# A GENERALIZED DELAUNAY TRIANGULATION AND THE SOLUTION OF CLOSEST NODE PROBLEMS IN THE PRESENCE OF BARRIERS

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## Abstract

A Delaunay triangulation of a set of nodes is a collection of triangles whose vertices are at the nodes and whose union fills the convex hull of the set of nodes. It also has several geometrical properties making it useful for solving closest point problems. The generalization presented here allows the triangulation to cover non-convex regions including those with holes. Although various such generalizations are possible, the one presented here is shown to retain important closest point characteristics. Thus it is useful for determining shortest paths within planar regions with polygonal boundaries.

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# A Generalized Delaunay Triangulation and the Solution of Closest Node Problems in the Presence of Barriers

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## 1. Introduction

Given a set of nodes arbitrarily distributed in the plane, the Delaunay triangulation provides an efficient means for treating closest node and other computational geometry problems. An extensive discussion and bibliography may be found in Preparata and Shamos [5]. In this paper, we present definitions, theorems, and algorithms for a generalization of the Delaunay triangulation: one that can deal with the presence of non-convex regions, holes in the regions, and predetermined edge constraints. This generalized Delaunay triangulation can often be used for modeling in situations where the standard Delaunay triangulation cannot be applied. Two important applications are the fitting of bivariate surfaces over non-convex regions and the fitting of discontinuous bivariate surfaces (e.g. the fitting of geological surfaces in regions with faults, Cline and Renka [2]) Another is the solving of closest node and shortest path problems in the presence of physical barriers (e.g., the shortest water route from Athens to Liverpool or the optimal path of a robot making its mail delivery route on the floor of a business establishment).

In the second section, we define the generalized Delaunay triangulation and present algorithms for obtaining its construction. The algorithms are based upon an alteration of a standard Delaunay triangulation. The third section is concerned with characterizing distances and shortest paths in the presence of barriers. The last section shows that the generalized Delaunay triangulation is the natural structure for solving closest node problems and that the closest node algorithm can easily be adapted to yield shortest paths.

## 2. Generalized Delaunay Triangulation

The standard Delaunay triangulation problem can be stated as follows:

Given a finite set $S$ of points in the plane (the *nodes*[1]), determine a set $T$ of triangles so that

[1] Throughout this discussion, *node* will refer to an element of $S$. We will use the term *point* in general reference to any element of the plane.

1. The vertices of the triangles are nodes.
2. No triangle contains a node other than its vertices.
3. The interiors of the triangles are pairwise disjoint.
4. The union of triangles is the convex hull[2] of $S$.
5. The interior of the circumcircle of each triangle contains no node.

Any set $T$ satisfying the first four properties is a *triangulation* of $S$. Property 5 sufficiently restricts such triangulations so that the solution to the problem is unique except in cases where the circumcircle of a triangle contains four or more nodes on its boundary. (In such cases, two or more closely related Delaunay triangulations exist.)

The intention of this section is to modify the problem by replacing property 4 regarding the convex hull by one that allows arbitrary polygonal boundaries. We will also allow certain triangle edges to be specified and will modify the circumcircle property appropriately.

To this end, let $l \geq 1$ and $B_1, \ldots, B_l$ be simple, closed polygonal curves in the plane. We assume that they are pairwise disjoint and that each line segment in each polygonal curve connects a pair of nodes but is otherwise disjoint from the set of nodes. The line segments are termed *boundary edges*. We will be concerned here only with problems for which each curve has a positive orientation such that no node is contained in its exterior. The closure of the mutual interiors of the curves is labeled $\Omega$; i.e.,

$$\Omega = \text{closure}\ (\ \text{interior}\ (B_1) \cap \cdots \cap \text{interior}\ (B_l))$$

and we assume $\Omega$ is bounded[3]. These assumptions guarantee that $\Omega$ is connected and closed and that $S \subset \Omega$. Those nodes that are contained in some $B_i$ are termed *boundary nodes*.

In addition to the boundary curves, we allow an additional set of line segments $E$ to be specified as required interior edges in the triangulation. We insist that the line segments connect pairs of nodes, that no other nodes lie on such segments, and that the line segments are contained in $\Omega$. Pairs of segments may have intersection only at the endpoints. The union of $E$ with the set of boundary edges is termed the set of *required edges* and denoted by $R$. A required edge *separates* two points if the interior of the line segment between those points intersects the required edge.

