and $TA(v_y)=TA(v_x)+(y_{1i}-x_{1i})*d_{1i}+(y_{1j}-x_{1j})*d_{1j}$

Definition 5.1-2: A diagnol in a mesh graph MG is a 2-tuple $D=\langle V_D,k\rangle$ where

1. V_D is a set of ordered computation vertices from MG and the ordering ranges from 1 to $|V_D|$ and if v_x and v_y are in V_D such that x_{1j} is less than y_{1j} then the index of v_x in the ordering is less than the index of v_y in the ordering

2. k is such that

   a. either for every v_x in V_D, $x_{1i}+x_{1j}=k$

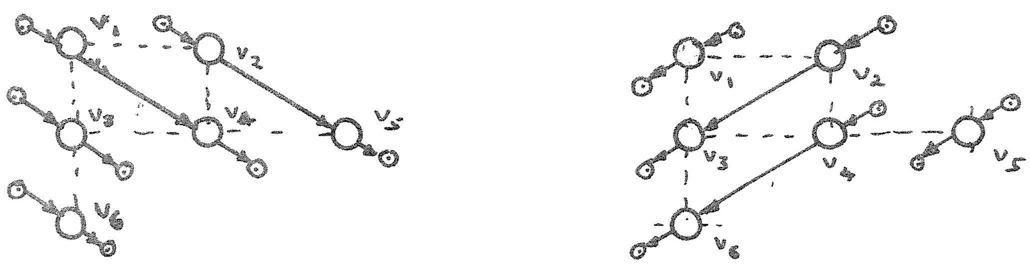
   b. or for every v_x in V_D, $x_{1i}-x_{1j}=k$

Let D_1={D such that every v_x in V_D satisfies (2a)} and D_r={D such that every v_x in V_D satisfies (2b)}. Augment V_D of every D in D_1 by a dummy[1] source vertex labelled 1l, a dummy sink vertex labelled 1l and a set of dummy directed edges labelled 1l. Assign the index Ø to the source vertex and the index $|V_D|+1$ to the sink vertex. For every pair of adjacent vertices v_x and v_y in the ordering of all the vertices in the augmented V_D if the index of v_y is greater than the index of v_x by 1 then direct a dummy edge labelled 1l from v_x to v_y. Consequently every D in D_1 is a major path labelled 1l. Similarly augment every D in D_r by a dummy source vertex labelled 1r, a dummy sink vertex labelled 1r and a

[1]not part of the original graph

set of dummy edges labelled lr and hence every D in D_r is a major path labelled lr.

Example 5.1-2: Consider the mesh graph MG of example 5.1-1. D_r and D_l in MG is shown in figure 5.1-2 below:



D_r D_l

Figure 5.1-2

In figure 5.1-2 the dummy source and sink vertices are denoted by '⊙'.

Let MG_{aug} denote the mesh graph MG augmented by the set of dummy edges, dummy source and sink vertices in D_l and D_r. Let A be the set of major paths in MG_{aug} such that for any pair of computation vertices v_x and v_y in any major path of A, $PA(v_x)=PA(v_y)$ and if v_x and v_y are in different major paths of A then $PA(v_x) \neq PA(v_y)$. Let $B=E_{li}$ if $A=E_{li}$ and $B=E_{lj}$ if $A=E_{li}$.

Lemma 5.1-1:

In any syntactically correct mapping of MG_{aug} the pair $\langle A,B \rangle$ must be one of $\{\langle E_{lj},E_{li}\rangle, \langle D_l,E_{li}\rangle, \langle D_r,E_{li}\rangle, \langle E_{li},E_{lj}\rangle\}$.

Proof: Let x and y be the value assumed by n_{1i} and n_{1j} respectively. Now by lemma 4.0-1 $n_{1i}=n_{1j}\neq\emptyset$ and hence $\langle x,y\rangle$ must be one of $\{\langle 1,\emptyset\rangle, \langle 1,1\rangle, \langle 1,-1\rangle, \langle\emptyset,1\rangle, \langle -1,\emptyset\rangle, \langle -1,1\rangle, \langle -1,-1\rangle, \langle\emptyset,-1\rangle\}$. If M_1 is the mapping that results from $\langle n_{1i},n_{1j}\rangle=\langle x,y\rangle$ then the mapping M_2 that results from $\langle n_{1i},n_{1j}\rangle=\langle -x,-y\rangle$ is the same as M_1 with the array in M_1 reversed from end to end. Hence we need to consider only the four different mappings that result from $\langle x,y\rangle$ assuming each of $\{\langle 1,\emptyset\rangle, \langle 1,1\rangle, \langle 1,-1\rangle, \langle\emptyset,1\rangle\}$. By forming A for each of these four different values the lemma follows.

Let 1a and 1b denote the label of edges, source and sink vertices in A and B respectively. Clearly 1a must be one of {1i, 1j, 1l, 1r} and 1b must be one of {1i, 1j}.

Lemma 5.1-2:

In any syntactically correct mapping of MG_{aug}, n_{1b} must be 1 and n_{1a} must be $\emptyset$.

Proof: n_{1a} must be $\emptyset$ follows from definiton of A. From lemma 5.1-1 $\langle A, B\rangle$ must be one of $\{\langle E_{1j},E_{1i}\rangle, \langle D_1,E_{1i}\rangle, \langle D_r,E_{1i}\rangle, \langle E_{1i},E_{1j}\rangle\}$. It can be easily verified that $B=E_{1i}$ if $n_{1i}=1$ and $B=E_{1j}$ if $n_{1j}=1$ and hence n_{1b} must be 1.

Now any maximally-connected subgraph SG of MG_{aug} with $L_{SG}=\{1a,1b\}$ is a mesh graph and consequently the relations r_{1a} and r_{1b} impose a linear chain on B and A respectively and hence the major paths in A and B are also totally ordered. Now for any computation vertex v_x, x_{1b} and x_{1a} are its horizontal and vertical co-ordinates. Also x_{1b} and x_{1a} are the indices of the major path in A and B respectively that pass through v_x.

