leftrightarrow$ | $\vdash pq \vee \vdash p \sim q$ | "p determines q" |
| Dcom : (Complete p) | $\leftrightarrow$ | $\forall q$(Det pq) | "p is complete" |
| Dwor : (World p) | $\leftrightarrow$ | $\Diamond p \wedge$(Complete p) | "p is a world" |

| | | | |
|---|---|---|---|
| Dval : (Valid p) | $\leftrightarrow$ | $\forall q(\text{World } q) \rightarrow \vdash qp$ | "p is valid" |
| Dsat : (Sat p) | $\leftrightarrow$ | $\exists q(\text{World } q) \wedge \vdash qp)$ | "p is satisfiable" |
| Duniq: (Uniq $\underline{p}\phi p$) | $\leftrightarrow$ | $\forall p \forall q \phi p \wedge \phi q \rightarrow p \equiv q$ | "p is unique in $\phi$" |
| Done : (one $\underline{p}\phi p$) | $\leftrightarrow$ | $\exists q \forall r(\phi r \leftrightarrow q \equiv r)$ | "p is one in $\phi$" |
| Dcat : (Cat p) | $\leftrightarrow$ | (One $\underline{q}$(World $q \wedge \vdash qp$)) | "p is categorical" |
| Dmaxp: (Maxpos p) | $\leftrightarrow$ | (One $\underline{q}$ ($\Diamond q \wedge \vdash qp$)) | "p is maximally possible" |
| Dmax : (Max p) | $\leftrightarrow$ | (One q($\vdash q$ p)) | "p is maximal" |

## 4. Kripke Semantics

The basic idea of Kripke Semantics [8] is to define the proposition that p is necessary holds in a world w as the proposition that p holds in all worlds u related to w by some binary relation R:

DK: $\vdash w\Box p \leftrightarrow \text{df}(\forall u.Rwu \rightarrow \vdash up)$

From this definition it is clear that the necessity symbol for various modal logics will be easily definable by assuming various axioms for the R-relation. The R-relation will be assumed to be an intensional symbol, and thus that the following axiom shall hold.

KA1: $p \equiv q \rightarrow (\forall rRpr \leftrightarrow Rqr) \wedge (\forall rRrp \leftrightarrow Rrq)$

In particulary this axiom is used to prove theorems KT1, KT4, KT5, and KT6 in the section 2.1. Also it does not seem unreasonable to interpret the R relation as being the same in all worlds.

KA2: $\Diamond Rxy \rightarrow \vdash Rxy$

Static Kripke Semantics is a degenerate case of Kripke Semantics that may be obtained by assuming either the intensional logic sentence:

$\vdash \forall p \Box p \rightarrow \vdash \Box p$

or the semantic condition of theorem KT11: $\forall u \forall v \forall w(Ruw \rightarrow Rvw)$

Essentially such an assumption makes the first argument of the relation R irrelavent to its meaning. For this reason; alternatively the same effect could be achieved by simply replacing the definition DK by the definition DSK:

DSK: $\vdash w\Box p \leftrightarrow (\forall u R u \rightarrow \vdash up)$

involving a unary predicate R whose argument is the second argument of the relation R. The utility of the DSK rule lies in the fact that if some intentional concept satisfies the axiom $\vdash \forall p \Box p \rightarrow \vdash \Box p$ then it will be more effecient to use the single law DSK as a rewrite rule rather than use both DK and that axiom.

We list in section 4.1 a number of theorems derviable from the axiom definition DK, and we list in section 4.2 a number of theorems derivable from the axiom definition DSK. It will be seen that the theorems which relate laws of intensional logic to their semantic conditions on the R relation in DK or the R predicate in DSK are generalizations of many well known results about Kripke Semantics. Proofs of all these theorems are given in [9].

4.1. Basic Theorems of Kripke Semantics

We first make a few definitions:

D1: $\Diamond p \leftrightarrow \sim\Box\sim p$

D2: $(\Box\text{-detpq}) \leftrightarrow \Box(p \rightarrow q) \vee \Box(p \rightarrow q)$

D3: $(\Box\text{-complete } p) \leftrightarrow \forall q(\Box\text{-detpq})$

D4: $(\Box\text{-world } p) \leftrightarrow \Diamond p \wedge (\Box\text{-complete } p)$

D5: $(\Box\text{-valid } p) \leftrightarrow \forall q(\Box\text{-world } q) \rightarrow \Box(p \rightarrow q)$

The main theorems of Kripke semantics are listed below: Variables u,v,w range over worlds.

KT0: $\forall p(\vdash p \rightarrow \vdash \Box p)$ $\leftrightarrow$ [redacted]

KT1: $\vdash (\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q))$ $\leftrightarrow$ [redacted]

KT2: $\vdash ((\Box\text{-valid } p) \rightarrow \Box p)$ $\leftrightarrow$ [redacted]

KT3:* $\vdash (\Box \forall x\ x \leftrightarrow \forall x \Box \phi x)$ $\leftrightarrow$ [redacted]

KT4: $\vdash (\exists x \Box\ x \rightarrow \Box \exists x \phi x)$ $\leftrightarrow$ [redacted]

*The reader should not be disturbed by the fact that KT3 holds in our Kripke Semantics where as in Kripke's [8] work it did not; because this is merely a consequence of Kripke's way of defining the classical logic operation of substitution. In [9] we show how to obtain quantifiers such that KT3 does not hold.