Reflecting the presence of required edges, the circumcircle test, property 5, is weakened to:

Modified circle test:

If any node is contained in the interior of the circumcircle of a triangle then every interior point of the triangle is separated from that node by a required edge.

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[2] The *convex hull of S* is the smallest convex set containing $S$.

[3] It follows that one boundary curve must have an unbounded exterior and any other boundary curves have bounded exteriors. The exteriors of these latter boundary curves are termed *holes*.

With this modification, a triangle may pass the test even if the interior of its circumcircle contains nodes. Any such node however must be "on the other side" of a required edge from the interior of the triangle.

We are now prepared to state the generalized Delaunay triangulation problem:

Given a finite set of nodes $S$, a collection of polygonal boundary curves, $B_1,...,B_l$, defining $\Omega$ (a superset of $S$), and a set of required edges $R$, determine a set of triangles $T$ so that

1. The vertices of the triangles are nodes.
2. No triangle contains a node other than its vertices.
3. The interiors of the triangles are pairwise disjoint.

4′. The union of the triangles is $\Omega$.

5′. If any node is contained in the interior of the circumcircle of a triangle, then every interior point of the triangle is separated from the node by an element of $R$.

6. Each element of $R$ is an edge of one of the triangles.

Figure 1 displays an example problem in which there are 29 nodes, two boundary curves, and two required edges not on the boundary. The standard Delaunay triangulation of $S$ is shown in Figure 2. Figure 3 shows the generalized Delaunay triangulation.

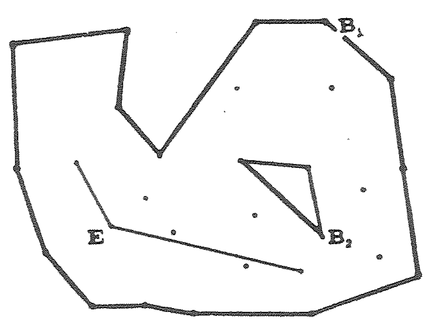


**Figure 1.** An example problem

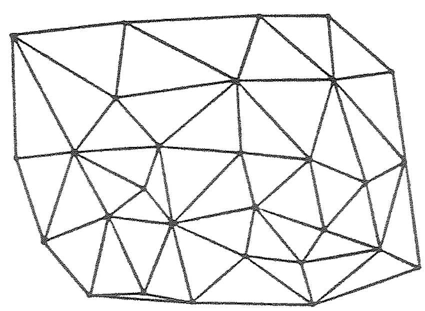

**Figure 2.** The standard Delaunay triangulation of the example

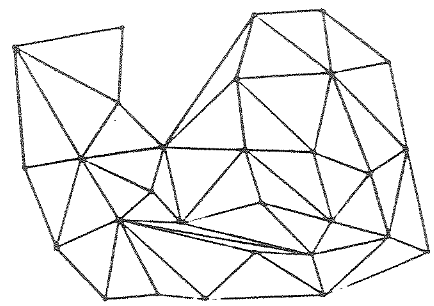

**Figure 3.** The generalized Delaunay triangulation of the example.

The generalized problem is an extension of the standard triangulation problem: by letting $\Omega$ be the convex hull of $S$, $B_1$ be its boundary, and $R = B_1$, a solution to the generalized problem is a solution to the standard problem. This is suggestive of our algorithm for solving the generalized problem: we shall first solve the standard problem, and then, one-by-one, make appropriate modifications to satisfy each edge constraint.

An important observation for the understanding of the algorithm for the generalized problem is the effect of adding a single required edge. Theorem 1 shows that the only necessary modification to the triangulation for the addition of a single required edge is the retriangulation of that region whose triangles intersect the new edge. Before stating the theorem, a definition, some notation, and two lemmas will be introduced.

