COMPACTNESS AND COMPLETENESS IN COMPONENT SEMANTICS

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In the paper which follows, we shall prove some generalized versions of some fundamental theorems in a form that has special interest to m-valued logic. Since some of the details will differ from the most conventional way in which things are expressed in model theory, a few preliminary explanations are desireable.

We will view what is normally called **satisfiability** as a relation between sets of well-formed formulae and members of a set of things we will call **model structures** such that if that relation holds we will say that model structure is a **model** of the set of wffs. In each case, we will regard the collection of model structures as describable by a set (or for the particular cases we are interested in here, a sequence) of items we will term **components,** with the specification that for each component, there exists a set of values which that component may take (not necessarily the same set for all components) and that the identity criterion for model structures is having the same value for each component, i.e. if x and y are model structures (in the same construction), x = y, if and only if, For every component $\mathcal{K}$, the value of $\mathcal{K}$ for x is the same as its value for y. In some cases, e.g. conventional propositional logic, the components may be the variables of the formal system; in others, they may be the predicates of the formal systems (and yet other possibilities occur). Similarly, in somme cases the values may be truth values, in others they may be individuals and in yet others they may be ordered couples of individuals and truth values (and again this does not exhaust the possibilities). In all cases, the crucial aspects are the identity criterion previously referred to, and the fact that the status of a particular model structure being a model of a particular set of wffs or not is completely determined by the values of its components.

In addition, a model structure will be called a model of a set of wffs iff it is a model of every element of the set.

Let us now assume we have a formal language $\mathfrak{L}$ with $\mathfrak{W}_{\mathfrak{L}}$ the set of wffs. Let $\mathfrak{K}$ be the set of components of a component-semantics for $\mathfrak{L}$.

We will call a component-semantics **quasi-independent** if there is a denumerable subset $\mathfrak{K}_I$ of $\mathfrak{K}$ such that for each combination of values for elements of $\mathfrak{K}_I$, there is at least one admissible model structure and for each $K_i \in \mathfrak{K}$, the set of values which K_i may take to assure the admissibility of the model structure depends on the values of a fixed finite subset of $\mathfrak{K}_I$.

We will now prove a number of theorems.

Theorem I : In any quasi-independent component-semantics, if:(1) $\mathfrak{W}$ and $\mathfrak{K}$ are denumerable, (2) every component has a finite range of values, and (3) for every wff, the status of a model structure (i.e. whether the structure is or is not a model), is dependent on a finite number of components; then: $\mathfrak{L}$ is compact (relative to $\mathfrak{K}$).

Proof: Let $\mathfrak{K} = K_1, K_2, K_3, \ldots$. Let $N_1, N_2, N_3, \cdots$ be the number of values in $K_1, K_2, K_3, \ldots$ respectively. For each component K_i, let its values be $v(i)_1, \cdots, v(i)_{n(i)}$. For each i, let f_i be a function from the values into the natural numbers such that $f_i[v(i)_j] = j - 1$. Let f be a function from the model structures into the real unit interval [0.1] such that if the values of m's components are $v(m)_1, v(m)_2, \cdots$, then

$$f(m) = \sum_{i=1}^{\infty} \frac{f(i)[v(m)_i]}{\prod_{j=1}^{i} (N_j + 1)}$$

It follows that $m_1 = m_2$ if and only if $f(m_1) = f(m_2)$. Note that not every real in the interval is in the range of f.

Let **A** be a finite set of wffs and let $\mathbf{R_A}$ be the set {f(m): m is a model of **A**}. Since $\mathbf{R_A}$ is bounded, it has a greatest lower bound. Let us represent f(m) as $f_1[v(m)_1], f_2[v(m)_2], \cdots$ (in analogy to the decimal expansion) and call this sequence the **mixed expansion** of f(m). Since there is a **k** such that m > **k** implies that K_m is not relevent to being a model of A, $.x_1x_2x_3\cdots \in \mathbf{R_A}$ implies $.x_1x_2x_3\cdots x_k y_{k+1} y_{k+2}\cdots$ is also, provided it is admissible.

