A SIMPLE DERIVATION OF THE MVA AND LBANC ALGORITHMS FROM THE CONVOLUTION ALGORITHM*

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ABSTRACT

The convolution algorithm, the Mean Value Analysis (MVA) algorithm and the LBANC algorithm are major algorithms for the solution of closed product-form queueing networks. For fixed-rate service centers, the efficiency of each algorithm is greatly improved by a recursive solution. We show that the recursive relations in all three algorithms are closely related so that each one can be easily derived from any of the others.

Keywords and Phrases: queueing networks, convolution algorithm, mean value analysis, local balance, recursive solutions.

1. INTRODUCTION

Three well-known computational algorithms for the numerical solution of closed multi-chain product-form queueing networks [1] are the convolution algorithm [2-4], the Mean Value Analysis (MVA) algorithm [5], and the LBANC algorithm [6]. The efficiency of each algorithm is greatly improved by a recursive relation when dealing with fixed-rate service centers. (The applicable service disciplines at a fixed-rate service center are described in [1].) In the next section, we define our notation and introduce the recursive relations in the three algorithms. A simple derivation of the recursive relations in the MVA and LBANC algorithms from the convolution algorithm's recursive relation is shown in Section 3.

2. PRELIMINARIES

Consider a network of M service centers and K closed routing chains. Let λ_{mk} be the mean number of visits of chain k customers to center m between successive visits to center m* (chosen arbitrarily), τ_{mk} be the mean service time of chain k customers at center m and define the traffic intensities

$$\rho_{mk} = \lambda_{mk} \tau_{mk}$$

for $m = 1,2,\ldots,M$ and $k = 1,2,\ldots,K$. Let N_k be the population size of chain k, for $k = 1,2,\ldots,K$. $\underline{N} = (N_1,N_2,\ldots,N_K)$ is said to be the population vector of the network.

Consider a closed multi-chain network with population vector $\underline{N}$. The equilibrium probability of the network state $(\underline{n}_1,\underline{n}_2,\ldots,\underline{n}_M)$ is given by the following product-form solution [1]

$$\frac{p_1(\underline{n}_1)p_2(\underline{n}_2)\cdots p_M(\underline{n}_M)}{G(\underline{N})} \qquad \text{for } \underline{0} \leq \underline{n}_m \leq \underline{N} \quad m = 1,2,\ldots,M$$

where $\underline{0}$ is a K-dimensional vector of zeroes, $\underline{n}_m$ is a K-dimensional vector of nonnegative integers, and $G(\underline{N})$ is the normalization constant. Each real-valued function p_m can be thought of as a K-dimensional array indexed between $\underline{0}$ and $\underline{N}$. The normalization constant $G(\underline{N})$ is simply an element of the following array (the element indexed by $\underline{N}$)

$$g_{\{1,2,\ldots,M\}} = p_1 \circledast p_2 \circledast \cdots \circledast p_M$$

where $\circledast$ denotes a convolution operation between 2 arrays [3,4]. In general, if SUBNET = $\{m_1,m_2,\ldots,m_s\}$ is a set of integers chosen from 1 to M, we define

$$g_{SUBNET} = g_{m_1} \circledast g_{m_2} \circledast \cdots \circledast g_{m_s}.$$

For a network with population vector $\underline{N}$, define

$L_{mk}(\underline{N})$ = mean number of chain k customers at center m

$T_{mk}(\underline{N})$ = throughput of chain k customers at center m

and

$D_{mk}(\underline{N})$ = mean delay of chain k customers at center m.