**Definition**: Given a pair of distinct points $a$ and $b$, let $\vec{ab}$ denote the directed line segment from point $a$ to point $b$. A third distinct point $c$ is *strictly left of* $\vec{ab}$ if the angle $<bac$ (measured counterclockwise from $\vec{ab}$ to $\vec{ac}$) is strictly between 0 and $\pi$ radians in measure. Notice that any point $c$ is either strictly left of $\vec{ab}$, strictly left of $\vec{ba}$, or collinear with $a$ and $b$.

**Notation**:

1. The closed line segment from point $a$ to point $b$ is denoted by $[a,b]$. (Notice that $[a,b] = [b,a]$.)
2. The open line segment from point $a$ to point $b$ is denoted by $(a,b)$. (Notice that $(a,b) = (b,a)$.)
3. For $c$ strictly left of $\vec{ab}$, the triangle with vertices $a$, $b$, and $c$ is denoted $\Delta abc$. The vertices of triangles are always specified in counterclockwise order.
4. The circumcircle of $\Delta abc$ is denoted by $\bigcirc\!\Delta abc$.

**Lemma 1**: Given $\Delta abc$, if $x$ is strictly left of $\vec{ba}$ and is interior to $\bigcirc\!\Delta abc$ and $d$ is strictly left of $\vec{ba}$ but is not interior to $\bigcirc\!\Delta abc$ then $x$ is interior to $\bigcirc\!\Delta adb$.

**Proof**: The segment $[a,b]$ is a chord of both $\bigcirc\!\Delta abc$ and $\bigcirc\!\Delta adb$ (Figure 4a). Since $x$ is interior to the first circle, $<bxa$ is strictly greater than half the arc of $\bigcirc\!\Delta abc$ subtended by $[a,b]$ and containing $c$. Since $d$ is not interior to the first circle, $<bda$ is less than or equal to half of this arc. Yet $<bda$ is half the arc of $\bigcirc\!\Delta adb$ subtended by $[a,b]$ (on the side opposite $d$), and thus $<bxa$ is strictly greater than half of this same arc of $\bigcirc\!\Delta adb$. We conclude that $x$ is interior to $\bigcirc\!\Delta adb$. □

**Lemma 2**: Given $\Delta uvw$ and $\Delta uwx$, if $x$ interior to $\bigcirc\Delta uvw$ then $v$ is interior to $\bigcirc\Delta uwx$.

**Proof**: Since $x$ is interior to $\bigcirc\Delta uvw$ and on $\bigcirc\Delta uwx$, the arc of $\bigcirc\Delta uvw$ subtended by $<wvu$ is larger than the arc of $\bigcirc\Delta wxu$ subtended by the same angle (Figure 4b). Since the first arc has twice the measure of the angle, $v$ must be interior to $\bigcirc\Delta uwx$. □


**Figures 4a-4b.**

**Theorem 1**: Given a generalized Delaunay triangulation $T$ of $S$ with required edges $R$ and given an second set of required edges $R' = R \cup \{e\}$ augmented by a single edge $e$, let $T^e$ be the set of those triangles whose interiors intersect $e$, $\Omega^e$ be the union of those triangles, $R^e$ be the set of required edges lying within $\Omega^e$, $B^e$ be the boundary of $\Omega^e$, and $S^e$ be the nodes contained in $\Omega^e$. If $T^*$ is a generalized Delaunay triangulation of $S^e$ over $\Omega^e$ with required edges $R^e \cup B^e \cup \{e\}$, then $T' = (T - T^e) \cup T^*$ is a triangulation of $S$ with required edges $R'$.

**Proof**: It is clear that the set of triangles $T'$ satisfies all properties of a generalized Delaunay triangulation with the possible exception of property 5′. Thus it must be shown that each triangle of $T'$ passes the modified circle test.

We proceed by assuming the property is violated for some $\Delta abc$. Let $x$ be a node interior to $\bigcirc\Delta abc$ and yet not separated from some interior point $p$ of $\Delta abc$ by any edge of $R'$ (Figure 5). By property 2, we may assume $x$ is exterior to $\Delta abc$. Without loss of generality, assume $x$ is strictly left of $\vec{ba}$. If edge $[a,b]$ were required (i. e. in $R'$), then $x$ would be separated from all interior points of $\Delta abc$ by $[a,b]$, contrary to assumption. Therefore, edge $[a,b]$ must not be required and thus could not be a boundary edge, since all boundary edges are required. It follows that a triangle sharing edge $[a,b]$ with $\Delta abc$ must exist in $T'$. Denote such a triangle by $\Delta adb$ and consider two cases: $d \neq x$ or $d = x$.