| | | | |
|---|---|---|---|
| existence | KT5: | $\vdash \forall p(\Box p \rightarrow \Diamond p) \leftrightarrow \forall w \exists r Rwu$ | "deontic" |
| uniqueness | KT6: | $\vdash \forall p(\Diamond p \rightarrow \Box p) \leftrightarrow \forall r \forall v \forall w Rwu \wedge Rwv \rightarrow u \equiv v$ | |
| reflexive | KT7: | $\vdash \forall p(\Box p \rightarrow p) \leftrightarrow \forall w Rww$ | "T" |
| symetric | KT8: | $\vdash \forall p(p \rightarrow \Box \Diamond p) \leftrightarrow \forall u \forall v Ruv \rightarrow Rvu$ | "B" |
| transitive | KT9: | $\vdash \forall p(\Box p \rightarrow \Box \Box p) \leftrightarrow \forall u \forall v \forall w Ruv \wedge Rvw \rightarrow Ruw$ | "S4" |
| | KT10: | $\vdash \forall p(\Box p \vee \Box \sim \Box p) \rightarrow \forall u \forall v \forall w Ruv \wedge Ruw \rightarrow Rvw$ | "S5" |
| | KT11: | $\vdash \forall p(\Box p \rightarrow \vdash \Box p) \leftrightarrow \forall u \forall v \forall w Ruw \rightarrow Rvw$ | "Static" |

## 4.2. Basic Theorems of Static Kripke Semantics

| | | | |
|---|---|---|---|
| SKT0: | $\forall p \vdash p \rightarrow \vdash \Box p$ | | |
| SKT1: | $\vdash (\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q))$ | | |
| SKT2: | $\vdash ((\Box\text{-valid } p) \rightarrow \Box p)$ | | |
| SKT3: | $\vdash (\Box \forall x \phi x \leftrightarrow \forall x \Box \phi x)$ | | |
| SKT4: | $\vdash (\exists x \Box \phi x \rightarrow \Box \exists x \phi x)$ | | |
| SKT5: | $\vdash \forall p(\Box p \rightarrow \Diamond p) \leftrightarrow \exists u Ru$ | "deontic" , | existence |
| SKT6: | $\vdash \forall p(\Diamond p \rightarrow \Box p) \leftrightarrow \forall u \forall v Ru \wedge Rv \rightarrow u \equiv v$ | | uniqueness |
| SKT7: | $\vdash \forall p(\Box p \rightarrow p) \leftrightarrow \forall w Rw$ | "T" | reflexive |
| SKT8: | $\vdash \forall p(p \rightarrow \Box \Diamond p) \leftrightarrow \exists v Rv \rightarrow \forall u Ru$ | "B" | symetric |
| SKT9: | $\vdash \forall p \Box p \rightarrow \Box \Box p) \leftrightarrow \blacksquare$ | "S4" | transitive |
| SKT10: | $\vdash \forall p(\Box p \vee \Box \sim \Box p) \leftrightarrow \blacksquare$ | "S5" | |
| SKT11: | $\vdash \forall p \Box p \rightarrow \vdash \Box p \leftrightarrow \blacksquare$ | "Static" | |

It is interesting to note that the modal laws of the SKT9 and SKT10 theorems become true without any restrictions on the R verb. Also it follows that the modal laws of the SKT7 become equivalent to the conjunction of the modal laws of SKT5 and SKT8

$$\vdash \forall p(\Box p \rightarrow p) \leftrightarrow (\vdash \forall p \Box p \rightarrow \Diamond p \wedge \vdash \forall p(p \rightarrow \Box \Diamond p))$$

$$\forall w Rw \leftrightarrow (\exists w \underline{Rw} \wedge (\exists w Rw \rightarrow \forall w Rw))$$

$$\forall w Rw \leftrightarrow (\exists w Rw \wedge \underline{\forall w Rw})$$

$$\forall w Rw \leftrightarrow \forall w Rw$$

reflexsitivity $\leftrightarrow$ existence$\wedge$symetry

5. B-Semantics

The basic idea of B semantics is to define the proposition that p is necessary holds in a world w as the proposition that p is entailed by the strongest proposition which is necessary in that world w. We let (fw) be the propositional function value which represents the strongest proposition which is necessary in that world. Thus we say:

DB: $\vdash w\Box p \leftrightarrow \vdash (fw)p$

From this definition it will be possible to define the necessity symbol for various modal logics by assuming various axioms for the fw function value.

Static B-Semantics is a degenerate case of B-Semantics that may be obtained by assuming either the intentional logic axiom:

$\forall p\Box p \rightarrow \vdash \Box p$

or by assuming the semantic condition

$\forall u\forall w \vdash fu\ fv$

Essentially such an assumption makes the propositional function f a propositional constant since it must then have the same meaning for all worlds. For this reason, alternatively the same effect could be achieved by simply replacing the definition DB by the definition DSB:

DSB! $\vdash w\Box p \leftrightarrow \vdash fp$

involving a propositional constant f. It should be noted that if a particular concept satisfies the axiom $\forall u\forall w \vdash fu\ fu$ then it will be easier to prove theorems about it using DSB instead of using both DB and that axiom.

We list in section 5.1 some theorems which can be derived from the axiom definition DB. We then list in section 5.2 some theorems which can be obtained from the axiom definition DSK. It should be noted that in each case the theorems are similar to those which can be obtained from the Kripke semantics described in section 4. The advantage of our new B-Semantics over Kripke Semantics is however that it is usually much easier to obtain proof in this new semantic system. Evidence supporting this assertion is given in section 6 where we present a few example proofs that could be obtained by a sequent calculus theorem prover such as the one described in [2,3,4,5,6,7]. Proofs of all these theorems are given in [9].

