

FREQUENCY DOMAIN CONSIDERATIONS OF
LSV DIGITAL FILTERS

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Abstract

The present paper develops a framework for the analysis and synthesis of linear shift-variant (LSV) digital filters in the frequency domain. First, LSV digital filters are theoretically modeled by the successive use of linear shift-invariant filters. On the basis of the model, we present an interpretation of shift-variant spectral modification or filtering. Further, shift-variant digital filtering is discussed in relation to the notions of the short-time spectrum and the generalized frequency function. In addition, we propose an efficient implementation procedure which reduces the number of filter coefficients and the amount of computation. The effectiveness of LSV digital filters in processing time-varying signals is demonstrated by experimental verification.

I. Introduction

Linear shift-invariant (LSI) digital filters have become important tools in a multitude of diverse fields of science and technology. Often, the use of LSI digital filters is insufficient to process various kinds of signals. In seismic data processing, for example, linear shift-variant (LSV) digital filters have been extensively used [1]-[4]. Thus, it is of practical and theoretical interest to study LSV digital filters [1]-[7].

Two of the most important applications of digital filters are system identification and modeling [8], and spectral modification [9]. In system identification, the objective is to find some parameters such as impulse responses and coefficients of difference equations to simulate the characteristics of practical systems. The representation of the system in terms of an LSI digital filter follows directly from the assumed stationarity of the system. In many applications, however, LSV digital filters may offer a more accurate representation of the system because of the presence of nonstationary components in some practical systems. As an example, LSV digital filters have been used to model the vocal tract in a speech analysis and synthesis system [10].

Alternatively, the objective of spectral modification or filtering is directed toward removing interference such as noise from the signal, or modifying the signal to present it in a form which is more easily interpreted by a human expert. Conventionally, the spectral modification is realized in terms of LSI digital filters either in time or frequency domains. In some practical applications, the frequency content of the desired signal changes significantly with time. Under these circumstances,

the use of an LSV digital filter is more effective than the use of LSI filters.

The purpose of this paper is to investigate several aspects of spectral modification using an LSV digital filter. To do so, we characterize a desired signal in terms of a time-dependent spectrum. Then we modify the spectrum with an LSV digital filter. In this way, the modification of the spectrum becomes a function of time. A schematic representation of a filtering problem is shown in Figure 1. The input signal $x(n)$ of the filter in Figure 1 can be expressed as $x(n) = s(n) + v(n)$, where $s(n)$ is the desired signal or useful information and $v(n)$ is the noise or unwanted information. The filter is required to produce an output which is some function of the signal with a delay of n_0 samples. The desired filter is then defined as the weighting function which minimizes the mean square of the error function.

$$\epsilon(n) = y(n) - g[x(n-n_0)], \quad (1)$$

where $\epsilon(n)$ is the difference between the actual and desired outputs.

Thus far, the LSI filter has played a dominant role in dealing with the problem. The desired characteristics of the optimal LSI filter can be determined either in time or in frequency domains. Often, it is easier to do so by working with the power spectra of the signal $s(n)$ and the noise $v(n)$.

Although several time-domain techniques for processing time-varying signals and synthesizing shift-variant filters have been proposed in the literature [4], little work has been done entirely in the frequency domain. Recent research [11]-[14] on the theory of short-time spectral

analysis has established a framework for efficient and accurate frequency domain analysis of time-varying signals. Since the short-time spectrum of a signal is a function of time, this suggests that shift-variant spectral modification is desirable and that the frequency characteristics of the optimal filter are shift-variant. In this paper, the concepts of shift-variant digital filters and short-time spectrum are brought together to provide a description of the effects of shift-variant spectral modification.

In Section II, LSV digital filters are theoretically modeled by the successive use of LSI digital filters. In the model, the effects of shift-variant spectral modification or filtering can be easily understood from the viewpoint of usual shift-invariant digital filtering. In Section III, shift-variant digital filtering is discussed in relation to the notions of the short-time spectrum and the generalized frequency function. It is shown that shift-variant spectral modification can be implemented as the convolution of the input signal with the impulse response of a nonrecursive LSV digital filter. In Section IV, we formulate an implementation procedure which is significantly more efficient than the shift-variant convolution as the computation and storage of filter coefficients are taken into account. In Section V, the effectiveness of spectral modification using an LSV digital filter is demonstrated by a simulation program with a synthetic time-varying signal.

II. A Model of LSV Digital Filters

A linear shift-variant digital filter is conveniently characterized in terms of the shift-variant convolution. Let $x(n)$ and $y(n)$ denote, respectively, the input and output signals of an LSV digital filter; they can be related by

$$y(n) = \sum_{m=-\infty}^{\infty} h(n, n-m) x(m), \quad (2)$$

where $h(n, m)$, the impulse response of the filter, is defined as the output measured at time n due to a unit impulse applied at time $n-m$.

Although the effects of spectral modification by LSI digital filters are widely known, the problem of applying an LSV digital filter to modify the spectrum of a signal has not received serious attention. Therefore, it is advantageous to utilize the theory of LSI filters in investigating LSV digital filters. Before presenting the general discussion of LSV digital filters, we first model them in terms of LSI filters. This model is depicted in Figure 2. What we do is multiply a sliding rectangular window function $w(k-n)$ to the input signal $x(n)$ and then move the window one sample ahead each time. As a result, it produces a sequence of overlapping sections as the window slides in time. Each section $u_k(n)$ is then convolved with a corresponding LSI filter with an impulse response $g_k(n)$. The final output $y(n)$ is obtained by fitting the filtered sections $z_k(n)$ together with a sequentially rotating switch. Our procedure of segmenting the input signal is slightly similar to the sectioned convolution technique in the shift-invariant convolution [15]. In our model of an LSV digital filter, however, each input section is different from the previous one

by one sample only. In addition, each section is applied respectively to a corresponding LSI filter associated with its unique characteristic.

Figure 2 illustrates the procedure of segmenting the input signal and fitting the filtered sections together. To be more specific, let us decompose the input signal $x(n)$ into a sequence of overlapping sections, each section having only $(2L+1)$ nonzero points, with the k^{th} section denoted by $u_k(n)$,

$$u_k(n) = \begin{cases} x(n), & k-L \leq n \leq k+L \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

Here, the time origin for each section $u_k(n)$ is defined to be at the origin of $x(n)$. The segmented section may be viewed as the multiplication of the input signal $x(n)$ and a sliding window function $w(k-n)$; i.e.,

$$u_k(n) = x(n) w(k-n), \quad (4)$$

where $w(\ell)$ is a $(2L+1)$ -point rectangular window function

$$w(\ell) = \begin{cases} 1, & -L \leq \ell \leq L \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