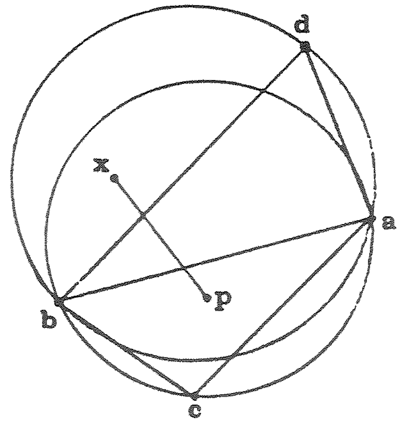


**Figure 5.**

In the first case, by virtue of Lemma 1, $x$ is also interior to $\bigcirc\Delta adb$. Again without loss of generality, we may assume $x$ is strictly left of $\vec{bd}$. We conclude that $<bxa$ is less than $<bxd$ and that $x$ is unseparated from some interior point of $\Delta adb$ (since the segment $(x,p)$ intersects the interior of $\Delta adb$). The situation of $x$ with respect to $\Delta adb$ is identical of that of $x$ with respect to $\Delta abc$ and the subtended angle at $x$ has strictly increased. As before, edge $[b,d]$ cannot be required or a contradiction would result. Also as before, a triangle sharing edge $[b,d]$ must exist and this new triangle must have $x$ as a vertex or not (i. e. the same cases as before).

The process of finding new triangles including $x$ in their circumcircles (yet having interior points unseparated from $x$) can continue as long as $x$ is not a vertex of an adjacent triangle. However, eventually it must terminate since only a finite number of triangles exist and the subtended angles monotonically increase. Thus a triangle $\Delta uvw$ of $T'$ must be found so that $x$ is not separated from some one of its interior points but is interior to its circumcircle, and x is the third vertex of a triangle sharing an edge with $\Delta uvw$. We may assume the shared edge is $[u,w]$ and, by virtue of Lemma 2, conclude that $v$ is interior to $\bigcirc\Delta uwx$. Notice that $v$ is unseparated from all interior points of $\Delta uwx$ (since $[u,w]$ is not required). However, if $\Delta uwx \in T - T^e$, then it satisfies the modified circle test with respect to $R$ since $T$ is a generalized Delaunay triangulation. This is a contradiction because $v$ is interior to $\bigcirc\Delta uwx$ and unseparated from the triangle's interior. Furthermore if $\Delta uvw \in T - T^e$, then the modified circle test for it would be violated by $x$, which is also contrary to hypothesis.

The final option is for both $\Delta uvw$ and $\Delta uwx$ to be in $T^*$. But the triangles of $T^*$ must satisfy the modified circle test with respect to $R^e \cup B^e \cup \{e\}$. The only edge that could separate $x$ from $\Delta uvw$ would be $[u,w]$. But $[u,w]$ cannot be either in $R^e$ or be the new edge e since $R'$ includes both $R^e$ and e. Finally, $[u,w]$ cannot be in $B^e$ either because it is a diagonal of quadrilateral $uvwx$ which is a subset of $\Omega^e$. We conclude that any violation of the circle test results in a contradiction. □

The following observations lead to an efficient procedure for determining the set of triangles that satisfy the modified circle test. First, we recognize that given any circle and a chord of that circle, that the angle subtended by the chord at any point on the circle (the "circumference angle") is constant. Furthermore, at points interior to the circle the subtended angles are larger and from points exterior to the circle the angles are smaller. (We assume throughout that the points in question are within the same half-plane determined by the chord.) Thus if a node $x$ maximizes the angle subtended by a line segment $[a,b]$ over all nodes strictly left of $\vec{ab}$, then no node also strictly left of $\vec{ab}$ is in $\bigcirc\Delta abx$. Alternatively stated, if $[a,b]$ is known to be an edge of a triangle in the triangulation (perhaps because it is required), and the third vertex of the triangle is sought, then this vertex can be located by maximizing the angle subtended by $[a,b]$. The resulting triangle may not satisfy the modified circle test because its interior might be intersected by a required edge; Theorem 2 shows how this maximization can be restricted to a certain subset that guarantees satisfaction of the modified circle test.