5.1. Basic Theorems of B-Semantics

We assume the definitions D1-D5 given in section 2.1. The main theorems of B semantics are listed below.

| | | | |
|---|---|---|---|
| BT0: | $\forall p \vdash p \rightarrow\ \vdash \Box p$ | "T" | |
| BT1: | $\vdash \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$ | "T" | |
| BT2: | $\vdash ((\Box\text{-valid } p) \rightarrow \Box p)$ | "atomic" | |
| BT3: | $\vdash (\Box \forall x\ x \rightarrow \forall x \Box \phi x)$ | "complete" | |
| BT4: | $\vdash (\exists x \Box\ x \rightarrow \Box \exists x \phi x)$ | | |
| BT5: | $(\vdash \forall p \Box p \rightarrow \Diamond p) \leftrightarrow (\forall w \Diamond fw)$ | "Deontic" | "existence" |
| BT5:* | $(\vdash \forall p \Box p \rightarrow \Diamond p) \leftrightarrow \forall w \exists u \vdash ufw$ | "Deontic" | "existence |
| BT6: | $(\vdash \forall p \Diamond p \rightarrow \Box p) \leftrightarrow (\forall w \text{ complete } (fw))$ | | "uniqueness |
| BT6:* | $(\vdash \forall p \Diamond p \rightarrow \Box p) \leftrightarrow \forall w \forall r \forall v \vdash ufw \wedge \vdash vfw \rightarrow u \equiv v$ | | |
| BT7: | $(\vdash \forall p \Box p \rightarrow p) \leftrightarrow \forall w \vdash wfw$ | "T" | "reflexive" |
| BT8: | $(\vdash \forall p p \rightarrow \Box \Diamond p) \leftrightarrow \forall a \forall b \vdash bfa \rightarrow\ \vdash afb$ | "B" | "symmetric" |
| BT9: | $(\vdash \forall p \Box p \rightarrow \Box \Box p) \leftrightarrow \forall a \forall b \vdash bfa \rightarrow\ \vdash fbfa$ | "S4" | "transitive" |
| BT9:* | $(\vdash \forall p \Box p \rightarrow \Box \Box p) \leftrightarrow \forall a \forall b \forall c \vdash cfb \wedge \vdash bfa \vdash cfa$ | "S4" | "transitive" |
| BT10: | $(\vdash \forall p \Box p \vee \Box \sim \Box p) \leftrightarrow \forall a \forall b \vdash bfa \rightarrow\ \vdash fafb$ | "S5" | |
| BT10:* | $\vdash \forall p (\Box p \vee \Box \sim \Box p) \leftrightarrow \forall a \forall b \forall c \vdash bfa \wedge \vdash cfa \rightarrow\ \vdash cfb$ | "S5" | |

BT11: $\vdash \forall p(\Box p \rightarrow \vdash \Box p) \leftrightarrow \forall u \forall v \vdash fu\ fv$ "static"

BT11:* $\vdash \forall p(\Box p \rightarrow \vdash \Box p) \leftrightarrow \forall u \forall v \forall w \vdash w\ fu \rightarrow \vdash w\ fv$ "static"

It is interesting to note that if $\Box$ is interpreted as $\vdash$ and if fw is interpreted as true then all these theorems will be theoresm of our original modal logic.

5.2. Basic Theorems of Static B-Semantics

SBT0: $\forall p \vdash p \rightarrow \vdash \Box p$

SBT1: $\vdash \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$

SBT2: $\vdash((\Box\text{-Valid } p) \rightarrow \Box p)$

SBT3: $\vdash(\Box \forall x \phi x \leftrightarrow \forall x \Box \phi y)$

SBT4: $\vdash(\exists x \Box \phi x \rightarrow \Box \exists x \phi x)$

SBT5: $\vdash \forall p(\Box p \rightarrow \Diamond p) \leftrightarrow \Diamond f$ "deontic" existence

SBT6: $\vdash \forall p(\Diamond p \rightarrow \Box p) \leftrightarrow (\text{complete } \blacksquare)$ uniqueness

SBT7: $\vdash \forall p(\Box p \rightarrow p) \leftrightarrow \vdash f$ "T" reflexive

SBT8: $\vdash \forall p(p \rightarrow \Box \Diamond p) \leftrightarrow (\Diamond f \rightarrow \vdash f)$ "B" symmetric

SBT9: $\vdash \forall p(\Box p \rightarrow \Box \Box p) \leftrightarrow \blacksquare$ "S4" transitive

SBT10: $\vdash \forall p(\Box p \vee \Box {\sim} \Box p) \leftrightarrow \blacksquare$ "S5"

SBT11: $\vdash \forall p(\Box p \rightarrow \vdash \Box p) \leftrightarrow \blacksquare$ "static"

6. Examples

We now give some example proofs of some theorems of Kripke and B Semantics that could be obtained by a sequent logic theorem prover such as the one described in [2,3,4,5,6,7] using the rewrite rules for our modal logic and the axioms of the semantical systems.