Lemma 5.1-3:

If v_x, v_y, v_w and v_u are any four computation vertices in MG_{aug} such that $y_{1b}-x_{1b}=w_{1b}-u_{1b}$ and if in any syntactically correct mapping of MG_{aug} $TA(v_y)-TA(v_x)=TA(v_w)-TA(v_u)$ then $y_{1a}-x_{1a}=w_{1a}-u_{1a}$.

Proof: Immediate from proposition 5.1-2.

Lemma 5.1-4:

If v_x and v_y are any two computation vertices in MG_{aug} such that $y_{1b}-x_{1b}=k*m$ and $y_{1a}-x_{1a}=k*n$ where k, m, n are integers then in any syntactically correct mapping of MG_{aug}, v_x $T^k_{\langle m,o \rangle}$ v_y where $o=\langle n*d_{1a}+m*d_{1b}\rangle$.

Proof: By proposition 5.1-2,

$$PA(v_y)=(y_{1b}-x_{1b})*n_{1b}+(y_{1a}-x_{1a})*n_{1a}+PA(v_x)$$

and $$TA(v_y)=(y_{1b}-x_{1b})*d_{1b}+(y_{1a}-x_{1a})*d_{1a}+TA(v_x)$$

By lemma 5.1-2 $n_{1a}=\emptyset$ and $n_{1b}=1$ and hence

$$PA(v_y)=k*m+PA(v_x)$$

and $$TA(v_y)=k*o+TA(v_x)$$

and hence the lemma.

5.2. Main Result

We will examine the syntactic properties of a class CL of program graphs such that there exists a syntactically correct mapping for every member of this class. In every program graph G in CL there exists at least a pair of labels li and lj such that the number of maximally--connected subgraphs SG with L_{SG}={li,lj} is 1. Hence all the major paths in E_{li} and E_{lj} are in this subgraph. By theorem 5.1-1 SG must be a mesh graph. Augment the subgraph SG with the two sets of diagnol paths D_l and D_r. Define the sets A and B as for the augmented SG as done earlier in this section . Let S_A={r_{lp}[1] | r_{lp} is a binary relation on the major paths in B and E_{lp} is in H} . Let S_B={r_{lp}|r_{lp} is a binary relation on major paths in B and E_{lp} is in H}. By lemma 5.1-1 the pair <A,B> must assume only one of {<E_{lj},E_{li}>, <D_l,E_{li}>, <D_r,E_{li}>, <E_{li},E_{lj}>}. Let us denote this set of four tuples as H.

Theorem 5.2-1:

There exists a syntactically correct mapping for any G in CL iff there exists at least one tuple in H assumed by <A,B> such that each of the following condition is satisfied:

1. S_A imposes a consistent order on the major paths in A and the consistency constant of every relation in S_A in the ordering imposed by S_A is one of {1, -1, Ø}.

2. S_B imposes a consistent order on the major paths in B.

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[1]refer definition 2.1-6, page 7

3. for any pair of computation vertices v_x and v_y in G and for any label if $y_{1b}-x_{1b}=k*a_{1p}$ and $y_{1a}-x_{1a}=k*b_{1p}$ where a_{1p} and b_{1p} are consistency constants of relations r_{1p} in the order imposed by S_A and S_B respectively then there must be a single major path labelled 1p that passes through v_x and v_y such that the distance from v_x to v_y in the major path is k.

4. for any relation r_{1p} in S_A if $a_{1p}=\emptyset$ then there cannot exist a transitive edge labelled 1p.

Proof:

Necessity:

1. Consider any relation r_{1p} in S_A. Now let v_x and v_y be any pair of computation vertices in any two major paths in A such that there is an edge labelled 1p directed from v_x to v_y. Now $PA(v_y)=PA(v_x)+n_{1p}$ where n_{1p} is one of $\{+1,-1,\emptyset\}$. Also $PA(v_y)=y_{1b}$ and $PA(v_x)=x_{1b}$ and hence $y_{1b}-x_{1b}=n_{1p}$ and hence condition (1).

2. Again consider any relation r_{1p} in S_A and S_B. Consider any two pair of computation vertices $\langle v_x,v_y\rangle$ and $\langle v_u,v_w\rangle$ such that there is an edge labelled 1p directed from v_x to v_y and also another edge labelled 1p directed from v_u to v_w. Now d_{1p} is a constant and hence $TA(v_w)-TA(v_u)=TA(v_y)-TA(v_x)$ and hence $(y_{1b}-x_{1b})*d_{1b}+(y_{1a}-x_{1a})*d_{1a}=(w_{1b}-u_{1b})*d_{1b}+(w_{1a}-u_{1a})*d_{1a}$. But by condition (1) $y_{1b}-x_{1b}=w_{1b}-u_{1b}$ and hence

$y_{1a}-x_{1a}=w_{1a}-u_{1a}$ and hence condition (2).

3. We first show that $n_{1p}=a_{1p}$ and $d_{1p}=a_{1p}*d_{1b}+b_{1p}*d_{1a}$. Consider any pair of computation vertices v_u and v_w such that there is an edge labelled 1p directed from v_u to v_w. Now by condition (1), $w_{1b}-u_{1b}=a_{1p}$. Also $PA(v_w)=w_{1b}$ and $PA(v_u)=u_{1b}$ and hence $PA(v_w)-PA(v_u)=n_{1p}=a_{1p}$. Also $TA(v_w)-TA(v_u)=d_{1p}=(w_{1b}-u_{1b})*d_{1b}+(w_{1a}-u_{1a})*d_{1a}$. From condition (2), $w_{1a}-u_{1a}=b_{1p}$ and hence $d_{1p}=a_{1p}*d_{1b}+b_{1p}*d_{1a}$. Now $n_{1b}=1$ and $n_{1a}=\emptyset$ and hence $PA(v_y)=PA(v_x)+k*a_{1p}$ and hence $PA(v_y)=PA(v_x)+k*n_{1p}$. From condition (2) $d_{1p}=a_{1p}*d_{1b}+b_{1p}*d_{1a}$ and hence $TA(v_y)=TA(v_x)+k*d_{1p}$ and hence $v_x\ T^k_{\langle m,n\rangle}\ v_y$ where $m=n_{1p}$ and $n=d_{1p}$. If $n_{1p}=\emptyset$ then by definition of a syntactically correct mapping condition (3) follows. If $n_{1p}\neq\emptyset$ then by lemma 4.0-3 condition (3) follows.