**Theorem 2**: Let $[a,b]$ be either a required edge or an edge of a triangle $\Delta acb$ in a generalized Delaunay triangulation, and let $m$ be the midpoint of this edge. If, over all nodes strictly left of $\vec{ab}$ and unseparated from $m$, the angle subtended by $[a,b]$ is maximized at node $x$, then $\Delta abx$ satisfies the modified circle test.

**Proof**: Let $Y$ be the set of points interior to $\bigcirc\Delta abx$, strictly left of $\vec{ab}$, and unseparated from $m$ by any required edge (Figure 6). This set clearly is connected as is its closure. It contains no nodes since at such a node a larger angle would be subtended than that at $x$. The boundary of $Y$ must consist only of the edge $[a,b]$, arcs of the circumcircle, and segments of required edges. Since the segments of the required edges on this boundary can have no nodes interior to $\bigcirc\Delta abx$, the region $Y$ is the intersection of the interior of the circumcircle with the open half-planes supported[4] by the the required edges and containing $[a,b]$ and the set of points strictly left of $\vec{ab}$. We conclude that $Y$ is convex.

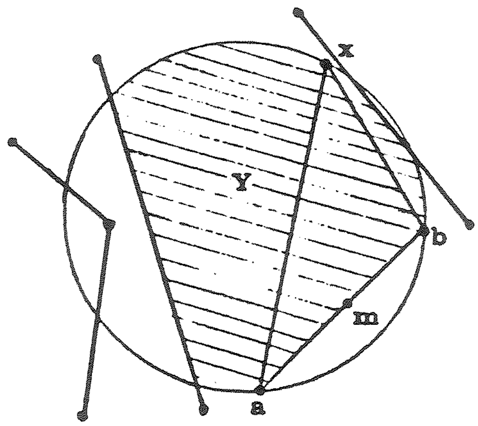


**Figure 6.**

By virtue of the convexity of $Y$ and the fact that $a$, $b$, and $x$ lie on its boundary, we may conclude that the interior of $\Delta abx$ is a subset of $Y$. This guarantees that no required edges intersect the interior of $\Delta abx$. To show that no interior point of $\Delta abx$ is unseparated from any node interior to $\bigcirc\Delta abx$ and left of $[a,b]$, we suppose such an interior point $p$ and node $z$ exist. Since $z \notin Y$, the segment $(p,z]$ must intersect the boundary of $Y$. The intersection cannot be at $z$ since then $m$ would be unseparated from $z$. The intersection cannot be on the boundary of $\bigcirc\Delta abx$ since both $p$ and $z$ are interior to this, nor can it be on $[a,b]$ since both are strictly left of $[a,b]$. From the earlier remarks about the boundary of $Y$, it follows that this intersection point must be on a required edge; but this contradicts the assumption that $p$ was unseparated from $z$ by any required edge. We conclude that no nodes unseparated from interior points of $\bigcirc\Delta abx$ lie strictly left of $[a,b]$.

If $[a,b]$ is required then it separates the interior of $\Delta abx$ from all nodes strictly left of $\vec{ba}$ (i. e. strictly right of $\vec{ab}$) and interior to $\bigcirc\Delta abx$. If $[a,b]$ is not required then it cannot be a boundary edge, and there must exist a triangle $\Delta acb$. Lemma 2 guarantees that $c$ is not in $\bigcirc\Delta abx$ (since $\Delta acb$ satisfies the modified circle test and $x$ must not be interior to $\bigcirc\Delta acb$). If a node $z$ strictly left of $\vec{ba}$ and interior to $\bigcirc\Delta abx$ exists, then by virtue of Lemma 1, $z$ must be interior to $\bigcirc\Delta acb$. By the modified circle test for $\Delta acb$, $z$ must be separated from the interior of $\Delta acb$ by a required edge and this edge must also separate $z$ from the interior of $\Delta abx$.