6.1. Reflexitivity implies factivity

Our first example is a proof that reflexitivity of the Kripke relation R in Kripke Semantics (i.e. DK) implies that the intensional symbol $\Box$ is a factive:

| | | |
|---|---|---|
| KT7a: | $\rightarrow \forall w Rww \supset \vdash \forall p \Box p \supset p$ | $: \rightarrow\supset$ |
| | $\forall w Rww \rightarrow \vdash \forall p \Box p \supset p$ | $: \vdash$ |
| | $\forall w Rww \rightarrow \forall w \vdash w^{*}(\forall p \Box p \supset p)$ | $: \rightarrow\forall$ |
| | $\forall w Rww \rightarrow \vdash^{*}a(\forall p \Box p \supset p)$ | $: \vdash \forall$ |
| | $\forall w Rww \rightarrow \forall p \vdash^{*}a \Box p \supset p$ | $: \rightarrow\forall$ |
| | $\forall w Rww \rightarrow \vdash^{*}a(\Box p \supset p)$ | $: \vdash \supset$ |
| | $\forall w Rww \rightarrow (\vdash^{*}a \Box p \supset \vdash^{*}ap)$ | $: \rightarrow\supset$ |
| | $\forall w Rww,\ \vdash^{*}a \Box p \rightarrow \vdash^{*}ap$ | $: \vdash \Box$ Def of $\Box$ |
| | $\forall w Rww,\ \forall u(Rau \supset \vdash^{*}up) \rightarrow \vdash^{*}ap$ | $: \forall \rightarrow$ |
| | $①,\ R^{*}_{1}{}^{*}_{1},\ \forall u(Rau \supset \vdash^{*}up) \rightarrow \vdash^{*}ap$ | $: \forall \rightarrow$ |
| | $①,\ R^{*}_{1}{}^{*}_{1},\ ②,\ (Ra^{*}_{2} \supset \vdash^{*}_{2}p) \rightarrow \vdash^{*}ap$ | $: \supset \rightarrow$ |

$①\ R^{*}_{1}{}^{*}_{1},\ ② \rightarrow \vdash^{*}ap,\ Ra^{*}_{2}$ $\qquad$ $①,\ R^{*}_{1}{}^{*}_{1},\ ②,\ \vdash^{*}{}_{*2}p \rightarrow \vdash^{*}ap$

$\begin{cases} {}^{*}_{1} := a \\ {}^{*}_{2} := {}^{*}_{1} \end{cases}$ $\qquad$ ${}^{*}_{2} := a$

## 6.2. The S4 law in Kripke Semantics

Our second example is a proof that transitivity of the Kripke relation R in Kripke semantics (ie. DK) implies that $\Box p \rightarrow \Box\Box p$.

KT9a $\forall u\forall v\forall w Ruv\wedge Rvw \supset Ruw \rightarrow\ \vdash \forall p(\Box p \supset \Box\Box p)$ $:\vdash$

$\forall u\forall v\forall w Ruv\wedge Rvw \supset Ruw \rightarrow \forall w \vdash^{*} w\forall p(\Box p \supset \Box\Box p)$ $:\rightarrow\forall$

$\forall u\forall v\forall w Ruv\wedge Rvw \supset Ruw \rightarrow\ \vdash^{*} a\forall p(\Box p \supset \Box\Box p)$ $:\vdash\forall$

$\forall u\forall v\forall w Ruv\wedge Rvw \supset Ruw \rightarrow \forall p \vdash^{*} a(\Box p \supset \Box\Box p)$ $:\rightarrow\forall$

$\forall u\forall v\forall w Ruv\wedge Rvw \supset Ruw \rightarrow\ \vdash^{*} a(\Box p \supset \Box\Box p)$ $:\vdash\supset$

$\forall u\forall v\forall w Ruv\wedge Rvw \supset Ruw \rightarrow (\vdash a\Box p \supset\ \vdash a\Box\Box p)$ $:\rightarrow\supset$

$\forall u\forall v\forall w Ruv\wedge Rvw \supset Ruw,\ \vdash a\Box p \rightarrow\ \vdash a\Box\Box p$ $:\vdash\Box$

$\forall u\forall v\forall w Ruv\wedge Rvw \supset Ruw, (\forall w Raw \supset\ \vdash wp) \rightarrow\ \vdash a\Box\Box p$ $:\vdash\Box$

$\forall u\forall v\forall w Ruv\wedge Rvw \supset Ruw, (\forall w Raw \supset\ \vdash wp) \rightarrow \forall w Raw \supset\ \vdash w\Box p$ $:\rightarrow\forall$

$\forall u\forall v\forall w Ruv\wedge Rvw \supset Ruw, (\forall w Raw \supset\ \vdash wp) \rightarrow Rab \supset\ \vdash b\Box p$ $:\rightarrow\supset$

$\forall u\forall v\forall w Ruv\wedge Rvw \supset Ruw, (\forall w Raw \supset\ \vdash wp), Rab \rightarrow\ \vdash b\Box p$ $:\vdash\Box$

$\forall u\forall v\forall w Ruv\wedge Rvw \supset Ruw, (\forall w Raw \supset\ \vdash wp), Rab \rightarrow (\forall w Rbw \supset\ \vdash wp)$ $:\rightarrow\forall$

$\forall u\forall v\forall w Ruv\wedge Rvw \supset Ruw, (\forall w Raw \supset\ \vdash wp), Rab, \rightarrow (Rbc \supset\ \vdash cp)$ $:\rightarrow\supset$

$\forall u\forall v\forall w Ruv\wedge Rvw \supset Ruw, \forall w Raw \supset\ \vdash wp,\ Rab,\ Rbc \rightarrow\ \vdash cp$ $:\rightarrow\forall$ (3 times)

(1), $R{*_1}{*_2} \wedge R{*_2}{*_3} \supset R{*_1}{*_3}, \forall w Raw \supset\ \vdash wp,\ Rab,\ Rbc \rightarrow\ \vdash cp$ $:\rightarrow\forall$