Let $x = \text{glb}[f(A)]$. Then $\epsilon > 0$ implies there is a $y \in \mathbf{R_A}$ such that $y-x < \epsilon$. Since $f(i)[v(m)_i] < m_i$ for every i, the i^{th} digit of x is between 0 and $n_i - 1$. Hence for any i, there exists a $y \in \mathbf{R_A}$ which agrees with x in the first i places. Hence for i greater than the index of the greatest such on which model status depends, we conclude that x cannot be an admissible non-model.But since for every component $\mathcal{K}_j$, there also exists a model which agrees with x in the j^{th} component and all the components on which it depends, x is also admissible. Hence if $\mathbf{A}$ is finite, glb $\mathbf{R_A} \in \mathbf{R_A}$.

Let $\mathbf{B} = \{w_1, w_2, \cdots\}$ be a set of wffs such that every finite subset of $\mathbf{B}$ has a model. Let $\mathbf{B_n} = \{w_1, \cdots, w_n\}$. Then since $\mathbf{B_n} \subset \mathbf{B_{n+1}}$, glb $f(\mathbf{B_i})$ is a monotonically non-decreasing function of i. Hence the first place in its mixed expansion is monotonically non-decreasing. But since it can increase only a finite number of times, there exists an $\mathbf{n}$ such that the first place of glb $f(\mathbf{B_m})$ for $\mathbf{m} > \mathbf{n}$. But if $\mathbf{n}$ is a number such that $\mathbf{m} > \mathbf{n}$ implies the first j places of glb $f(\mathbf{B_n})$ are constant, the $(j + 1)^{st}$ place is monotonically non-decreasing for $\mathbf{m} > \mathbf{n}$ and hence there is an $\mathbf{n_0}$ such that $\mathbf{m} > \mathbf{n_0}$ implies the first $j + 1$ positions are constant. Hence for every k, there is an $\mathbf{n}$ such that $\mathbf{m} > \mathbf{n}$ implies the first k places of glb $f(\mathbf{B_m})$ are constant. It follows that there is a model structure $\mathbf{m_0}$ such that lub (glb $\mathbf{B_i}$)) = $f(\mathbf{m_0})$. By the same argument that we applied to the glb, $f(\mathbf{m_0})$ must be admissible. For any i, there is a k such that only the first k components are relevent to being a model of $\mathbf{B_i}$. But then there is an $\mathbf{n} > \mathbf{i}$ such that the first k places of glb($\mathbf{B_n}$). agrees with $f(\mathbf{m_0})$ in the first k places. Hence $\mathbf{m_0}$ is a model of $\mathbf{B_i}$ if and only if the $\mathbf{m_i}$ such that $f(\mathbf{m_i})$ = glb $f(\mathbf{B_n})$ is. Since $\mathbf{B_i} \subset \mathbf{B_n}$, every model of $\mathbf{B_n}$ is a model of $\mathbf{B_i}$ so that $\mathbf{m_1}$ and hence $\mathbf{m_0}$ must be a model of $\mathbf{B_i}$. Hence $\mathbf{m_0}$ is a model of each $\mathbf{B_i}$, and hence of each w_i, i.e. $\mathbf{m_0}$ is a model of $\mathbf{B}$.

Since the conditions of theorem I are satisfied by a propositional logic with the components being the variables and the values the truth-values, theorem I gives us compactness for normal two-valued propositional logic and with $m = n_i$ for all i, for any extensional m-valued propositional logic. Since dependence on a finite number of components is a weaker condition than extensional dependence, in principle a variety of m-valued context dependent logics ought to satisfy our conditions, though it is not clear whether any interesting ones do. In addition, it should be noted that our argument does not require total independenceof the components, so that taking as components the one-place predicates and there negations closed under conjunction, our result also applies to two and m - valued monadic predicate logic. This case illustrates a peculiarity of our method since it shows that for the same system different sets of components components can be defined some of which may while others may not satisfy the conditions of our theorem. This does however raise the difficulty of not not knowing that our theorem fails to apply (short of being able to show that compactness fails). Thus for example we are not aware of any component-semantics for classical predicate logic which satisfy our conditions. But we know no proof of non-existence and so, since compactness does hold (as is well-known), one may nevertheless exist.

Now let call a set of wffs $\mathbb{D} = \{D_{ij}$: i and j are positive integers) a **D-family** of wffs. We will call a set of wffs α a **D-set** (relative to $\mathbb{D}$) provided for every i there is a unique j such that $D_{ij} \in \alpha$.