In Figure 2, the resulting output signal $z_k(n)$ of the LSI filter is a linear convolution of the impulse response and segmented signal, i.e.,

$$z_k(n) = \sum_{m=-\infty}^{\infty} g_k(n-m) u_k(m). \quad (6)$$

The final output signal $y(n)$ is constructed from the filtered sections by a sequentially rotating switch such that the switch is connected with the n^{th} section at time n . Then we have the following relation:

$$y(n) = z_n(n). \quad (7)$$

Substituting (6) into (7), we obtain

$$\begin{aligned} y(n) &= \sum_{m=-\infty}^{\infty} g_n(n-m) w(n-m) x(m) \\ &= \sum_{m=-\infty}^{\infty} h(n,n-m) x(m). \end{aligned} \quad (8)$$

The above discussion shows that the configuration of Figure 2 corresponds to a nonrecursive LSV digital filter with an impulse response determined from the window function and the impulse responses of LSI filters. According to (8), the impulse response $h(n,m)$ of the corresponding LSV digital filter is defined as

$$h(n,m) = g_n(m) w(m). \quad (9)$$

Figure 3 shows an LSV digital filter having the same characteristics as a bank of LSI filters illustrated in Figure 2. It follows from (9) that the impulse response $h(n,m)$ of the LSV digital filter at time n is equivalent to the impulse response of the n^{th} LSI filter multiplied by the window function.

The objective of the model is to investigate the effects of shift-variant spectral modification. To this end, the linear convolution in (6) is transformed into a multiplication in the frequency domain. Upon application of the concept of the Fourier transform, it is straightforward to show that

$$Z_k(e^{j\phi}) = G_k(e^{j\phi}) U_k(e^{j\phi}), \quad (10)$$

where

$$Z_k(e^{j\phi}) = \sum_{m=-\infty}^{\infty} z_k(m) e^{-jm\phi}, \quad (11)$$

$$G_k(e^{j\phi}) = \sum_{m=-\infty}^{\infty} g_k(m) e^{-jm\phi}, \quad (12)$$

and

$$U_k(e^{j\phi}) = \sum_{m=-\infty}^{\infty} u_k(m) e^{-jm\phi}. \quad (13)$$

$Z_k(e^{j\phi})$, $G_k(e^{j\phi})$ and $U_k(e^{j\phi})$ are Fourier transforms of sequences $z_k(n)$, $g_k(n)$ and $u_k(n)$ respectively. Substituting (4) into (13), we have

$$U_k(e^{j\phi}) = \sum_{m=-\infty}^{\infty} x(m) w(k-m) e^{-jm\phi} = \sum_{m=k-L}^{k+L} x(m) e^{-jm\phi}. \quad (14)$$

Equation (14) shows that $U_k(e^{j\phi})$ can be viewed as the spectrum of the input signal at time k . At each time, the signal spectrum is modified by the frequency response of an LSI filter as seen in (10). Thus the spectral modification becomes a function of time.

By taking the inverse Fourier transform of (10) and substituting it into (7), the final output $y(n)$ is

$$y(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} G_n(e^{j\phi}) U_n(e^{j\phi}) e^{j\phi n} d\phi. \quad (15)$$

As a result, we conclude that the shift-variant convolution as presented in Figure 2 is equivalent to the shift-variant modification of the signal spectrum in the frequency domain. The shift-variant filtering process can be implemented either by a bank of LSI filters as depicted in Figure 2 or by a single LSV digital filter as shown in Figure 3. Thus far, the

measurement of the signal spectrum is achieved by using a rectangular data window. It is unnecessary to restrict the data window to the form (5). Moreover, the rectangular data window usually gives unsatisfactory results because of its sharp edges. In the following section, the discussion of shift-variant digital filtering will be generalized to include the notions of the short-time spectrum as well as the generalized frequency function.

III. Spectral Modification Using LSV Digital Filters

The Fourier representation of a signal has proved to be useful in linear system theory. For an arbitrary signal $x(n)$ which is absolutely summable, it can be represented by its Fourier transform $X(e^{j\phi})$, which is usually interpreted as a measure of the frequency content of the signal. For example, slowly varying signals have Fourier transforms which are mostly concentrated near the origin of the frequency axis and rapidly varying signals have more frequency content near the Nyquist frequency.

For a time-varying signal, its Fourier transform is not so meaningful as the frequency content of the signal changes significantly with time. Recently, several techniques have been developed to characterize a time-varying signal in the frequency domain. In [16], a method is proposed to derive the instantaneous frequency of a time-varying signal using an adaptive linear prediction filter. Another method which is of particular interest to this paper is the short-time spectrum [11]-[14], defined as

$$X(e^{j\phi}, n) = \sum_{m=-\infty}^{\infty} x(m) w(n-m) e^{-jm\phi} \quad (16)$$

where $w(\ell)$ is an appropriately chosen window function. According to (16), $X(e^{j\phi}, n)$ is obtained by weighting the input signal $x(n)$ with a sliding window function $w(n-m)$ and then Fourier transforming the windowed segment. Without loss of generality, the window function $w(\ell)$ can be normalized so as to satisfy the condition

$$w(0) = 1. \quad (17)$$

Multiplying (16) by $(1/2\pi) \exp(j\phi m)$ and integrating over $-\pi \leq \phi \leq \pi$, we obtain

$$x(m)w(n-m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\phi}, n) e^{j\phi m} d\phi. \quad (18)$$

Thus the original signal can also be exactly recoverable from its short-time spectrum defined in (16). The technique of short-time spectrum has been widely used in the analysis and synthesis of time-varying signals. In this section, we are interested in being able to modify the short-time spectrum of a signal using an LSV digital filter.

Now we illustrate the frequency representations of a time-varying signal with the following example. The example will also serve to illustrate the ideas to be presented in this paper. Consider a finite duration time-varying signal of the form:

$$x(n) = \begin{cases} \exp[-\pi\gamma n^2 T^2] \cdot \cos[2\pi(\nu nT + \beta n^2 T^2/2)], & -128 \leq n \leq 127 \\ 0, & \text{elsewhere.} \end{cases} \quad (19)$$

Expression (19) gives an example of a signal with a varying frequency and a varying amplitude. The frequency of the signal varies linearly with time. In order to visualize the frequency representations of the time-varying signal, we have chosen the following numerical values:

$$\begin{aligned} \gamma &= 0.01 \\ \beta &= 1.5 \\ \nu &= 0 \\ T &= 0.05 \end{aligned} \quad (20)$$

Notice that the signal specified in (19) and (20) is symmetrical about the time axis. It varies slowly near the center of the time axis and gradually becomes a rapidly varying signal as time shifts in both directions. Figure 4 shows the amplitude of the Fourier transform of the signal specified in (19) and (20). As shown in Figure 4, the Fourier

transform of the signal is not a useful measure of the frequency content of the time-varying signal. Through the Fourier transform, frequency is defined over an entire time interval. For a time-varying signal, however, we are particularly interested in measuring its frequency content at various values of time.