4. Supposing there existed a E_{1p} in H such that $n_{1p}=\emptyset$. By condition (3) E_{1p} must be identical to A and hence $d_{1p}=d_{1a}$. We show that if E_{1p} exists then d_{1b} must be some multiple of d_{1p}.

Consider two consecutively indexed major paths q_1 and q_2 in E_{1p} as shown in figure 5.2-1 below:

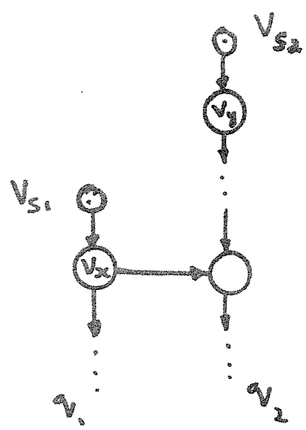

Figure 5.2-1

Let $PA(v_x)=p$ and hence $PA(v_y)=p+1$. Let $TA(v_x)=t_1$ and $TA(v_y)=t_2$ and hence $t_2=d_{1b}+(y_{1a}-x_{1a})*d_{1p}+t_1$. Let $y_{1a}-x_{1a}=k$ and hence $t_2=t_1+k*d_{1p}+d_{1b}$. The start time $t_s<t_1$. Besides there must be two tuples $\langle v_{s1},t_s\rangle$ and $\langle v_{s2},t_s\rangle$ in SO'_G such that $IOA\langle v_{s1},t_s\rangle=p$ and $IOA\langle v_{s2},t_s\rangle=p+1$ and hence $t_1-k_1*d_{1p}=t_s=t_1+k*d_{1p}+d_{1b}-k_2*d_{1p}$ and hence $d_{1b}=(k_2-k_1-k)*d_{1p}$

Next we will show that d_{1b} cannot be a integral multiple of d_{1p}. Consider two consecutively indexed paths q_1 and q_2 in E_{1p} and a transitive edge e_a in q_1 as shown in figure 5.2-2 below:

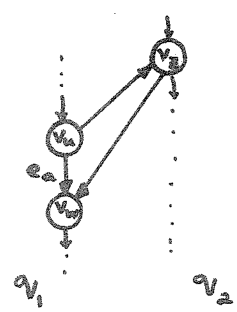

Figure 5.2-2

Let $u_{1a}-z_{1a}=k$ where k is some integer and hence $w_{1a}-z_{1a}=k+1$. Let $PA(v_u)=p$ and hence $PA(v_w)=p$ and $PA(v_z)=p+1$. Let $TA(v_u)=t_u$, $TA(v_w)=t_w$ and $TA(v_z)=t_z$. In any syntactically correct mapping these times must be related by the following set of five equations:

a. $t_z>t_u$

b. $t_w>t_z$

c. $t_z-t_u=-k*d_{1p}+d_{1b}$

d. $t_w-t_u=d_{1p}$

e. $t_w-t_z=(k+1)*d_{1p}-d_{1b}$

Using the above equations we can show that $k*d_{1p}<d_{1b}<(k+1)*d_{1p}$ and hence d_{1b} cannot be an integral multiple of d_{1p}.

Sufficiency:

Let Ψ be the program transformed to G. Let $|A|=n$ and so the major paths in A are totally ordered with indices of the major paths in A ranging from $\emptyset$ to (n-1). Construct a linear-array LA with $\Psi=\Psi_P$, $L_G=L_P$ and $|PR|=n$ and construct a mapping as follows:

Set $n_{1a}=\emptyset$ and $n_{1b}=1$. For every r_{1p} in S_A set $n_{1p}=a_{1p}$. If $A=D_1$ then set $d_{1a}=2$ else set $d_{1a}=1$. If the consistency constants of all the relations in S_B are $> \emptyset$ then set $d_{1b}=1$ else set $d_{1b}=d_{1a}*|b_{1p}|+1$ where b_{1p} is the minimum among all the consistency constants of all the relations in S_B. For every E_{1p} in H set $d_{1p}=a_{1p}*d_{1b}+b_{1p}*d_{1a}$. This completes assignment of n_{1p} and d_{1p} for every label 1p such that E_{1p} is in H. Next for every E_{1p} in F if E_{1p} in H is identical to some E_{1q} in H then set $n_{1p}=n_{1q}$ and $d_{1p}=d_{1q}$. For every relation E_{1p} in S set B_{1p} as empty. This completes assignment of n_{1p} and d_{1p} for every label in G. Next construct the functions PA and TA. For every vertex v_x in a path in A whose index is i set $PA(v_x)=i$. For every computation vertex v_y set $TA(v_y)=t_{\emptyset}+y_{1b}*d_{1b}+y_{1a}*d_{1a}$ where $t_{\emptyset}$ is the time at which the computation vertex whose co-ordinates are $<\emptyset,\emptyset>$ (i.e., the computation vertex on the major paths in A and B whose indices in the ordering of the major paths in A and B are $\emptyset$.) is mapped. This completes the construction of the mapping. We next show that this mapping is syntactically correct.

$d_{1a}>\emptyset$. Hence no two computation vertices in any major path in A are simultaneously mapped onto the same processor. The computation vertices in distinct major paths in A are mapped onto distinct processors. Hence no two computation vertices in G are mapped simultaneously onto the same processor.

Next consider any two computation vertices v_x and v_y such that there is an edge labelled 1p directed from v_x to v_y.