Finally, since no nodes are on the interior of $[a,b]$, we conclude that no nodes interior to $\bigcirc\Delta abx$ are unseparated from any interior point of $\Delta abx$. □

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4 An open half-plane is *supported* by an edge if that edge is on the boundary of the half-plane.

Maximization of the subtended angle is equivalent to maximizing the negative of its cotangent since all candidate angles are less then $\pi$. The negative of the cotangent can be calculated with 15 arithmetic operations by using cross products and scalar products.

From Theorem 1, to update a triangulation with the addition of a required edge $e = [a,b]$ only the region $\Omega^e$ must be retriangulated. According to Theorem 2, one new triangle on the left side of $\vec{ab}$ is obtained by maximizing the subtended angle over those nodes in $S^e$ that are also left of $\vec{ab}$ and unseparated from the midpoint of the edge. Labeling $x$ a maximizer, the triangle $\Delta abx$ is added to the triangulation. Another new triangle can be obtained by maximizing the angle subtended by $[a,x]$ over all nodes in $S^e$ that are left of $\vec{ax}$ and unseparated from the midpoint of $[a,x]$. Still another new triangle can be obtained by maximizing the angle subtended by $[x,b]$ over all nodes in $S^e$ that are left of $\vec{xb}$ and unseparated from the midpoint of $[x,b]$. This process can be applied recursively to triangulate that portion of $\Omega^e$ in the left half plane defined by $\vec{ab}$. Since the right half plane defined by $\vec{ab}$ is the left half plane defined by $\vec{ba}$, the same approach can be used with the roles of $a$ and $b$ reversed to obtain the retriangulation of that portion of $\Omega^e$ on the right of $\vec{ab}$. The result is a retriangulation of $\Omega^e$ with the edge $[a,b]$ included.

The following two lemmas guarantee that the retriangulation process does not result in overlapping triangle interiors.

**Lemma 3**: Suppose we are given a set of required edges $R$ and any two triangles $\Delta abc$ and $\Delta def$ whose interiors are not intersected by any element of $R$. Assume both triangles satisfy the modified circle test with respect to the required edges $R$ and the set of nodes $\{a,b,c,d,e,f\}$ and all endpoints of elements of $R$. If the nodes are not all cocircular then the interiors of $\Delta abc$ and $\Delta def$ must be disjoint.

**Proof**: Consider the interiors of $\bigcirc\Delta abc$ and $\bigcirc\Delta def$. Either the interiors are disjoint, one is a subset of the other, or the boundaries of $\bigcirc\Delta abc$ and $\bigcirc\Delta def$ intersect in exactly two points. In the first case, the triangle interiors themselves must be disjoint since they are respective subsets of the circumcircle interiors. For the second case, we may assume without loss of generality that $\bigcirc\Delta abc \subset \bigcirc\Delta def$. Suppose there exists a point $x$ common to both triangle interiors. Since the circumcircles are not identical, at least one vertex of $\Delta abc$ is interior to $\bigcirc\Delta def$. Assume it to be $a$. By virtue of the modified circle test, $(x,a)$ must intersect a required edge, but this segment is interior to $\Delta abc$ contrary to the hypothesis on the required edges. Thus no such point can exist in the second case. Finally, consider the third case and assume the circumcircle boundaries intersect at exactly two points $g$ and $h$. In this case, the line through $g$ and $h$ separates the plane into two open half-planes. In one half-plane a portion of the interior of $\bigcirc\Delta abc$ is a subset of the interior of $\bigcirc\Delta def$. In the other half-plane, the reverse is true. If $a$, $b$, and $c$ lie in the closure of the first half-plane, and $d$, $e$, and $f$ lie in the closure of the other half-plane, then the triangle interiors must be disjoint. Otherwise, either one of $a$, $b$, or $c$ lies in the second open half-plane or one of $d$, $e$, or $f$ lies in the first open half-plane. Any such vertex must be interior to the other triangle's circumcircle. Without loss of generality, assume $a$ is such a vertex and $x$ is interior to both triangles. Since $(x,a)$ must intersect a required edge and be interior to $\Delta abc$, a contradiction results just as in the second case. □

Using the notation of Theorem 2, if $x$ along with a another node $y$ unseparated from $m$ attain the maximum subtended angle, then $y$ lies on the boundary of $\bigcirc\Delta abx$. Alternatively stated, $a$, $b$, $x$, and $y$ are cocircular. In fact, all such maximizers are cocircular with $a$, $b$, and $x$. Any choice among the maximizers for the third vertex of a triangle would preserve the modified circle test (that was the conclusion of Theorem 2). However, some choices could produce overlapping triangles. In these cocircular cases, the following rule removes the ambiguity of choice in such a manner that overlapping triangles are avoided. Lemma 4 is a statement of this.