Theorem II: If $\mathbb{D} = \{D_{ij}\}$ is a D-family satsfying:

1. **For every i,j and k and wff A, $j \neq k \Rightarrow D_{ij}, D_{ik} \vdash A$**
2. **For every i, every wff A and every set of wffs α, $(\alpha, D_{ij} \vdash A$, for every $j) \Rightarrow \alpha \vdash A$**
3. **If α is a D-set, $[\text{not}(\alpha \vdash A)] \Rightarrow \alpha \cup \{A\}$ is (syntactically) inconsistent.**
4. **For any set of wffs α and any D_{ij}, if $\alpha \vdash D_{ij}$, there exists a finite set β such that $\beta \subset \alpha$ and $\beta \vdash D_{ij}$.**
5. **For each i, the number of distinct D_{ij} is finite.**
6. **The set of wffs is denumerable.**

Then: Every (syntactically) consistent set of wffs is contained in a maximally consistent set.

Proof: Let $\{w_1, w_2, \cdots\}$ be the set of all wffs. Let α be a consistent set of wffs. Let $\alpha_0 = \alpha$ and $\alpha_{n+1} = \alpha_n \cap \{w_{n+1}\}$, if that is consistent and $\alpha_{n+1} = \alpha_n$ otherwise. Let $\beta = \cup \alpha_i$.(i a positive integer).

a. **For every α, α_i is consistent.** α_0 is consistent since it is α. For every n, if α_n is consistent, either either $\alpha_n \cup \{w_{n+1}\}$ is consistent and equal to α_{n+1}, or else $\alpha_n \cup \{w_{n+1}\}$ is inconsistent $\alpha_{n+1} = \alpha_n$ and hence is consistent.

b.**For every i, β conts no more than one D_{ij}.** Since α_i is consistent, for every i, then for every i and n, α_n has no more than one D_{ij}. But since the α_n are mononically non-decreasing with increasing n, $[D_{ij} \in \alpha_n$ and $m > n] \Rightarrow D_{ij} \in \alpha_m$. Hence $\cup\alpha_n$ contains no more than one D_{ij}.

c.**For each i there is a j such that $D_{ij} \in \beta$.** For every i and j, there is a **K**(i,j) such that $D_{ij} = w_{\mathbf{K}(i,j)}$. Let **K**(i) = max(**K**(i,j)). If not-($D_{ij} \in \beta$) for every i, $\alpha_{\mathbf{K}(i)} \cup \{D_{ij}\}$ is inconsistent, for every j. Hence $\alpha_{\mathbf{K}(i)}$ is inconsistent by condition 2, which contradicts a).

d.**β is consistent.** Suppose not. Then for some i, there exists a j and a k such that $j \neq k$, $D_{ij} \in \beta$ and $\beta \vdash D_{ik}$. Then by condition 4, there exists a finite set β' such that $\beta' \subset \beta$ and $\beta' \vdash D_{ik}$. Hence, there exists an n such that for every $m > n$, $\beta' \subset \alpha_m$ and hence $\alpha_m \vdash D_{ik}$. Since however $D_{ij} \in \beta$, there exists an n' such that $D_{ij} \in \alpha_{n'}$. Hence for sufficiently large m $\alpha_m \vdash$ Dij and $\alpha_m \vdash D_{ik}$ and hence is inconsistent by condition 1, contrary to a).

e. **β is maximal.** Suppose not($A \in \beta$). But there is an n such that $A = w_n$. Hence $\alpha_{n-1} \cup \{A\}$ is inconsistent. Hence $\beta \cup \{A\}$ is also, since $\alpha_{n-1} \subset \beta$. Hence β is maximally consistent.

Theorem III: Under the conditions of theorem II, for every maximally consistent set α, there exists a unique D-set, of which α is its unique

maximally consistent extension.

Proof: Let α be maximally consistent, then it is its own maximally consistent extension. Then by b) and c) of theorem II, it contains a D-set. Let α and β be maximally consistent and contain the same D-set μ. Suppose $A \in \alpha$ and not($A \in \beta$). Since β is maximally consistent, we have not($\beta \vdash A$). Since $\mu \subset \beta$, not($\mu \vdash A$). Since μ is a D-set, $\mu \cup \{A\}$ is inconsistent, and by condition 3, since $\mu \subset \alpha$, $\alpha \cup \{A\}$ is inconsistent, contrary to assumption.