Now,
$$\begin{aligned} PA(v_y) &= PA(v_x)+(y_{1b}-x_{1b})*n_{1b}+(y_{1a}-x_{1a})*n_{1a} \\ &= PA(v_x)+(y_{1b}-x_{1b}) \\ &= PA(v_x)+a_{1p} \end{aligned}$$

$=PA(v_x)+n_{1p}$

and n_{1p} is one of $\{1,-1,\emptyset\}$ as a_{1p} is one of $\{1,-1,\emptyset\}$.

Also $$TA(v_y)=TA(v_x)+(y_{1b}-x_{1b})*d_{1b}+(y_{1a}-x_{1a})*d_{1a}$$
$$=TA(v_x)+a_{1p}*d_{1b}+b_{1p}*d_{1a}$$
$$=TA(v_x)+d_{1p}$$

Next consider any two computation vertices v_x and v_y and any label 1p such that $y_{1b}-x_{1b}=k*a_{1p}$ and $y_{1a}-x_{1a}=k*b_{1p}$. It can be verified that $v_x\ T^k_{\langle m,n\rangle} v_y$ where $m=a_{1p}$ and $n=d_{1p}$. If $n_{1p}=\emptyset$ then by condition (3) E_{1p} is identical to A. In our construction no two distinct paths in A are mapped onto the same processor and hence no two distinct paths in E_{1p} are mapped onto the same processor. If $n_{1p}\neq\emptyset$ then by lemma 4.0-3 there must be a single path labelled 1j passing through v_x and v_y which is condition (3) .Hence no two source or sink vertices are simultaneously mapped onto the input port labelled 1p of any processor in the array.

6. Semantic Issues

In the previous section we characterized a class CL of program graphs such that for every program graph in this class there exists a syntactically correct mapping. Recall that a syntactically correct mapping of a program graph G computes the program ψ represented by G correctly iff the contents of every source and sink vertex in G that is mapped onto the input ports or output ports of processors more than once remains invariant. More formally in a syntactically correct mapping of G if there exists a pair of tuples $\langle v_x,t_m\rangle$ and $\langle v_x,t_n\rangle$ either in SI'_G or SO'_G then the contents of v_x at t_m must be the same as the contents of v_x

at t_n. For the contents to remain invariant the program Ψ represented by the program graph G which is the same as the function Ψ_P computed by every processor in the linear array needs interpretation. However in a syntactically correct mapping of G if there does not exist any pair of tuples $\langle v_x, t_m \rangle$ and $\langle v_x, t_n \rangle$ either in SI_G' or SO_G' such that $t_m \neq t_n$ then clearly the program Ψ needs no interpretation. But we demonstrate in theorem 6.1-1 that this sub-class of program graphs in CL is very limited. We then examine some semantic properties required of programs in CL for them to be correctly executable.

6.1. Uniterpreted Characterization

Herein we will examine the structure of program graphs in CL that can be computed correctly without interpretation of the programs represented by the program graphs. We establish a few preliminary results of a general nature which we subsequently use to prove the main result, .i.e., theorem 6.1-1.

Lemma 6.1-1:

If a syntactically correct mapping of G (not necessarily in CL) computes the program Ψ transformed to G correctly without interpretation of Ψ then for any label lp if E_{1p} is in H and $n_{1p} \neq \emptyset$ then all the major paths in E_{1p} must have the same path-length.

Proof: Let q_1 and q_2 be the two major paths in E_{1p} such that the path-length of q_1 is greater than that of q_2. Let the path-length of q_1 be k. Now $n_{1p} \neq \emptyset$ and hence there must be at least k processors in the linear-array. Let v_s and v_f be the source and sink vertices in q_2. Let v_x and v_y be the first[1] and the last[2] computation vertices respectively in q_2. Let $PA(v_x)=p_1$, $PA(v_y)=p_2$, $TA(v_x)=t_1$, $TA(v_y)=t_2$. Since the path-length of q_2 is less than the path-length of q_1 either one of the following cases must be true:

1. there exist atleast two tuples $\langle v_s, t_1-d_{1p}\rangle$ and $\langle v_s, t_1\rangle$ in SO_G' such that $IOA\langle v_s, t_1-d_{1p}\rangle=p_1$ and $IOA\langle v_s, t_1\rangle=p_1-1$

2. there exist at least two tuples $\langle v_f, t_2+d_{1p}\rangle$ and $\langle v_f, t_2+2*d_{1p}\rangle$ in SI_G' such that $IOA\langle v_f, t_2+d_{1p}\rangle=p_2$ and $IOA\langle v_f, t_2+2*d_{1p}\rangle=p_2+1$

In both cases interpretation is needed.

Lemma 6.1-2:

If a syntactically correct mapping of G (not necessarily in CL) computes the program Ψ transformed to G correctly without interpretation of Ψ then for any pair of labels lp and lq if E_{1p} and E_{1q} are both in H then $n_{1p}=1 \Longrightarrow n_{1q}=\emptyset$ and $n_{1q}=1 \Longrightarrow n_{1p}=\emptyset$ (.i.e., n_{1p} and n_{1q} cannot both be non-zero simultaneously).

Proof: Without loss of generality let $n_{1p}=n_{1q}=1$. By theorem 5.1-1 any maximally-connected subgraph SG with $L_{SG}=\{1p,1q\}$ must be a mesh graph and hence the major paths labelled lp in SG are

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[1] the edge labelled li from v_s is directed into v_x

[2] the edge labelled li from v_y is directed into v_f

totally ordered and similarly the major paths labelled 1q in SG are also totally ordered. Let q_r denote the major path with index $\emptyset$ in E_{1p} and q_w be the major path with index $\emptyset$ in E_{1q}. Let v_x be a computation vertex in q_w and q_r as shown in figure 6.1-1 below:

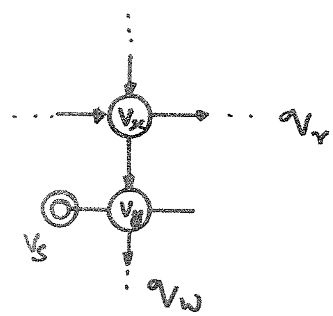

Figure 6.1-1

Let v_y be a computation vertex in q_w and v_s a source vertex labelled 1p. Let $PA(v_x)=p$ and $TA(v_x)=t$. $n_{1p}=n_{1q}=1$ and hence $PA(v_y)=p+1$ and $TA(v_y)=t+d_{1q}$. Consequently there must exist a tuple $\langle v_s,t+d_{1q}\rangle$ and $\langle v_s,t\rangle$ in SO_G' such that $IOA\langle v_s,t\rangle=p$ and $IOA\langle v_s,t+d_{1q}\rangle=p+1$ and hence interpretation is needed.