(1), $R{*_1}{*_2} \wedge R{*_2}{*_3} \supset R{*_1}{*_3}$, (2), $Ra{*_4} \supset\ \vdash {*_4}p,\ Rab,\ Rbc \rightarrow\ \vdash cp$ $:\supset\rightarrow$

(1), (2), $Ra{*_4} \supset\ \vdash {*_4}p,\ Rab,\ Rbc,\ R{*_1}{*_3} \rightarrow\ \vdash cp$ $:\supset\rightarrow$

(1), (2), $\vdash {*_4}p,\ Rab,\ Rbc,\ R{*_1}{*_3} \rightarrow\ \vdash cp$ : Unify

$*_4 := c$

(1), (2), $Rab,\ Rbc,\ R{*_1}{*_3} \rightarrow Ra{*_4}$ : Unify

$[*_4 := b]$ or $[*_1 := a$ and $*_3 := *_4]$

fail

①, ②, $Ra*_4 \supset \vdash *_4 p$, Rab, Rbc $\rightarrow \vdash cp$, $R*_1*_2 \wedge R*_2*_3$ :→ ∧

①, ②, $Ra*_4 \supset \vdash *_4 p$, Rab, Rbc $\rightarrow \vdash cp$, $R*_1*_2$ :⊃ →

①, ②, $\vdash *_4 p$, Rab, Rbc $\rightarrow \vdash cp$, $R*_1*_2$

$*_4 = c$ or [$*_1=a$, $*_2=b$] or [$*_1=b$ $*_2=c$]
■ ■ fail

①, ②, Rab, Rbc $\rightarrow \vdash cp$, $R*_1*_2$, $Ra*_4$

$*_4=b$ or [$*_1=a$ $*_2=b$] or [$*_1=b$ $*_2=c$]
■ fail

①, ②, $Ra*_4 \supset \vdash *_4 p$, Rab, Rbc $\rightarrow \vdash cp$, $R*_2*_3$ ⊃ →

①, ②, $\vdash *_4 p$, Rab, Rbc $\rightarrow \vdash cp$, $R*_2*_3$

$*_4=c$ or [$*_2=a$, $*_3=b$] or [$*_2=b$, $*_3=c$]
fail ■

①, ②, Rab, Rbc $\rightarrow \vdash cp$, $R*_2*_3$, $Ra*_4$

$*_4=b$ or [$*_2=a$, $*_3=b$] or [$*_2=b$, $*_3=c$]
fail fail ■

final bindings:

$*_1=a$

$*_2=b$

$*_3=c$

$*_4=c$

6.3. The S5 law is true in Static Kripke Semantics.

$\rightarrow\ \vdash \forall p \Box p \vee \Box\sim\Box p$ : $\vdash$

$\rightarrow\ \forall w \vdash w \forall p \Box p \vee \Box\sim\Box p$ : $\rightarrow \forall$

$\rightarrow\ \vdash a \forall p \Box p \vee \Box\sim\Box p$ : $\vdash \forall$

$\rightarrow\ \forall p \vdash a(\Box p \vee \Box\sim\Box p)$ : $\rightarrow \forall$

$\rightarrow\ \vdash a(\Box p \vee \Box\sim\Box p)$ : $\vdash \vee$

$\rightarrow\ \vdash a\Box p \vee \vdash a\Box\sim\Box p$ : $\rightarrow \vee$

$\rightarrow\ \vdash a\Box p,\ \vdash a\Box\sim\Box p$ : DSK

$\rightarrow\ \forall b Rb \supset \vdash bp,\ \vdash a\Box\sim\Box p$ : $\rightarrow\forall$

$\rightarrow\ Rb \supset \vdash bp,\ \vdash a\Box\sim\Box p$ : $\rightarrow\supset$

$Rb \rightarrow \vdash bp,\ \vdash a\Box\sim\Box p$ : DSK

$Rb \rightarrow \vdash bp,\ \forall c Rc \supset \vdash c\sim\Box p$ : $\rightarrow\forall$

$Rb \rightarrow \vdash bp,\ Rc \supset \vdash c\sim\Box p$ : $\rightarrow\supset$

$Rb,\ Rc \rightarrow \vdash bp,\ \vdash c\sim\Box p$ : $\vdash\sim$

$Rb,\ Rc \rightarrow \vdash bp,\ \sim\vdash c\Box p$ : $\rightarrow\sim$

$Rb,\ Rc \vdash c\Box p \rightarrow \vdash bp$ : DSK

$Rb,\ Rc,\ \underbrace{(\forall d Rd \supset \vdash dp)}_{} \rightarrow \vdash bp$ : $\forall\rightarrow$

$Rb,\ Rc,\ (1),\ (R*_1 \supset \vdash *_1 p) \rightarrow \vdash bp$ : $\supset\rightarrow$

$Rb,\ Rc,\ (1),\ \vdash *_1 p \rightarrow \vdash bp$

$*_1 := b$

$Rb,\ Rc,\ (1) \supset R*_1,\ \vdash bp$

$[*_1 := b]$ or $[*_1 := c]$ fail

## 6.4. The S4 law in B-Semantics

This example is a proof that $\vdash bfa \to \vdash fbfa$ implies $\Box p \to \Box\Box p$ in B-Semantics (ie. DB). Note that the entailment symbol $(\vdash p\ q)$ is replaced by its definition $\vdash (p \supset q)$ only if its first arguement p is not a world variable. Otherwise it is treated as the $(\vdash {*p}\ q)$ symbol in our modal sequent calculus.