To illustrate the short-time spectrum of the signal specified in (19) and (20), an appropriate window function has to be selected. In the remainder of the section, we choose a Hamming window of the form

$$w(\ell) = \begin{cases} 0.54 + 0.46 \cos\left(\frac{2\pi\ell}{2L+1}\right), & -L \leq \ell \leq L \\ 0.0 & \text{elsewhere,} \end{cases} \quad (21)$$

with $L = 16$. Figure 5 shows the amplitude of the short-time spectrum as a function of ϕ , with n as a parameter. As shown in Figure 5, the time-varying frequency content of the signal can be observed from $|X(e^{j\phi}, n)|$. Therefore, the short-time spectrum is a more useful measure of the frequency content of a time-varying signal.

To demonstrate the need for shift-variant digital filters, we now consider a simple filtering problem as illustrated in Figure 6, where the filter input $x(n)$ is represented as a combination of a desired time-varying signal $s(n)$ and an unwanted interference $v(n)$. Here the objective of the filtering is to remove the interference from the input signal. Suppose that the Fourier transform of the desired signal is concentrated within a particular frequency band, then an appropriate frequency selective shift-invariant filter can be chosen to remove undesirable frequency components outside this band. For a time-varying signal as shown in Figures 4 and 5,

however, the frequency content of the signal changes with time. To achieve better results, it may be desirable to employ a shift-variant filter where the frequency characteristics change with time in a prescribed manner. In this aspect, our concept is similar to that of the dynamic tracking filter used to demodulate an f.m. (frequency modulated) continuous signal [17],[18]. However, the discussions in [17] and [18] are restricted to the analysis of particular circuits which can perform dynamic filtering operation.

In an attempt to generalize the discussion of shift-variant filtering in the frequency domain, we characterize the LSV digital filter by its generalized frequency function $H(e^{j\phi}, n)$, which is defined as [5]

$$H(e^{j\phi}, n) = \sum_{m=-\infty}^{\infty} h(n, m) e^{-jm\phi}. \quad (22)$$

From (22), it can be observed that $H(e^{j\phi}, n)$ is the Fourier transform of the sequence $h(n, m)$ with respect to the variable m . With this observation, it follows that the impulse response $h(n, m)$ can be evaluated from $H(e^{j\phi}, n)$ by means of inverse Fourier transform, regarding n as a constant,

$$h(n, m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\phi}, n) e^{jm\phi} d\phi. \quad (23)$$

In case that the filter is shift-invariant, both the generalized frequency function $H(e^{j\phi}, n)$ and the impulse response $h(n, m)$ would be independent of n .

As shown in Figure 5, $X(e^{j\phi}, n)$ represents the spectrum of the input signal at time n . Suppose that the spectrum is subjected to a shift-variant modification $H(e^{j\phi}, n)$, then let us define an output spectrum $Y(e^{j\phi}, n)$ as

$$Y(e^{j\phi}, n) = X(e^{j\phi}, n) H(e^{j\phi}, n). \quad (24)$$

By comparing (24) with (10), it is clear that $Y(e^{j\phi}, n)$ corresponds to the output spectrum $Z_n(e^{j\phi})$ of an LSI filter as shown in Figure 2. With the help of the relation (15), the output signal $y(n)$ can be exactly recoverable from $Y(e^{j\phi}, n)$.

Then we have

$$y(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j\phi}, n) e^{j\phi n} d\phi. \quad (25)$$

Substituting (24), (16) into (25) and using (23), we show that

$$y(n) = \sum_{m=-\infty}^{\infty} x(m) w(n-m) h(n, n-m) \quad (26)$$

$$= \sum_{m=-\infty}^{\infty} x(m) \hat{h}(n, n-m) \quad (27)$$

where $\hat{h}(n, m)$ is defined as

$$\hat{h}(n, m) = h(n, m) w(m). \quad (28)$$

In the proposed approach to the shift-variant filtering process, the output signal $y(n)$ is obtained by convolving a windowed segment of the input signal centered at time n with the impulse response $h(n, m)$ as shown in (26). This process can also be realized as the shift-variant convolution of the input signal $x(n)$ with an impulse response $\hat{h}(n, m)$ which is the product of the sequences $h(n, m)$ and $w(m)$ as derived in (27). It is also possible to implement the shift-variant convolution in terms of the recursive shift-variant difference equation, as is discussed in [5].

With respect to the frequency domain, this approach is equivalent to a shift-variant modification of the signal spectrum by a generalized frequency function as shown in (24). By modifying the spectrum of the input signal, the filter can remove undesirable frequency components from the input signal. The proposed technique allows the spectral modification to be a function of time abreast with the changing frequency content of the signal. The overall advantage is that the resultant bandwidth of the

shift-variant digital filter is, in general, much narrower than that of the shift-invariant filter. Consequently, an LSV digital filter can contribute to removing more undesirable frequency components to obtain better result [17],[18].

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IV. Implementation of LSV Digital Filters

In previous sections we have pointed out that two approaches can be employed to directly implement the shift-variant filtering process. Although these two approaches have been proved to be useful for theoretical considerations, they may not be efficient for practical implementation. The major disadvantage of direct implementation is that a great number of filter coefficients need to be computed and stored. In some practical applications, it is desirable to filter an input signal with very long duration. While theoretically we can compute and store the filter coefficients for each sampling instant and then implement the procedure as discussed in Sections II and III, such a shift-variant filter is not feasible in practical use. In this section we formulate an implementation procedure which is significantly more efficient than the direct procedure.

By reference to the model shown in Figure 2, we note that the window function $w(\ell)$ is moved one sample ahead each time. From the viewpoint of implementation, the redundancy in the amount of computation and storage of filter coefficients is obvious due to the overlap between two successive sections. To reduce the overlap, one would expect to move the window N samples each time. In this way, it is important to determine an optimum value of N so that the output signals of LSI filters can completely specify the desired output signals. In addition, an appropriate technique is required for constructing the desired output signals from those of LSI filters. A conventional approach to achieve this aim is to apply a linear merging technique to the overlapping region of two successive sections [1],[2]. With this approach, however, the transitions in overlapping

regions may be apparent due to the stepwise changes in the characteristics of the corresponding LSI filters. Based on the preceding analysis, we now explore an implementation procedure which allows the outputs of LSV digital filters to be computed from those of LSI filters using an interpolation scheme.