Let G be any program graph in CL. Let SG be the only maximally-connected subgraph of G with $L_{SG}=\{1i,1j\}$.

Lemma 6.1-3:

If a syntactically correct mapping of G in CL computes the program ψ transformed to G correctly without interpretation of ψ then there must be at most two sets of major paths in H (.i.e., H can have atmost E_{1i} and E_{1j}) and the path-lengths of the major paths in the same set must be identical to one another (.i.e., the path-lengths of major paths in E_{1i}

must be identical to one another and similarly the path-lengths of major paths in E_{1j} must also be identical to one another).

Proof: We first show that H must have atmost E_{1i} and E_{1j}. Suppose $H=\{E_{1i}, E_{1j}, E_{1p}\}$. By lemma 6.1-2 n_{1i} and n_{1j} cannot both be non-zero. Let $n_{1i}\neq\emptyset$ and $n_{1j}=\emptyset$. Also by lemma 6.1-2 n_{1i} and n_{1p} cannot both be non-zero and hence $n_{1p}=\emptyset$. But by lemma 4.0-1 $n_{1j}=n_{1p}\neq\emptyset$.

We next show that the path-lengths of the major paths in E_{1i} are identical and the path lengths of the major paths in E_{1j} are also identical. By lemma 6.1-2 n_{1i} and n_{1j} cannot both be non-zero. Without loss of generality let $n_{1i}\neq\emptyset$ and so $n_{1j}=\emptyset$. By lemma 6.1-1 the path lengths of the major paths in E_{1i} are identical.

Suppose there exist major paths in E_{1j} such that the path lengths are not same. Let q_1 and q_2 be two consecutively indexed major paths in the total ordering of the major paths in E_{1j} whose path lengths differ. Without loss of generality let the path length of q_1 be greater than that of q_2 as shown in figure 6.1-2 below.

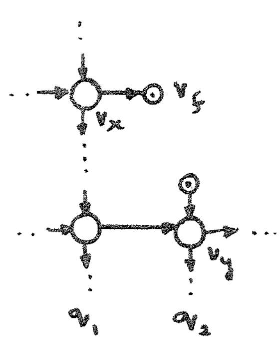

Figure 6.1-2

In figure 6.1-2 v_f is a sink vertex labelled li and v_x and v_y are computation vertices. Let $PA(v_x)=p$ and $TA(v_x)=t$. Now $n_{1i}=\emptyset$ and $n_{1j}=\emptyset$. Without loss of generality let $n_{1i}=1$ and hence $PA(v_y)=p+1$. Consequently there must exist two tuples $\langle v_f, t+d_{1i}\rangle$ and $\langle v_f, t+2*d_{1i}\rangle$ in SI_G' such that $IOA\langle v_f, t+d_{1i}\rangle=p$ and $IOA\langle v_f, t+2*d_{1i}\rangle=p+1$ and hence interpretation is needed.

Theorem 6.1-1:

A syntactically correct mapping of G in CL computes the program Ψ transformed to G correctly without interpretation of Ψ iff there is only one set of major paths in E (.i.e., E is either E_{1i} or E_{1j}) and the path length of all the major paths in this set are identical to one another.

Proof: Necessity:

Suppose $H=\{E_{1i}, E_{1j}\}$. Let $n_{1i}=1$ and hence by lemma 6.1-2 $n_{1j}=\emptyset$. By lemma 6.1-1 the path lengths of all major paths in E_{1i} are the same and the path lengths of all major paths in E_{1j} are the same. Now let q_1 and q_2 be two major paths in E_{1j} indexed $\emptyset$ and 1 respectively as shown in figure 6.2-2 below:

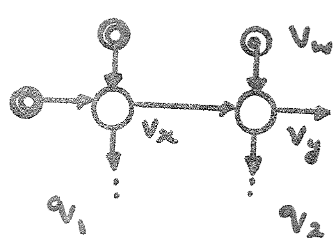

Figure 6.2-2

In figure 6.2-2 v_x and v_y are computation vertices. v_w is a source vertex labelled 1j. Let $PA(v_x)=p$ and $TA(v_x)=t$. Consequently $PA(v_y)=p+1$ and $TA(v_y)=t+d_{1i}$. Now $t\leq t_s$ where t_s is the start time. Hence $t+d_{1i}>t_s$ and hence there must be at least the two tuples $\langle v_w,t_s\rangle$ and $\langle v_w,t+d_{1i}\rangle$ in SO_G' such that $IOA\langle v_w,t_s\rangle=IOA\langle v_w,t+d_{1i}\rangle=p+1$ and this needs interpretation.

Sufficiency:

Let ψ be the program transformed to G. Assign unique indices to the major paths in E_{1i}. ($H=\{E_{1i}\}$ and hence there is no edge directed from a computation vertex in some major path in E_{1i} to another computation vertex in some other distinct major path in E_{1i} and hence the indices can be assigned unique indices arbitrarily). Let $E_{1i}=\{q_\emptyset, q_1, .., q_m\}$ where q_i is the major path assigned index i. Let $v_{\langle r,w\rangle}$ denote the computation vertex at a distance w from the first computation vertex in the major path q_r. Let the the number of computation vertices in any major path be n. Construct a linear array LA with $\psi=\psi_p$, $|PR|=n$ and $L_G=L_p$. Construct a mapping MP_G as follows:

For every E_{1p} in F set $n_{1p}=n_{1i}=1$. For every E_{1p} in S set B_{1p} as empty. Define $PA(v_{\langle\emptyset,\emptyset\rangle})=\emptyset$, $TA(v_{\langle\emptyset,\emptyset\rangle})=t_\emptyset$, $PA(v_{\langle r,w\rangle})=w$ and $TA(v_{\langle r,w\rangle})=t_\emptyset+r+w$. It can be easily verified that such a mapping is syntactically correct and every tuple in SO_G' and SI_G' are mapped only onto the appropriate distinguished input ports and output ports.