**Cocircular Selection Rule**: Order the maximizers in counterclockwise fashion from $b$ on their common circle. If no maximizers share a triangle edge with $b$, let $x$ be the first maximizer counterclockwise from $b$. Otherwise, let $x$ be the most counterclockwise maximizer from $b$ that shares a triangle edge with $b$.

**Lemma 4**: Let $[a,b]$ be either a required edge or an edge of a triangle in a generalized Delaunay triangulation, and let $m$ be the midpoint of this edge. If, over nodes strictly left of $\vec{ab}$ and unseparated from $m$, the angle subtended by $[a,b]$ is maximized at more than one node, then the node $x$ chosen by the cocircular selection rule guarantees that $\Delta abx$ has interior disjoint from the interior of any other triangle in the triangulation.

**Proof**: Applying Lemma 3, we recognize that the only triangles that might intersect the interior of $\Delta abx$ are ones that have identical circumcircles. First we notice that if a $\Delta abz$ already exists in the triangulation then the node $z$ is exactly $x$ as chosen by the rule. Henceforth, we shall assume no such triangle exists in the triangulation. Suppose $\Delta def$ is a triangle whose interior intersects that of $\Delta abx$ and thus $a$, $b$, $x$, $d$, $e$, and $f$ are cocircular. If all of $d$, $e$, and $f$ are on the closed arc counterclockwise from $a$ to $b$ then no intersection is possible. In fact, if any of the three are on this arc without all three being such, then the edge $[a,b]$ must intersect the interior of $\Delta def$. Whether $[a,b]$ is a required edge or the edge of $\Delta acb$, a contradiction results. Thus we conclude that all of $d$, $e$, and $f$ are on the closed arc from $b$ counterclockwise to $a$. Without loss of generality, we assume that in counterclockwise order from $b$ they occur: $d$, $e$, $f$. To have any intersection with the interior of $\Delta abc$, $d$ must precede $x$ in the counterclockwise order and $f$ must follow $x$. If $d$ is identical with $b$, then by the rule, all three vertices of $\Delta def$ lie on the sub-arc from $b$ to $x$ and hence no intersection with $\Delta abx$ is possible. If $d$ is not $b$ but precedes $x$ in counterclockwise order $x$ cannot be the first maximizer counterclockwise from $b$. The selection rule then guarantees a triangle edge $[x,b]$ exists and the associated triangle must have an intersection with $\Delta def$. □

Having guaranteed that the maximization procedure allows only triangles that satisfy the modified circle test and that do not intersect the interiors of existing triangles, we may formalize the recursion suggested by the results of Theorems 1 and 2. We assume that *nodeset* is some set of nodes, $a$ and $b$ are nodes not in *nodeset*, $T$ is a set of triangles, and $R$ is a set of required edges.

**Algorithm 1**: *retriangulate* (*nodeset*, $a$, $b$, $T$, $R$)

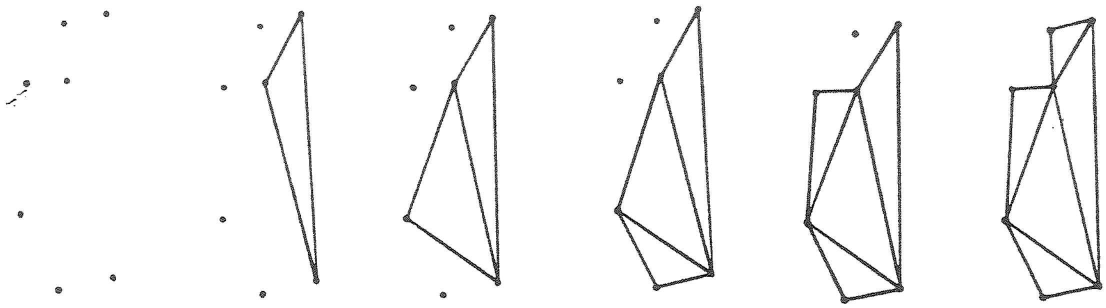
1. Define $X$ as the set of elements of *nodeset* strictly left of $\vec{ab}$ and unseparated from the midpoint of $a$ and $b$ by any element of $R$.