BT9: $\to (\forall a \forall b \vdash bfa \supset \vdash fbfa) \supset (\vdash \forall p \Box p \supset \Box\Box p)$ :→ ⊃

$(\forall a \forall b \vdash bfa \supset \vdash fbfa) \to (\vdash \forall p \Box p \supset \Box\Box p)$ :⊢

$(\forall a \forall b \vdash bfa \supset \vdash fbfa) \to (\forall a \vdash a \forall p \Box p \supset \Box\Box p)$ : → ∀

$\forall a \forall b \vdash bfa \supset \vdash fbfa \to (\vdash a \forall p \Box p \supset \Box\Box p)$ :⊢ ∀

$\forall a \forall b \vdash bfa \supset \vdash fbfa \to \forall p \vdash a(\Box p \supset \Box\Box p)$ : → ∀

$\forall a \forall b \vdash bfa \supset \vdash fbfa \to \vdash a(\Box p \supset \Box\Box p)$ :⊢ ⊃

$\forall a \forall b \vdash bfa \supset \vdash fbfa \to \vdash a\Box p \supset \vdash a\Box\Box p$ : → ⊃

$\forall a \forall b \vdash bfa \supset \vdash fbfa, \vdash a\Box p \to \vdash a\Box\Box p$ :DB

$\forall a \forall b \vdash bfa \supset \vdash fbfa, \vdash fap \to \vdash a\Box\Box p$ :Entail

$\forall a \forall b \vdash bfa \supset \vdash fbfa, \vdash (fa \supset p) \to \vdash a\Box\Box p$ :⊢

$\forall a \forall b \vdash bfa \supset \vdash fbfa, \forall b \vdash b(fa \supset p) \to \vdash a\Box\Box p$ :DB

$\forall a \forall b \vdash bfa \supset \vdash fbfa, \forall b \vdash b(fa \supset p) \to \vdash fa\Box p$ :Entail

$\forall a \forall b \vdash bfa \supset \vdash fbfa, \forall b \vdash b(fa \supset p) \to \vdash (fa \supset \Box p)$ :⊢

$\forall a \forall b \vdash bfa \supset \vdash fbfa, \forall b \vdash b(fa \supset p) \to \forall b \vdash b(fa \supset \Box p)$ : → ∀

$\forall a \forall b \vdash bfa \supset \vdash fbfa, \forall b \vdash b(fa \supset p) \to \vdash b(fa \supset \Box p)$ :⊢ ⊃

$(\forall a \forall b \vdash bfa \supset \vdash fbfa), \forall b \vdash b(fa \supset p), \to \vdash b\ fa \supset \vdash b\Box p)$ : → ⊃

$(\forall a \forall b \vdash bfa \supset \vdash fbfa), (\forall b \vdash bfa \supset p), \vdash bfa \to \vdash b\Box p$ :DB

$(\forall a \forall b \vdash bfa \supset \vdash fbfa), (\forall b \vdash bfa \supset p), \vdash bfa \to \vdash fbp$ :Entail

$(\forall a \forall b \vdash bfa \supset \vdash fbfa), (\forall b \vdash bfa \supset p), \vdash bfa \to \vdash (fb \supset p)$ :⊢

$(\forall a \forall b \vdash bfa \supset \vdash fbfa), (\forall b \vdash bfa \supset p), \vdash bfa \to \forall c \vdash c(fb \supset p)$ :∀

$(\forall a \forall b \vdash bfa \supset \vdash fbfa), (\forall b \vdash bfa \supset p), \vdash bfa \to \vdash c(fb \supset p)$ :⊢⊃

$(\forall a \forall b \vdash bfa \supset \vdash fbfa), (\forall b \vdash bfa \supset p), \vdash bfa \to \vdash cfb \supset \vdash cp$ : → ⊃

$(\forall a \forall b \vdash bfa \supset \vdash fbfa), (\forall b \vdash bfa \supset p), \vdash bfa, \vdash cfb \rightarrow \vdash cp$    $:\forall \rightarrow$ twice

①, $\vdash *_2 f*_1 \supset \vdash f*_2 f*_1, \forall b \vdash bfa \supset p, \vdash bfa, \vdash cfb \rightarrow \vdash cp$    $:\forall \rightarrow$

①, $\vdash *_2 f*_1 \supset \vdash f*_2 f*_1$, ② $\vdash *_3(fa \supset p), \vdash bfa, \vdash cfb \rightarrow \vdash cp$    $:\supset \rightarrow$

①, $\vdash f*_2 f*_1$, ②, $\vdash *_3(fa \supset p), \vdash bfa, \vdash cfb \rightarrow \vdash cp$    :Entail

①, ②, $\vdash (f*_2 \supset f*_1), \vdash *_3(fa \supset p), \vdash bfa, \vdash cfb \rightarrow \vdash cp$    $:\vdash$

①, ②, $\forall c \vdash c(f*_2 \supset f*_1), \vdash *_3(fa \supset p), \vdash bfa, \vdash cfb \rightarrow \vdash cp$    $:\vdash \supset$

①, ②, $\forall c \vdash c(f*_2 \supset f*_1), (\vdash *_3 fa \supset \vdash *_3 p), \vdash bfa, \vdash cfb \rightarrow \vdash cp$    $:\supset \rightarrow$

①, ②, $\forall c \vdash c(f*_2 \supset f*_1), \vdash *_3 p, \vdash bfa, \vdash cfb \rightarrow \vdash cp$    $:\forall \rightarrow$

①, ②, ③, $\vdash *_4(f*_2 \supset f*_1), \vdash *_3 p, \vdash bfa, \vdash cfb \rightarrow \vdash cp$    $:\vdash \supset$