6.2. Semantic Properties

From theorem 6.1-1 it is clear that the subclass of program graphs that can can be computed without any semantic information is very limited. Herein we will examine three simple semantic properties of a program and demonstrate their significance in ensuring the correctness of computation of a syntactically correct mapping of any program graph in CL. The notations used are similar and hence have the same meaning as those used in definition 2.2-1 and definition 2.2-2 of a program and its transformation to a program graph.

Definition 6.2-1: Ψ_i is an identity function iff for any x and z in Y if $\Psi(x)=z$ then $z_i=x_i$.

Example 6.2-1: In example 2.2-1 Ψ_2 and Ψ_3 are identity functions.

Example 6.2-2: Let $W=\langle w_1, w_2, .., w_{(i-1)}, w_{(i+1)}, .., w_{(k)}\rangle$ denote a (k-1)-tuple where $\forall$ $j, 1 \leq j \leq k$ and $j \neq i$, $w_j \in Y_j$. W is an identity vector for Ψ_i iff for any $x_i \in Y_i$ and for any z in Y if $\Psi(w_1, w_2, .., w_{(i-1)}, x_i, w_{(i+1)}, .., w_{(k)})=z$ then $z_i=x_i$.

Example 6.2-3: In example 2.2-1 $w=\langle\emptyset,\emptyset\rangle$ is an identity vector for 1.

Definition 6.2-2: Let $w_i \in Y_i$. w_i is a fixpoint element for Ψ_i iff for any z in Y, any x_j in Y_j and $\forall$ j, $1 \leq j \leq k$ and $j \neq i$, if $\Psi(x_1, x_2, .., x_{(i-1)}, w_i, x_{(i+1)}, .., x_k)=z$ then $z_i=w_i$.

Example 6.2-4: Let $Y=Y_1 \times Y_2$. For any x and z in Y if $\Psi(x)=z$ then let $z_1=\min(x_1, x_2)$ and $z_2=\max(x_1, x_2)$.

$+\infty$ is a fixpoint element for min and $-\infty$ is a fixpoint element for max.

Let $ID=ID_1 \cup ID_2 \cup .. \cup ID_k$ where $ID_i \subseteq Y_i \ \forall \ i | \ 1 \leq i \leq k$. ID_i contains elements that are fixpoints of the function Ψ_i or some elements from Y_i that are components of an identity vector for some Ψ_j, $1 \leq j \leq k$ and $j \neq i$.

Let MID be a function from ID to a subset of T (T has the same significance as used in definition 3.0-1[1]) such that if MID(x)=t and if x is in ID_i then the element x is mapped onto the distinguished input port for label li at time t.

Lemma 6.2-1: If MID(x)=t and x is in ID_i and x is a fixpoint element of Ψ_i then the element at the input port of processor indexed $p+k*n_{li}$ at $t+k*d_{li}$ is x where p is the processor index whose input port labelled li is the distinguished input port for label li.

A transformation of a program to a program graph assigns to every element in each of the sets Y_1, Y_2, .., Y_k either a labelled edge or a labelled source vertex or a labelled sink vertex. Consider a syntactically correct mapping of a program graph G. For every source or sink vertex v_x let $S_x=\{\langle v_x,t\rangle | \ \langle v_x,t\rangle \ SI'_G$ or $\langle v_x,t\rangle \ SO'_G \}$. In every S_x there is a tuple $\langle v_x,t_{max}\rangle$ such that for every other tuple $\langle v_x,t\rangle$ in S_x $t<t_{max}$. Let $IOA\langle v_x,t\rangle=p$. Let $ID_{\langle x,t\rangle}$ denote the elements at the other input ports of p at time t. Let $ID_x=\{ID_{\langle x,t\rangle}\}$

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Theorem 6.2-1: A syntactically correct mapping of a program graph G computes correctly if for any label li one of the following holds:

1. Ψ_i is an identity function

2. for any source aor sink vertex labelled the element in Y_i that has been transformed to v_x is a fixpoint for Ψ_i

3. every $ID_{\langle x,t\rangle}$ in ID_x is a identity vector for Ψ_i

7. Illustration

In this section we illustrate the syntactic and semantic characterization developed in the previous sections by two examples. In the first example we consider the problem of multiplying a band matrix by a vector. We will use our characterization to synthesize the solution to this problem proposed in [5]. In the second example we consider the problem of sorting and we will again use our characterization to synthesize the rebound sorter [1].

Example 7.0-1: Multiplication of a Band-Matrix by a Vector

Herein we will use matrix M and vector X used in example 2.2-1. For the program graph (figure 2.2-2) in example 2.2-1 $H=\{E_{11}, E_{13}\}$, $F=\{\emptyset\}$ and $S=\{E_{12}\}$. The number of maximally-connected subgraphs SG with $L_{SG}=\{11,13\}$ is 1 and hence the program graph is in CL. Augment SG by the set D_1 of diagnol paths. The resulting augmented program graph is shown in figure 7.∅-1 below:

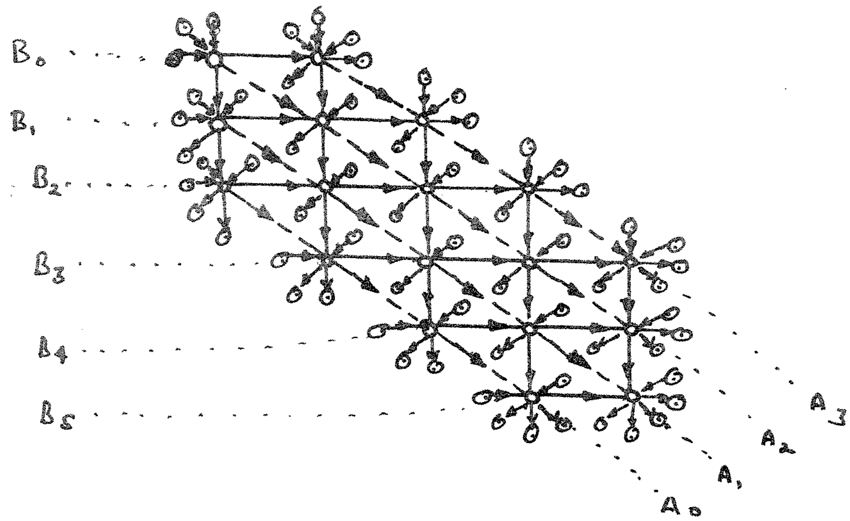

Figure 7.∅-1

In figure 7.∅-1 the dashed lines denote the diagnol paths. D_1={$A_\emptyset$, A_1, A_2, A_3} and E_{11}={$B_\emptyset$, B_1, B_2, B_3, B_4, B_5}. Let A=D_1 and hence B=E_{11}. So S_A={E_{11}, E_{13} } and S_B={D_1, E_{13} }. It can be verified that this program graph satisfies theorem 5.2-1. The consistency constants a_{11} and a_{13} for E_{11} and E_{13} in S_A are 1 and -1 respectively. The consistency constants b_{11} and b_{13} for D_1 and E_{13} in S_B are both 1.

We demonstrate a mapping of this program graph using the construction used in the sufficiency of theorem 5.2-1. n_{1a}=∅ and n_{1b}=1. B is E_{11} and hence n_{11}=1. Set n_{13}=a_{13}=1. A is D_1 and hence set d_{1a}=2. Set d_{1b}=1 and hence d_{11}=1. d_{13}=d_{1b}*a_{13}+d_{1a}*b_{13} and hence d_{13}=1. Set $t_\emptyset$=∅. Map every computation vertex in A_i and B_j onto processor i at time i+2*j. The mapping is shown in figure 7.∅-2 below:

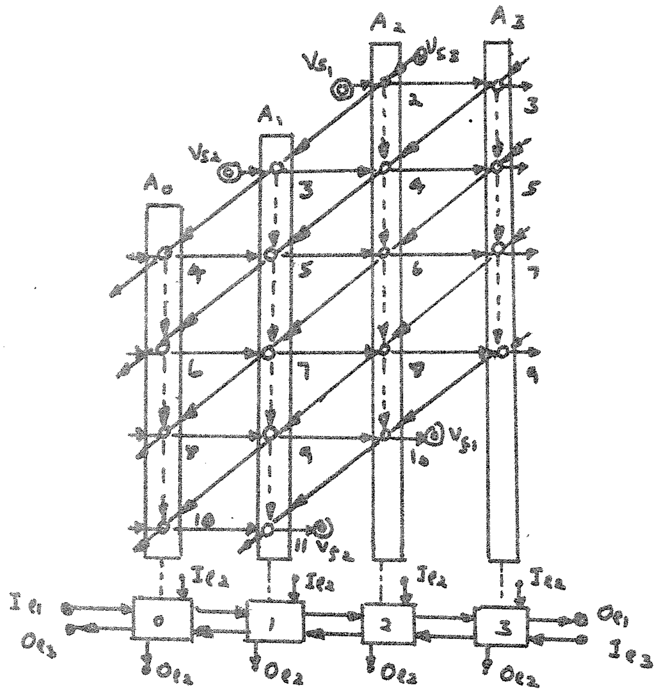

Figure 7.Ø-2

In figure 7.Ø-2 we have omitted most of the source and sink vertices from the program graph for clarity. I_{11} and O_{11} are the distinguished input port and output port for label 11, I_{12} and O_{12} are the distinguished input port and output port respectively for label 12. and I_{13} and O_{13} are the distinguished input ports and output ports respectively for label 13. All the computation vertices in any shaded box are mapped onto the same processor to which the shaded box is connected. The time at which a computation vertex is mapped is shown by the side of the computation vertex. v_{s3} is the only source vertex labelled 12 that gets mapped onto the input ports labelled 12 more than once. However $_3$ is an identity function and hence the contents of v_{s3} remains invariant till it reaches processor 2. v_{s1}, v_{s2}, v_{f1} and v_{f2} are the only source and sink vertices labelled 11 that get mapped onto input ports labelled 11 of processors more than once. Now $_1$ has the vector <x,Ø> as its identity vector where x can be any element from Y_3. Map the element Ø onto port I_{12} of processor Ø at times Ø and 2 and onto port I_{12} of processor 1 at time 1. This ensures that the contents of v_{s1} and v_{s2} remain invariant till it reaches processor 2 and 1 respectively. Similarly map the element Ø onto the input port I_{12} of processor 2 at time 12 and input port I_{12} of processor 3 at times 11 and 13. This ensures that the contents of v_{f1} and v_{f2} remain invariant till they reach O_{11}.