To start with the discussion of the implementation procedure, let the short-time spectrum $X(e^{j\phi}, n)$ defined in (16) be rewritten as the linear convolution of the signal $x(n)e^{-j\phi n}$ with impulse response $w(n)$:

$$X(e^{j\phi}, n) = [x(n)e^{-j\phi n}] * w(n), \quad (29)$$

where $*$ is used to denote discrete convolution, and the window function $w(\ell)$ is generally chosen to approximate the impulse response of an ideal lowpass filter with cutoff frequency W_1 . By considering ϕ as a parameter, the short-time spectrum $X(e^{j\phi}, n)$ defined in (29) may be viewed as an output sequence of a lowpass filter. From (29), it is clear that $X(e^{j\phi}, n)$ is an approximately bandlimited sequence in n with bandwidth $2W_1$ [12]. In several practical situations, the desired characteristics of LSV filters usually change slowly with time. In this case, it is reasonable to assume that the generalized frequency function $H(e^{j\phi}, n)$ approximates a bandlimited sequence for any constant value of ϕ . Under this condition, let the bandwidth of the sequence $H(e^{j\phi}, n)$ be $2W_2$. Then, with the value of ϕ fixed, it is easy to show that the output spectrum $Y(e^{j\phi}, n)$ as defined in (24) will have a bandwidth $2W$ where

$$W = W_1 + W_2. \quad (30)$$

According to the sampling theorem, $Y(e^{j\phi}, n)$ can be expanded in terms of its sampled function $Y(e^{j\phi}, mN)$ with respect to the variable n as follows:

$$Y(e^{j\phi}, n) = \sum_{m=-\infty}^{\infty} Y(e^{j\phi}, mN) \frac{\sin \pi(n-mN)}{\pi(n-mN)}, \quad (31)$$

where N is an integer less than or equal to π/W . It is clear from (31) that the output spectrum $Y(e^{j\phi}, n)$ is completely characterized by its sampled function $Y(e^{j\phi}, mN)$. In addition, the equation provides an interpolation formula for recovering the output spectrum from its samples. By substituting (31) into (25), the output signal $y(n)$ is

$$y(n) = \sum_{m=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j\phi}, mN) \frac{\sin \pi(n-mN)}{\pi(n-mN)} e^{j\phi n} d\phi. \quad (32)$$

As mentioned in Section III, $Y(e^{j\phi}, mN)$ corresponds to the output spectrum $z_{mN}(e^{j\phi})$ of an LSI filter in Figure 2. Interchanging the order of summation and integration in (32), we have

$$y(n) = \sum_{m=-\infty}^{\infty} z_{mN}(n) \frac{\sin \pi(n-mN)}{\pi(n-mN)}, \quad (33)$$

where $z_{mN}(n)$ are the output signals of an LSI filter applied at time mN as shown in Figure 2. Thus $y(n)$ can be uniquely determined from the output signals of LSI filters applied at any time which is an integer multiple of N . However, it is impossible to evaluate $y(n)$ from (33) because the interpolating function $\sin(\pi n)/\pi n$ has infinite duration. Rather than simply truncate the interpolating function, it is more reasonable to design a finite duration interpolator. Notice that the function $\sin(\pi n)/\pi n$ is the impulse response of an ideal lowpass filter $F_{\text{ideal}}(e^{j\phi})$

$$F_{\text{ideal}}(e^{j\phi}) = \begin{cases} N, & |\phi| \leq \frac{\pi}{N} \\ 0, & \frac{\pi}{N} < |\phi| \leq \pi. \end{cases} \quad (34)$$

Therefore, we may employ one of the techniques in [19] for designing the finite impulse response (FIR) lowpass filters to design the finite duration interpolator. Let $f(i)$ be a $(2M+1)$ -point sequence which approximates the impulse response of the ideal lowpass filter $F_{ideal}(e^{j\phi})$, then equation (33) can be approximated by

$$\hat{y}(n) = \sum_{m=-M}^M z_{mN}(n) f(n-mN), \quad (35)$$

where $\hat{y}(n)$ is an approximated value of $y(n)$ and the approximation is attributed to the substitution of a finite duration sequence for the infinite duration interpolating function. By proper choice of the duration of the interpolator, the difference between $y(n)$ and $\hat{y}(n)$ can be made negligible. Notice that the number of multiplications and additions required in (35) is proportional to M/N .

The notion of the implementation technique is best explained with the use of the example illustrated in Figure 7. Figure 7a shows a finite duration interpolating function $f(i)$ and Figure 7b shows an array on which the output signals $z_k(n)$ of LSI filters are computed. In considering $z_k(n)$ as an array, we will refer to n as a row index and k as a column index. In this fashion, the k^{th} row of the array represents the output signals of the k^{th} LSI filter. It is clear from (7) that the output signals $y(n)$ of an LSV digital filter are identical to the diagonal elements of the array. In this section, we present a technique for interpolating $z_{mN}(n)$ to obtain $y(n)$. For the illustration of the interpolating procedure, it may be instructive to define the concept of a mask as in Figure 7b. Let the duration of the mask be the same as that of the interpolating function. To compute the output value $\hat{y}(n)$, the

mask is placed over the appropriate samples of the array. Each sample $z_{mN}(n)$ in the mask is then multiplied by the corresponding sample of the interpolating function and the products are summed to give $\hat{y}(n)$. Successive output values of $\hat{y}(n)$ can be obtained by shifting the mask one sample ahead in both horizontal and vertical directions, and repeating the process.

One major advantage of the present approach is that the number of filter coefficients computed and stored is reduced by a factor of N . To conclude this section, we summarize the present implementation procedure as follows. First, the input signal $x(n)$ is decomposed into a number of overlapping sections $u_{mN}(n)$ where the beginning of each input section is separated from that of its neighbors by N samples. Each input section $u_{mN}(n)$ is then filtered with an impulse response $[g_{mN}(n) \cdot w(n)]$ and thus we obtain an output section $z_{mN}(n)$. The final output $\hat{y}(n)$ is constructed from $z_{mN}(n)$ using a finite duration interpolator.

V. Numerical Results

To investigate the result as discussed above, we apply the time-varying signal specified in (19) and (20) to the filter illustrated in Figure 6. In general, the frequency characteristics of a linear filter are specified from the characteristics of both signal and noise. In the following examples, we assume that $v(n) = 0$, i.e., the noise is not taken into account. In this situation, the objective is to design an ideal shift-variant bandpass digital filter which allows the signal passing without any distortion. The cutoff frequencies of the ideal bandpass filter are determined from the short-time spectrum of the input signal. To do so, we define

$$E(n) = \int_0^{\pi} |X(e^{j\phi}, n)|^2 d\phi \quad (36)$$

and

$$\hat{E}(n) = \int_{\phi_L(n)}^{\phi_H(n)} |X(e^{j\phi}, n)|^2 d\phi, \quad (37)$$

i.e., $E(n)$ is the integration of frequency components of $|X(e^{j\phi}, n)|^2$ from 0 to π , while $\hat{E}(n)$ is that of frequency components from lower to higher cutoff frequencies. In our example, we specify the cutoff frequencies $\phi_H(n)$ and $\phi_L(n)$ so that at each time n ,

$$\hat{E}(n) \geq 0.98 E(n) \quad (38)$$

and the bandwidth

$$BW(n) = \phi_H(n) - \phi_L(n) \quad (39)$$

is minimized. Figure 8 presents the resultant cutoff frequencies which change with time n . With the cutoff frequencies of the ideal bandpass filter

specified, the impulse response $\hat{h}(n,m)$ can be derived from (23) and (28). The output signal $y(n)$ is implemented as a shift-variant convolution of the input signal with $\hat{h}(n,m)$ as shown in (27).