①, ②, ③, $\vdash *_4 f*_2 \supset \vdash *_4 f*_1 \ \vdash *_3 p, \vdash bfa, \vdash cfb \rightarrow \vdash cp$    $:\supset \rightarrow$

①, ②, ③, $\vdash *_4 f*_1, \vdash \underline{*_3 p}, \vdash bfa, \vdash cfb \rightarrow \vdash \underline{cp}$

$*_3 := c$ ■

①, ②, ③, $\vdash *_3 p, \vdash bfa, \vdash cfb \rightarrow \vdash cp, \vdash *_4 f*_2$

$*_3 := c$ ■ or $[*_4 = b, *_2 = a]$ or $[*_4 = c \ *_2 = b]$

①, ②, $\forall c \vdash c(f*_2 \supset f*_1), \vdash bfa, \vdash cfb \rightarrow \vdash cp, \vdash *_3 fa$    $:\forall \rightarrow$

①, ②, ④, $\vdash *_5(f*_2 \supset f*_1), \vdash bfa, \vdash cfb \rightarrow \vdash cp, \vdash *_3 fa$    $:\vdash \supset$

①, ②, ④, $\vdash *_5 f*_2 \supset \vdash *_5 f*_1, \vdash \underline{bfa}, \vdash cfb \rightarrow \vdash cp, \vdash *_3 fa$    $:\supset \rightarrow$

①, ②, ④, $\vdash *_5 f*_1, \vdash bfa, \vdash cfb \rightarrow \vdash cp, \vdash *_3 fa$

$[*_3 = *_5, *_1 := a]$ or $*_3 := b$ fail

①, ②, ④, $\vdash bfa, \vdash cfb \rightarrow \vdash cp, \vdash *_3 fa, \vdash *_5 f*_2$

$*_3 = b$ (fail) or $[*_5 := b, *_2 := a]$ (fail) or $[*_5 := c, *_2 := b]$ ■

↓

$(1), (2), \vdash *_3(fa \supset p), \vdash bfa, \vdash cfb \rightarrow \vdash cp, \vdash *_2 f *_1 \quad : \vdash \supset$

$(1), (2), \vdash *_3 fa \supset \vdash *_3 p, \vdash bfa, \vdash cfb \rightarrow \vdash cp, \vdash *_2 f *_1 \quad : \supset \rightarrow$

$(1), (2), \vdash *_3 p, \vdash bfa, \vdash cfb \rightarrow \vdash cp, \vdash *_2 f *_1$

$*_3 := c$ ■ or $[*_2 = b, *_1 = a]$ ■ or $[*_2 = c, *_1 = b]$ (fail)

$(1), (2), \vdash bfa, \vdash cfb \rightarrow \vdash cp, \vdash *_2 f *_1, \vdash *_3 fa$

$[*_2 := b, *_1 = a]$ ■ or $[*_2 := c, *_1 = b]$ (fail) or $*_3 = b$ (fail)

final substitutions

$*_1 = a$

$*_2 = b$

$*_3, *_5 = c$

6.5. <u>The deontic existnece law in Static B-Semantics</u>.

We now show that if the f constant is consistent then necessity implies possibility .

| | | |
|---|---|---|
| SBT5 | $\Diamond f \to\ \vdash \forall p(\Box p \supset \langle\Box\rangle p)$ | $: \Diamond$ |
| | $\sim\vdash\sim f \to\ \vdash \forall p \Box p \supset \langle\Box\rangle p$ | $: \sim \to$ |
| | $\to\ \vdash \sim f, \vdash \forall p \Box p \supset \langle\Box\rangle p$ | $: \vdash$ |
| | $\to \forall a \vdash a\sim f, (\vdash \forall p \Box p \supset \langle\Box\rangle p)$ | $: \to \forall$ |
| | $\to\ \vdash a(\sim f), \vdash \forall p \Box p \supset \langle\Box\rangle p$ | $: \vdash \sim$ |
| | $\to \sim\vdash af, \vdash \forall p \Box p \supset \langle\Box\rangle p$ | $: \to \sim$ |
| | $\vdash af \to\ \vdash \forall p \Box p \supset \langle\Box\rangle p$ | $: \vdash$ |
| | $\vdash af \to \forall b \vdash b(\forall p \Box p \supset \langle\Box\rangle p)$ | $: \to \forall$ |
| | $\vdash af \to\ \vdash b(\forall p \Box p \supset \langle\Box\rangle p)$ | $: \vdash \forall$ |
| | $\vdash af \to \forall p \vdash b(\Box p \supset \langle\Box\rangle p)$ | $: \to \forall$ |
| | $\vdash af \to\ \vdash b(\Box p \supset \langle\Box\rangle p)$ | $: \vdash \supset$ |
| | $\vdash af \to\ \vdash b\Box p \supset\ \vdash b\langle\Box\rangle p$ | $: \to \supset$ |
| | $\vdash af,\ \vdash b\Box p \to\ \vdash b\langle\Box\rangle p$ | :DSB |
| | $\vdash af,\ \vdash fp \to\ \vdash b\langle\Box\rangle p$ | :Entail |
| | $\vdash af,\ \vdash(f \supset p) \to\ \vdash b\langle\Box\rangle p$ | $: \vdash$ |
| | $\vdash af, \forall w \vdash w(f \supset p) \to\ \vdash b\langle\Box\rangle p$ | $: \langle\Box\rangle$ |
| | $\vdash af, \forall w \vdash w(f \supset p) \to\ \vdash b\sim\Box\sim p$ | $: \vdash \sim$ |
| | $\vdash af, \forall w \vdash w(f \supset p) \to\ \vdash b\Box\sim p$ | $: \to \sim$ |
| | $\vdash af, \forall w \vdash w(f \supset p),\ \vdash b\Box\sim p \to$ | :DSB |
| | $\vdash af, \forall w \vdash w(f \supset p),\ \vdash f\sim p$ | :Entail |
| | $\vdash af, \forall w \vdash wf \supset p,\ \vdash(f \supset \sim p) \to$ | $: \vdash$ |
| | $\vdash af, \forall w \vdash wf \supset p, \forall u \vdash u(f \supset \sim p) \to$ | $: \forall \to$ |
| | $\vdash af$, ①, $\vdash *_1(f \supset p), \forall u \vdash u(f \supset \sim p) \to$ | $: \forall \to$ |
| | $\vdash af$, ①, $\vdash *_1(f \supset p)$, ② $\vdash *_2(f \supset \sim p) \to$ | $: \vdash \supset$ |