The Convolution Algorithm

The convolution algorithm first computes a set of normalization constants and then computes performance measures in terms of the normalization constants. Thus, throughputs are given by

$$T_{mk}(\underline{N}) = \lambda_{mk} \frac{G(\underline{N} - \underline{1}_k)}{G(\underline{N})} \qquad (1)$$

where $\underline{1}_k$ is a unit vector with its kth component equal to one and all others equal to zero, and $G(\underline{N} - \underline{1}_k)$ is the normalization constant of a network with population vector $\underline{N} - \underline{1}_k$. The mean queue lengths in a fixed-rate service center are given by [4]

$$L_{mk}(\underline{N}) = \rho_{mk} \frac{G_{m+}(\underline{N} - \underline{1}_k)}{G(\underline{N})} \qquad (2)$$

where $G_{m+}(\underline{N} - \underline{1}_k)$ is the normalization constant of a network with population vector $\underline{N} - \underline{1}_k$ and M + 1 centers where the extra center has the same set of traffic intensities that center m has. By Little's law [7], we have

$$D_{mk}(\underline{N}) = T_{mk}(\underline{N})\ L_{mk}(\underline{N}).$$

The convolution algorithm obtains the array $g_{\{1,2,\ldots,M\}}$ by the following procedure

$$g_{\{1\}} = p_1$$
$$g_{\{1,2,\ldots,m\}} = g_{\{1,2,\ldots,m-1\}} \otimes p_m \qquad (3)$$

If center m is a fixed-rate center, then the array $g_{\{1,2,\ldots,m\}}$ can be computed efficiently using the following recursive relation [2,4]

$$g_{\{1,2,\ldots,m\}}(\underline{i}) = g_{\{1,2,\ldots,m-1\}}(\underline{i}) + \sum_{k=1}^{K} \rho_{mk}\ g_{\{1,2,\ldots,m\}}(\underline{i}-\underline{1}_k) \qquad (4)$$
$$\text{for } \underline{0} \leq \underline{i} \leq \underline{N}.$$

(Note that throughout this paper, we adopt the convention that any quantity, such as $g_{\{1,2,\ldots,m\}}(\underline{i} - \underline{1}_k)$ above, whose argument has one or more negative components is equal to zero.)

The MVA Algorithm

The MVA algorithm skips the normalization constants and solves for the performance measures directly using the following recursive relation given that center m is a fixed-rate center [5]

$$D_{mk}(\underline{N}) = \tau_{mk}[1 + \sum_{h=1}^{K} L_{mh}(\underline{N} - \underline{1}_k)] \tag{5}$$

$$T_k(\underline{N}) = \frac{N_k}{\sum_{m=1}^{M} D_{mk}(\underline{N})\ \lambda_{mk}} \tag{6}$$

and

$$L_{mk}(\underline{N}) = \lambda_{mk}\ T_k(\underline{N})\ D_{mk}(\underline{N}) \tag{7}$$

where $T_k(\underline{N})$ is the throughput of chain k customers at center m*. Both equations (6) and (7) are based upon Little's law. Eq. (5) is the key relation in the MVA algorithm and is related to the Arrival Theorem [8,9]. The initial condition for the MVA recursion is:

$$L_{mk}(\underline{0}) = 0 \qquad \text{for } m = 1,2,\ldots,M \text{ and } k = 1,2,\ldots,K.$$

The LBANC Algorithm

Define the unnormalized mean queue lengths

$$q_{mk}(\underline{N}) = G(\underline{N})\ L_{mk}(\underline{N}) \qquad \text{for all m and k} \tag{8}$$

For center m begin a fixed-rate center, the LBANC algorithm uses the following recursion to obtain the unnormalized mean queue lengths and normalization constants [6]:

$$q_{mk}(\underline{N}) = \rho_{mk}[G(\underline{N} - \underline{1}_k) + \sum_{h=1}^{K} q_{mh}(\underline{N} - \underline{1}_k)] \tag{9}$$

and

$$G(\underline{N}) = \frac{\sum_{m=1}^{M} q_{mk}(\underline{N})}{N_k} \tag{10}$$

This last equation is a consequence of the observation

$$\sum_{m=1}^{M} L_{mk}(\underline{N}) = N_k \qquad \text{for any } k.$$

The initial condition for the LBANC recursion is:

$$G(\underline{0}) = 1 \text{ and } q_{mk}(\underline{0}) = 0 \qquad \text{for all } m \text{ and } k.$$