To evaluate the performance of this new spectral technique for shift-variant filtering process, we define a measure criterion as

$$ASE = \frac{\sum_n [y(n)-s(n)]^2}{\sum_n [s(n)]^2}, \quad (40)$$

which is obtained as the summation of the squared error divided by the summation of the square of the signal. Examining the input and output signals, we compute the value of ASE for direct implementation and obtain

$$ASE = 0.00451177 \quad (41)$$

Equation (41) demonstrates that our proposed spectral technique for the shift-variant filtering process is promising and significant. The resultant small error is attributed to the specification of the cutoff frequencies in (38) because only 98% of the frequency components of the signal are allowed to pass the filter.

To verify the simplified implementation and to compare its result with that of the direct implementation, we implement the above example with the procedure presented in Section IV. The interpolating function $f(n)$ used in (35) is a 33-point nonrecursive lowpass filter designed by using the McClellan algorithm [20]. The value of N in (35) is chosen equal to 8. Following the implementation procedure and using (40), we compute the value of ASE for the simplified implementation and obtain

$$ASE = 0.00348408. \quad (42)$$

The simplified implementation is particularly useful for those signals where the frequency content changes very slowly with time.

In this example, although a shift-invariant digital filter can also be designed to pass the time-varying signal without much distortion, its bandwidth is much wider than that of the shift-variant filter, as can be observed from the signal spectra shown in Figures 4 and 5. Therefore, we conclude that the shift-variant filter is more effective in processing time-varying signals.

VI. Discussion and Conclusion

In conventional signal processing applications, the signal spectrum is measured through Fourier transforming a large number of samples. The desired frequency characteristics of the LSI digital filter are then specified from the resultant spectrum. However, as discussed in this paper, this technique may not work under the circumstances where the frequency content of the signal varies with time. In our approach, the time-varying signal is first analyzed in terms of its short-time spectrum. Thus the desired frequency characteristics of the LSV digital filter are determined from the short-time spectrum. In the present technique, it is essential that the performance of an LSV digital filter be closely related to the extent that the short-time spectrum represents the frequency content of a time-varying signal.

The short-time spectrum may be viewed as measuring the infinite-time Fourier transform of the signal, seen through a window. In general, the width of the window function exerts an influence on the characteristics of the short-time spectrum. As we increase the width of the window, or equivalently, the number of signal samples, the short-time spectrum will encompass more frequency components of the time-varying signal. In this way, the outcome is undesirable due to the increase of the bandwidth of the corresponding LSV digital filter. Decreasing the width of the window, on the other hand, may raise another issue concerning the resolution of frequencies. This phenomenon is a consequence of Gabor's uncertainty relation [21], which states that if the number of signal samples used in the spectral analysis is small, its spectrum will be obscured and diluted.

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This will also result in an increase of the bandwidth of the corresponding LSV digital filter. From the intuitive concept, one would expect a narrower window when the signal frequency is changing rapidly, and a wider window when the signal frequency is changing slowly. Therefore, techniques may be available by which the width of the window could be chosen on the basis of a priori knowledge of the signals being studied. The characteristics of the short-time spectrum will also depend on the window shape [22]. Up to now, however, the question of what the optimal window is in a particular situation is not well understood. In the absence of systematic techniques for choosing the window function, it is necessary to rely on the subjective judgment of the investigator. In most cases one would expect that the width and shape of the optimal window are functions of time since the frequency variation rate of the signal may be distinct at different instants of time. Developing a technique for selecting a proper window function is a matter of further research.

In summary, the objective of this paper has been to develop a framework for considering LSV digital filters in the frequency domain. It has been shown that the shift-variant modification of the short-time spectrum can be implemented by an LSV digital filter. From our point of view, LSV digital filters show promise for successfully processing time-varying signals, and they deserve further investigation.

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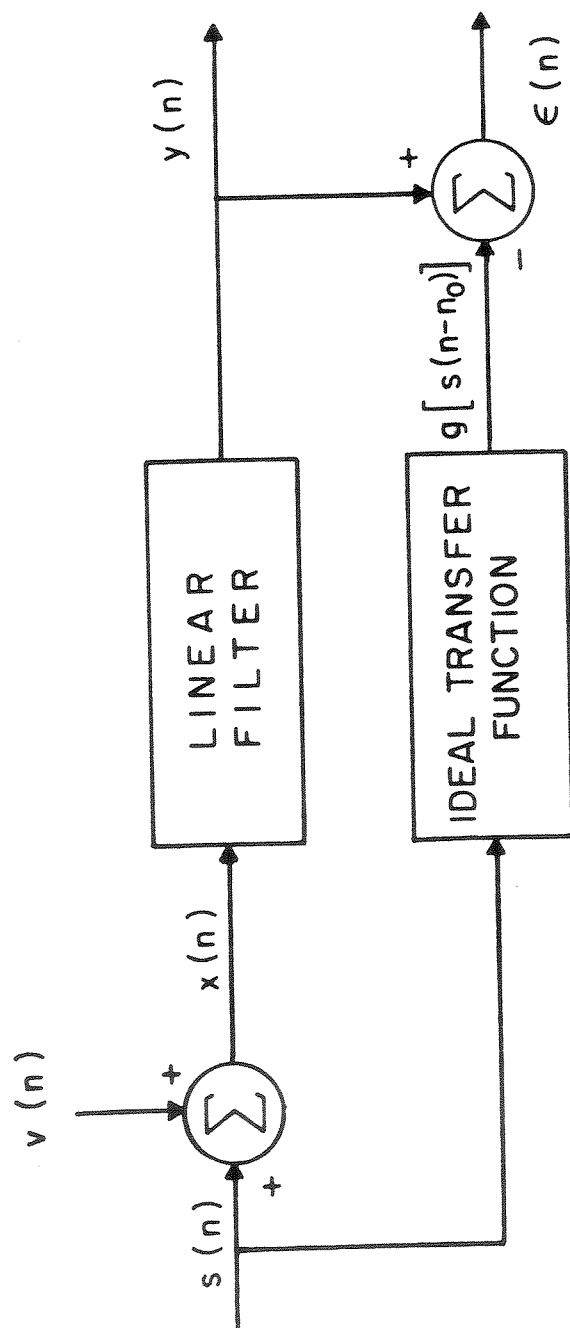


Figure 1. A schematic representation of a filtering problem.