Note that a consequence of theorem III is that under the conditions of theorem II, there is a one-one correspondence between consistent D-sets and the maximally consistent sets which are their extensions. Note also that a collection of D-sets under those conditions provide a component-semantics for the formal systems. That this component-semantics is a "natural" one can be seen from the following theorems.

Theorem IV: Under the conditions of theorem II, a formal system is sound relative to the component-semantics generated by its D-sets.

Proof: Assume $\alpha \vdash A$. If α is inconsistent, every one of its extensions is also. Hence trivially every complete consistent extension of α contains A. If α is consistent, then every consistent extension of α is consistent with A, so that every complete consistent extension of α contains A. If β is a complete consistent extension of α and μ its D-set, $A \in \beta$. Hence $\mu \cup \{A\}$ is consistent also. Hence, by condition 3, $\mu \models A$. Hence $\alpha \models A$.

Theorem V. Let $\mathcal{L}$ be a formal language such that the conditions of theorem II are satisfied and for every wff A, there is a finite set of components (in the component-semantics generated by its D-sets) β such that for every D-set μ there is a set ν such that $\nu \subset \mu$ and $D_{ij} \in \nu$ implies $K_i \in \beta$, and not($\nu \vdash A$) implies $\mu \cup \{A\}$ is inconsistent.

Then $\mathcal{L}$ is complete relative to the component-semantics generated by its D-sets.

Proof Let A be a wff. Then there is a finite set of components such that the status of a model structure being a model of A depends only on the value of these components. Hence by theorem I, $\mathcal{L}$ is compact. Hence $\alpha \models A$ implies there is a finite set $\beta \subset \alpha$ such that $\beta \models A$. Hence there is a $\mathbf{k}$ such that $m > \mathbf{k}$ implies the value of K_m has no effect on a model structure being a model. Hence for any sequence $D_{1w(1)}, \ldots, D_{\mathbf{k}w(\mathbf{k})}$, we have $\beta, D_{1w(1)}, \ldots, D_{\mathbf{k}w(\mathbf{k})} \models A$ since either $D_{1w(1)}, \ldots, D_{\mathbf{k}w(\mathbf{k})}$ is an initial sequence of the D-set μ and hence $D_{1w(1)}, \ldots, D_{\mathbf{k}w(\mathbf{k})} \vdash A$, or else $\beta \cup \{D_{1w(1)}, \ldots, D_{\mathbf{k}w(\mathbf{k})}\}$ is inconsistent. Consequently by $\mathbf{k}$ applications of condition 2, $\beta \vdash A$ and hence $\alpha \vdash A$.

Theorem UI: If $\mathcal{L}$ is a formal language such that the conditions of theorem II are satisfied and for every wff A, there exists a finite set of wffs $\{A_2, \cdots, A_n\}$ such that for every set α of wffs and every wff B [for every i,$(\alpha, A_i \vdash B)$] $\Rightarrow \alpha \vdash B$, and [$i \neq j \Rightarrow \{A_i, A_j\}$ is inconsistent], then $\mathcal{L}$ is complete relative to the component-semantics generated by its D-sets.

Proof Suppose not ($\alpha \vdash A_1$). Then there exist an A and a set of n-1 wffs $\{A_2, \cdots, A_n\}$ such that [for every i,$(\alpha, A_i \vdash A_1) \Rightarrow \alpha \vdash A_1$. Hence there exists an i ($1 \leq i \leq n$) such that not ($\alpha, A_i \vdash A_1$). Hence there exists an i ($1 \leq i \leq n$) such that $\alpha \cup \{A_i\}$ is consistent. Then by theorem II, there is a cosistent complete set β such that $\alpha \cup \{A_i\} \subset \beta$. Since β is consistent and contains A_i for some i ($2 \leq i \leq n$), β does not contain A_1. Hence, by theorem III, there is a D-set μ of which β is its unique complete consistent extension. Hence $\mu \vdash C$ for every $C \in \alpha$ and $\mu \vdash A_i$. Hence, not($\mu \vdash A_1$). Therefore, not ($\alpha \models A_1$).