$\vdash af, \textcircled{1}, \vdash *_1 f \supset \vdash *_1 p, \textcircled{2}, \vdash *_2 (f \supset \sim p) \rightarrow$     $: \supset \rightarrow$

$\vdash af, \textcircled{1}, \vdash *_1 p, \textcircled{2}, \vdash *_2 (f \supset \sim p) \rightarrow$     $: \vdash \supset$

$\vdash af, \textcircled{1}, \vdash *_1 p, \textcircled{2}, \vdash *_2 f \supset \vdash *_2 \sim p \rightarrow$     $: \supset \rightarrow$

$\vdash af, \textcircled{1}, \vdash *_1 p, \textcircled{2}, \vdash *_2 \sim p \rightarrow$     $: \vdash \sim$

$\vdash af, \textcircled{1}, \vdash *_1 p, \textcircled{2}, \sim \vdash *_2 p \rightarrow$     $: \sim \rightarrow$

$\vdash af, \textcircled{1}, \vdash *_1 p, \textcircled{2}, \rightarrow \vdash *_2 p$     :Unify

$*_1 := *_2$

$\vdash af, \textcircled{1}, \vdash *_1 p, \textcircled{2} \rightarrow \vdash *_2 f$     :Unify

$*_2 := a$

$\vdash af, \textcircled{1}, \textcircled{2}, \vdash *_2 (f \supset \sim p) \rightarrow \vdash *_1 f$     $: \vdash \supset$

$\vdash af, \textcircled{1}, \textcircled{2}, \vdash *_2 f \supset \vdash *_2 \sim p \rightarrow \vdash *_1 f$     $: \supset \rightarrow$

$\vdash af, \textcircled{1}, \textcircled{2}, \vdash *_2 \sim p \quad \vdash *_1 f$     $: \vdash \sim$

$\vdash af, \textcircled{1}, \textcircled{2}, \sim \vdash *_2 p \quad \vdash *_1 f$     $: \sim \rightarrow$

$\vdash af, \textcircled{1}, \textcircled{2}, \rightarrow \vdash *_2 p, \vdash *_1 f$     :Unify

$*_1 := a$

$\vdash af, \textcircled{1}, \textcircled{2} \rightarrow \vdash *_1 f, \vdash *_2 f$     :Unify

[$*_1 := a$] or [$*_2 := a$]

## 7. Applications to Artificial Intelligence

The modal logic and semantical systems described in this paper can form the basis of a rich intensional logic containing concepts of belief, knowledge, obligation, desire, expectation, ability, action, and tense. For example, the static B-Semantics is especially effective in modeling the traditional epistomological concepts of belief and knowledge.

If we let (beliefs E) represent the beliefs of a robot E expressed as a conjunction of axioms, then we can say that a robot E believes something iff it is entailed by his beliefs:

(Believe E x) = df $\vdash$ (Beliefs E)x

Furthermore we can say that a robot E knows something iff E believes it and it is the case.

(Know E x) = df (Believes E x) $\wedge$ x

From these definitions it follows that Believe possesses properties of a deontic S5 modal logic, [10] and Know posseses the properties of an S4 modal logic. [10]

Furthermore since these concepts are essentially defined in our modal logic using a rewrite rule such as DK, DSK, DB, or DSB and since our modal logic is essentially a system of rewrite rules proving theorems about these concepts is straightforward, as we have demonstrated in the example proofs in section six.

Our Modal Logic is also potentially useful in any scientific theory, which one intends to implement as a program, and which involves both the expressions of a language, and the meanings of those expressions. The reason for this is that our modal logic, because it captures the notion of logical truth, when supplemented by a meaning function [1] is both adequate to represent reasoning about expressions and their meanings, and is computationally better than any other known system which is capable of such reasoning. The reason our modal logic is computationally better than other systems of metatheoretic reasoning such as model theory, is due to its inherent simplicity and the existence of a sequent calculus for it. Model Theory, which depends on set theory, by comparison is extremely complex.

One particular area in which such metatheoretic reasoning seems especially important, is in constructing theories of natural language understanding. It is easy to see that such a theory must involve both natural language expressions and their meanings. An initial proposal for the use of metatheoretic reasoning in natural language understanding is given in [12, 13].

## 8. Conclusion

We have described a sequent calculus for the modal logic which captures the notion of logical truth. We have shown how several semantical systmes for representing intensional concepts can be defined in this modal logic in a direct manner. This then provides an efficient sequent calculus for proving theorems about these intensional concepts.

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