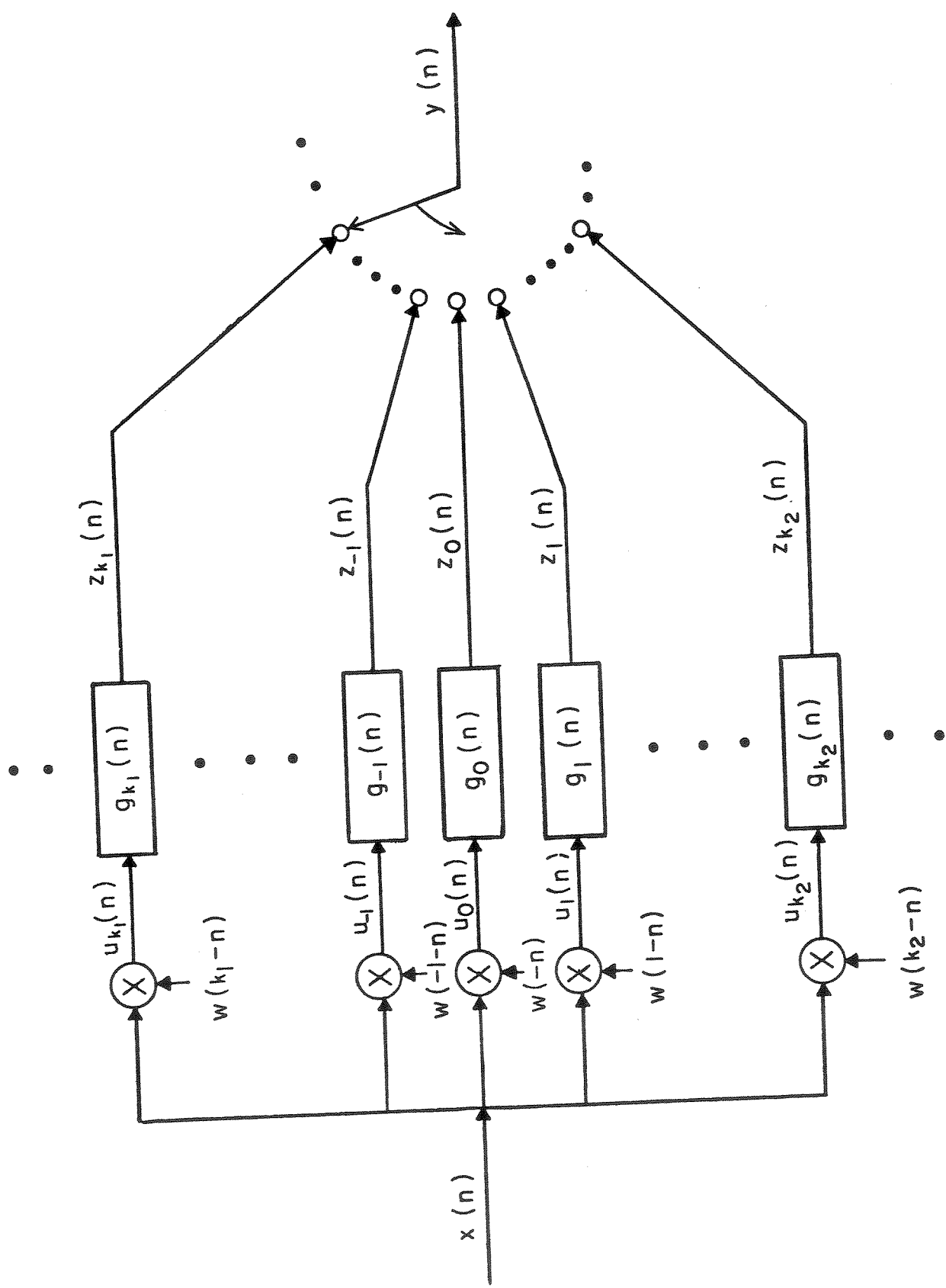


Figure 2. A model of LSV digital filters by successive use of LSI filters.

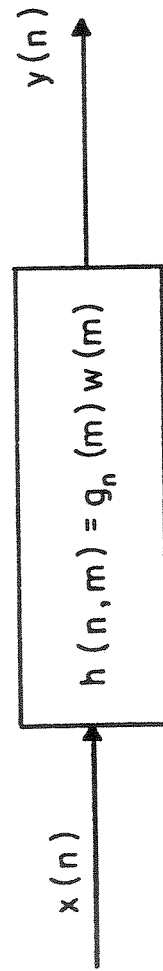


Figure 3. An LSV digital filter which is equivalent to a bank of LSI filters illustrated in Figure 2.