Example 7.0-2: Sorting

We wish to sort the set of elements {2, 1Ø, 5, 6}. A program graph that performs sorting is shown in figure 7.Ø-3 below:

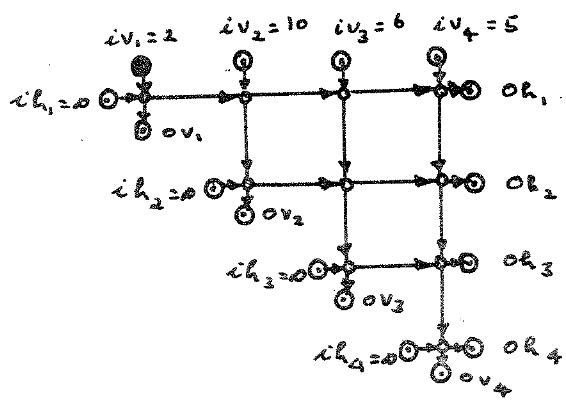

Figure 7.Ø-3: Program Graph for Sorting

In figure 7.Ø-3 'o' denote computation vertices and '⊙' denote either source or sink vertices. Each computation vertex computes the minimum and maximum of the two input elements denoted by the incoming horizontal and vertical edges. The outgoing horizontal and vertical edges denote the minimum and maximum respectively of the two input elements computed by the computation vertex. The horizontal edges are labelled 11 and the source and sink vertices connected to horizontal edges are all labelled 11. The vertical edges are labelled 12 and the source and sink vertices connected to vertical edges are all labelled 12. The set of horizontal source vertices is {ih_1, ih_2, ih_3, ih_4} and the set of horizontal sink vertices is {oh_1, oh_2, oh_3, oh_4}. Similarly the set of vertical source vertices is {iv_1, iv_2, iv_3, iv_4} and the set of vertical sink vertices is {ov_1, ov_2, ov_3, ov_4}. The source vertices iv_1, iv_2, iv_3 and iv_4 are initialized to 2, 1Ø, 6 and 5 respectively. The source vertices ih_1, ih_2, ih_3 and ih_4 are all initialized to oo. It can be verified that oh_1, oh_2, oh_3 and oh_4 are 2, 5, 6 and 1Ø respectively.

E_{11}={set of horizontal paths} and E_{12}={set of vertical paths}. H={E_{11}, E_{12}}, F={ɸ} and S={ɸ}. The number of maximally-connected subgraphs SG with L_{SG}={11, 12} is 1 and hence the program graph is in CL. Augment SG by the set D_1 of diagnol paths. The resulting program graph is shown in figure 7.Ø-4 below:

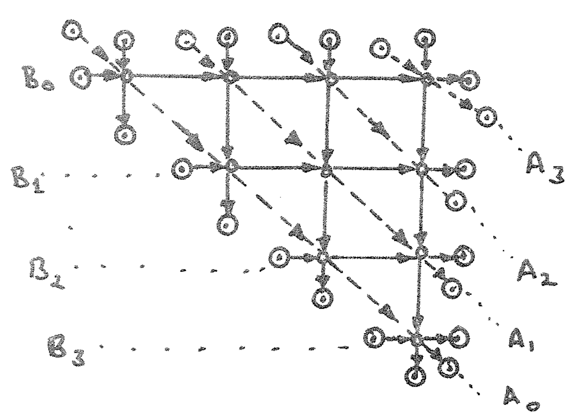

Figure 7.Ø-4: Sorting Graph

In figure 7.Ø-4 the dashed lines denote the diagnol paths. D_1={$A_\emptyset$, A_1, A_2, A_3} and E_{11}={$B_\emptyset$, B_1, B_2, B_3}. Let A=D_1 and hence B=E_{11}. So

$S_A=\{E_{11}, E_{12}\}$ and $S_B=\{D_1, E_{12}\}$. It can be verified that this program graph satisfies theorem 5.2-1. The consistency constants a_{11} and a_{12} for E_{11} and E_{12} in S_A are 1 and -1 respectively. The consistency constants b_{11} and b_{12} for D_1 and E_{12} in S_B are both 1.

We next construct a mapping of this program graph. Set $n_{1a}=Ø$ and $n_{1b}=1$. B is E_{11} and hence $n_{11}=1$. Set $n_{12}=a_{12}=1$. A is D_1 and hence set $d_{1a}=2$. Set $d_{1b}=1$ and hence $d_{11}=1$. $d_{12}=d_{1b}*a_{12}+d_{1a}*b_{12}$ and hence $d_{12}=1$. Set $t_Ø=7$. Map every computation vertex in A_i and B_j onto processor i at time 7+i+2*j. The mapping is shown in figure 7.Ø-5 below:

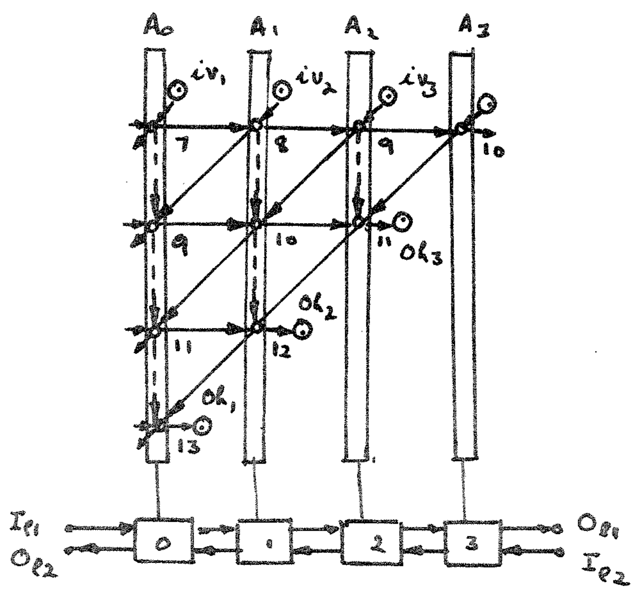

Figure 7.Ø-5

The time at which a computation vertex is mapped onto a processor is shown by the side of the computation vertex. We have to ensure that the contents of iv_1, iv_2 and iv_3 remain unchanged till they reach processors Ø, 1 and 2 respectively. Ψ_1 is the min function and Ψ_2 is the max function. ∞ is the identity element for $_1$ and $-\infty$ is the identity element for max function. Map $-\infty$ onto I_{11} at times 1, 3 and 5. Similarly to ensure invariance of the contents of oh_1, oh_2 and oh_3 map ∞ onto I_{12} at times 12, 14 and 16. This mapping is a synthesis of the rebound sorter [1].

8. Conclusions

In this paper we characterized a class of uniform graphs that are correctly computable on a model of linear array processors for VLSI. We illustrated our characterization by synthesizing published algorithms for two important computational problems. Obviously not all program graphs belong to this class. In [7] the characterization of arbitrary program graphs is being explored.

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