For the sake of completeness, we note that the recursive solutions described above for the MVA and LBANC algorithms are still applicable if the network consists of both fixed-rate centers and infinite-servers (IS) centers [5,6]. If center m is an IS center, then Eq. (5) in the MVA algorithm is simply replaced by

$$D_{mk}(\underline{N}) = \tau_{mk} \qquad \text{for all } k$$

and Eq. (9) in the LBANC algorithm is simply replaced by

$$q_{mk}(\underline{N}) = \rho_{mk}\, G(\underline{N} - \underline{1}_k) \qquad \text{for all } k.$$

3. THE DERIVATION

We next proceed to derive Eq. (5) for the MVA recursion starting with the convolution algorithm's recursive relation in Eq. (4). We note that

$$G_{m+}(\underline{N} - \underline{1}_k) = g_{\{1,2,\ldots,M\} \cup \{m\}}\,(\underline{N} - \underline{1}_k)$$

By Eq. (4), we have

$$G_{m+}(\underline{N} - \underline{1}_k) = G(\underline{N} - \underline{1}_k) + \sum_{h=1}^{K} \rho_{mh}\, G_{m+}(\underline{N} - \underline{1}_k - \underline{1}_h)$$

Divide both sides by $G(\underline{N})$, multiply by ρ_{mk} and applying Eq. (2), we get

$$L_{mk}(\underline{N}) = \rho_{mk} \left[\frac{G(\underline{N} - \underline{1}_k)}{G(\underline{N})} + \sum_{h=1}^{K} \rho_{mh} \frac{G_{m+}(\underline{N} - \underline{1}_k - \underline{1}_h)}{G(\underline{N})} \right]$$

$$= \tau_{mk} \lambda_{mk} \frac{G(\underline{N} - \underline{1}_k)}{G(\underline{N})} \left[1 + \sum_{h=1}^{K} \rho_{mh} \frac{G_{m+}(\underline{N} - \underline{1}_k - \underline{1}_h)}{G(\underline{N} - \underline{1}_k)} \right]$$

$$= \tau_{mk} T_{mk}(\underline{N}) \; [1 + \sum_{h=1}^{K} L_{mh}(\underline{N} - \underline{1}_k)]$$

where Eq. (1) has been applied. Dividing both sides by $T_{mk}(\underline{N})$ and applying Little's law, Eq. (5) is obtained. Conversely, it should be obvious that starting from Eq. (5) in the MVA algorithm, the recursive relation Eq. (4) in the convolution algorithm can be derived.

To derive Eq. (9) that is the key of the LBANC recursion, we again consider

$$G_{m+}(\underline{N} - \underline{1}_k) = G(\underline{N} - \underline{1}_k) + \sum_{h=1}^{K} \rho_{mh} G_{m+}(\underline{N} - \underline{1}_k - \underline{1}_h)$$

based upon the convolution algorithm's recursion. Multiply both sides of the above equation by ρ_{mk}, we get

$$\rho_{mk} G_{m+}(\underline{N} - \underline{1}_k) = \rho_{mk} \left[G(\underline{N} - \underline{1}_k) + \sum_{h=1}^{K} \rho_{mh} G_{m+}(\underline{N} - \underline{1}_k - \underline{1}_h) \right]$$

which becomes Eq. (9) in the LBANC recursion by observing from Eqs. (2) and (8) that

$$q_{mk}(\underline{N}) = \rho_{mk} G_{m+}(\underline{N} - \underline{1}_k)$$

for all k and center m being a fixed-rate center.

It is interesting to note that the LBANC recursion is an intermediate step in the sequence of steps that transform the convolution recursion into the MVA recursion. The LBANC recursion involves both normalization constants and mean values, while the convolution recursion involves only normalization constants and the MVA recursion involves only mean values.

4. CONCLUSIONS

The convolution, MVA and LBANC algorithms provide recursive solutions for product-form queueing networks consisting of fixed-rate and infinite-servers centers. We have shown that the recursive relations in all three algorithms are closely related so that each one can be easily derived from any of the others.

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