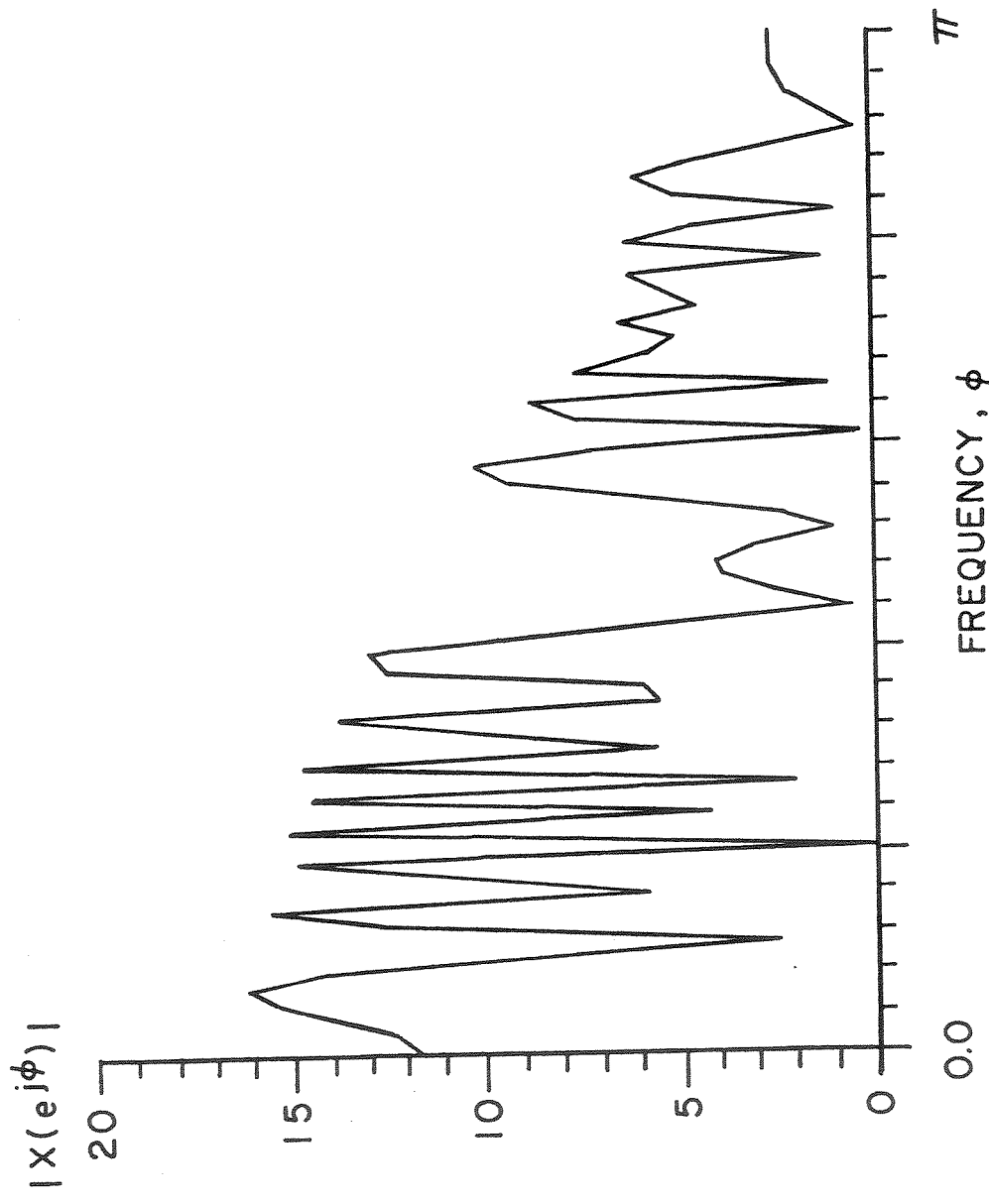


Figure 4. Amplitude of the Fourier transform of the signal specified in (19) and (20).

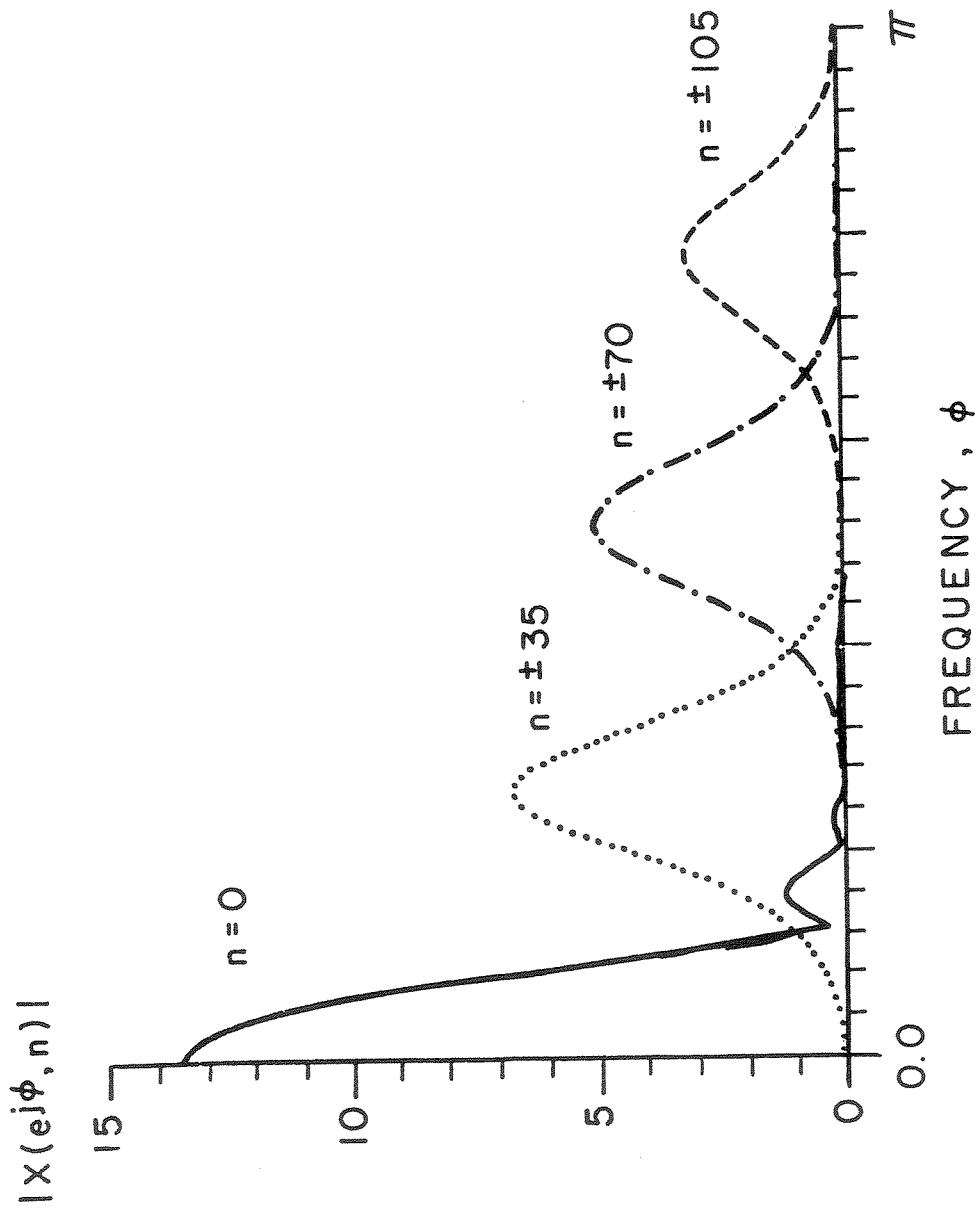


Figure 5. Amplitude of the short-time spectrum of the signal specified in (19) and (20) at $n = 0, \pm 35, \pm 70, \pm 105$.

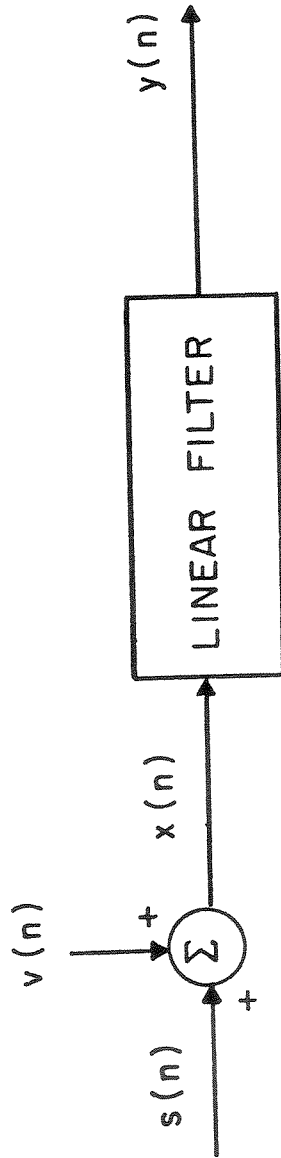


Figure 6. A simple linear filter.

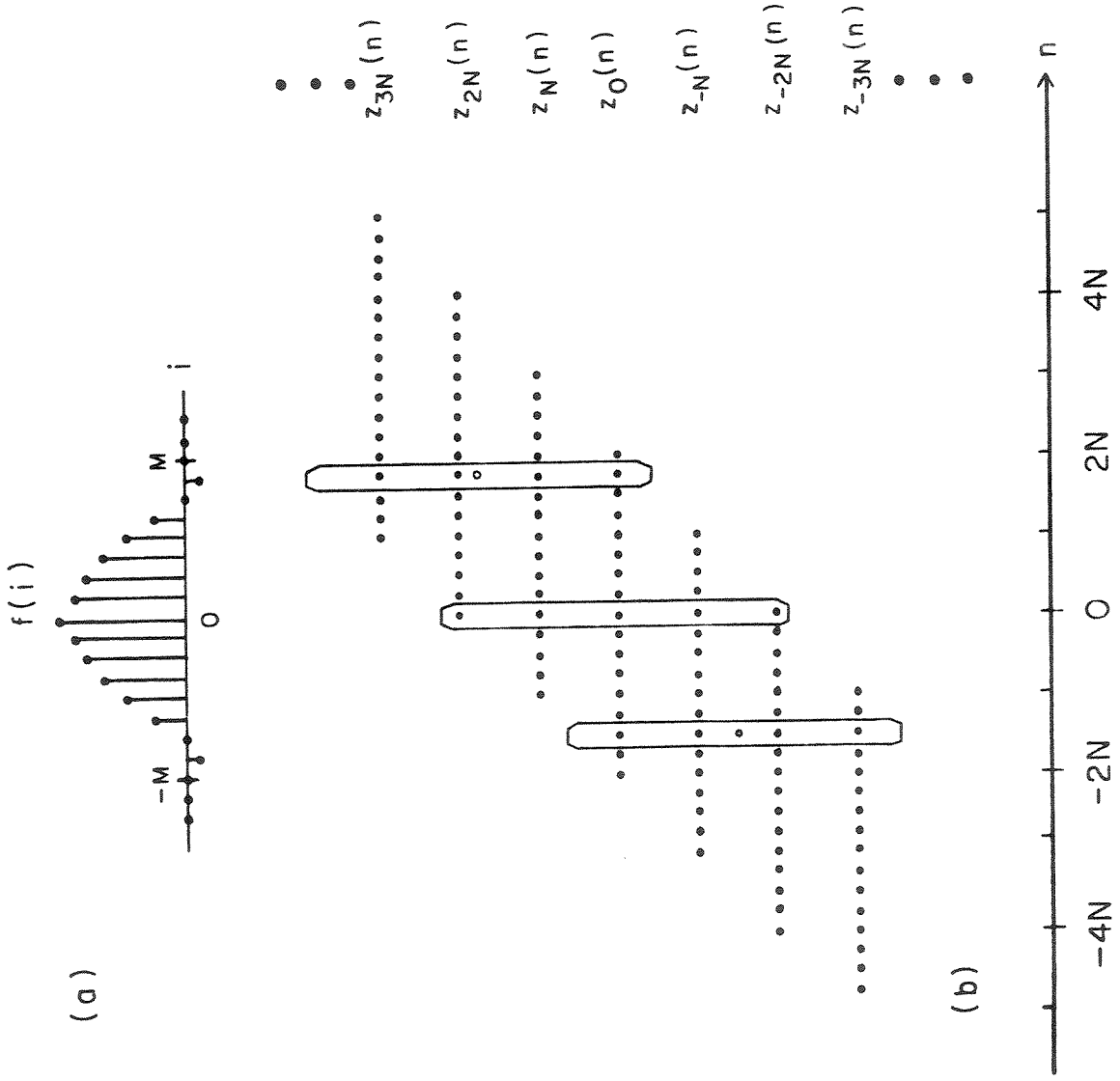


Figure 7. (a) A finite duration interpolating function $f(i)$; (b) An array on which the output signals $z_{mN}(n)$ of LSI filters are computed, and the mask that extracts $z_{mN}(n)$ which are required for interpolation using $f(i)$.

required for interpolation using $f(i)$.

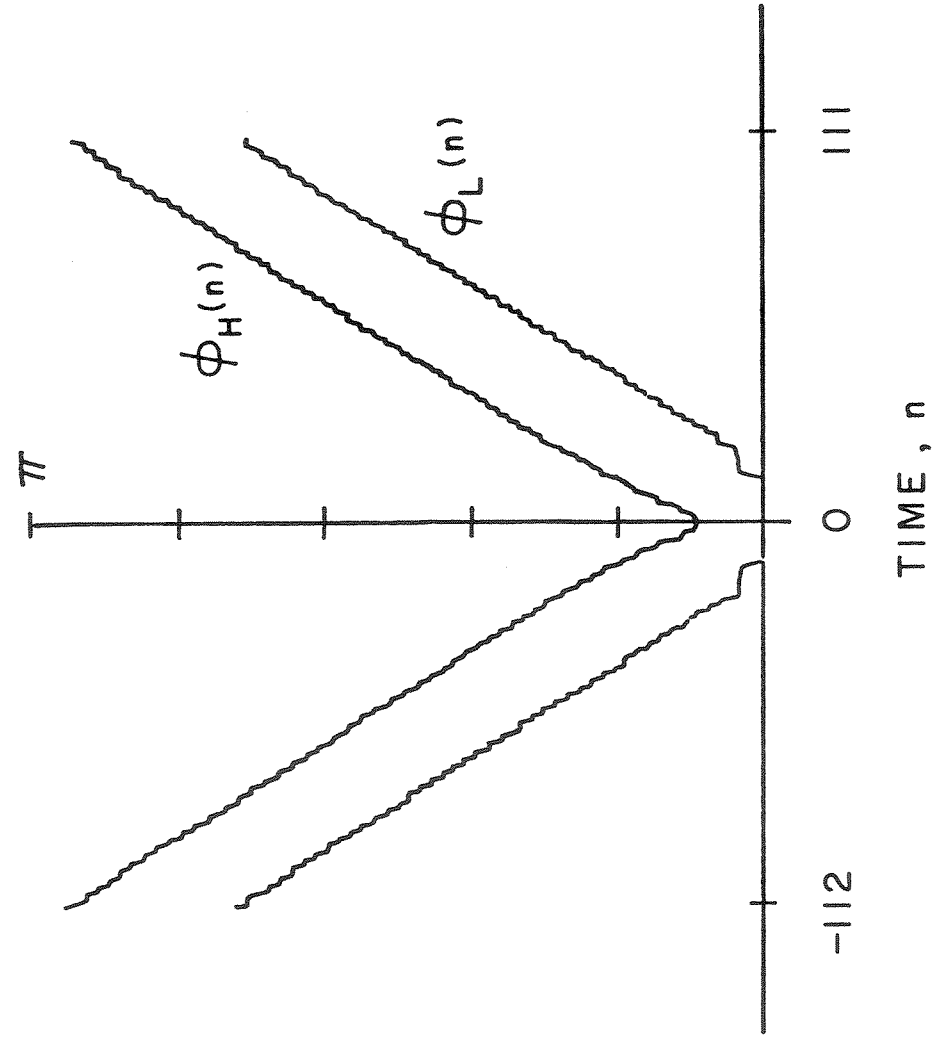


Figure 8. Cutoff frequencies of a shift-variant bandpass filter for the signal specified in (19) and (20). $\phi_H(n)$ and $\phi_L(n)$ are higher and lower cutoff frequencies, respectively.

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