   If $X$ is not empty:

2. Determine $x \in X$ that maximizes $\angle axb$.
3. Add $\Delta abx$ to $T$.

4. Delete $x$ from *nodeset*.
5. If $[a,x] \notin R$, *retriangulate* ( *nodeset*, $a$, $x$, $T$, $R$ ).
6. If $[x,b] \notin R$, *retriangulate* ( *nodeset*, $x$, $b$, $T$, $R$ ).

Figures 7a-7f show the application of Algorithm 1 to a simple seven point example.



a b c d e f

Figures 7a-7f. An example of the application of Algorithm 1

The algorithm to implement the construction described by Theorem 1 can now be presented. We assume $e = [a,b]$ and $T$ is a generalized Delaunay triangulation with required edges $R$.

**Algorithm 2**: *addedge* ( e, T, R )

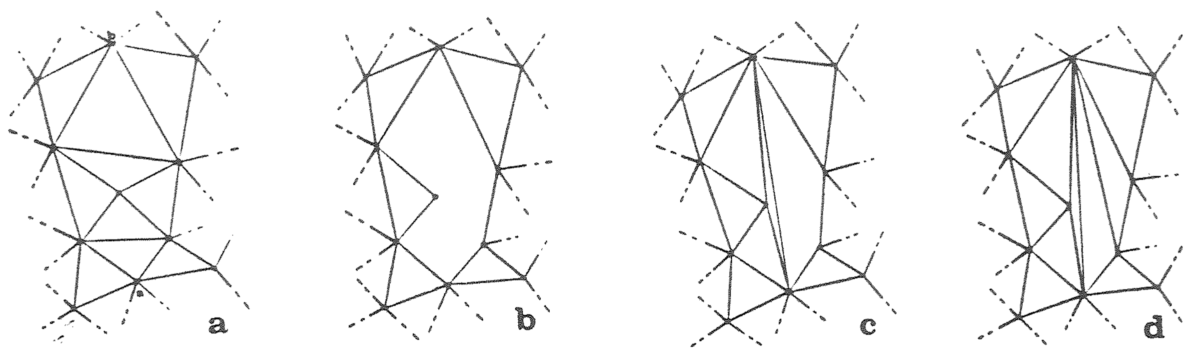
1. Find those triangles in $T$ whose interiors intersect e. If no such triangles exist, then stop; otherwise remove each of them from $T$.
2. Set $\Omega^e$ to the union of the triangles found in step 1, $B^e$ to the boundary of $\Omega^e$, $R^e$ to the set of required edges in $\Omega^e$, and *nodeset* to the set of nodes in $\Omega^e$ other than $a$ and $b$. Copy *nodeset* into *nodesetp*.
3. Triangulate left portion: *retriangulate* ( *nodeset*, $a$, $b$, $T$, $R^e \cup B^e$ ).
4. Triangulate right portion: *retriangulate* ( *nodesetp*, $b$, $a$, $T$, $R^e \cup B^e$ ).
5. Replace $R$ with $R \cup e$.

An application of Algorithm 2 is shown in Figures 8a-8d. Figure 8a displays the existing triangulation. In Figure 8b, we see the results of steps 1 and 2 of the algorithm: triangle removal and boundary building. Figure 8c shows the retriangulation of the left portion (step 3), and Figure 8d shows the result after the right portion has been retriangulated (step 4).

We prove that this algorithm properly adapts the triangulation to include the single new edge.

**Theorem 3**: Given a generalized Delaunay triangulation $T$ of $S$ with required edges $R$ and given an augmented set of required edges $R \cup e$, Algorithm 2 produces a generalized Delaunay triangulation of $S$ with required edges $R \cup e$ in variable $T$.

**Proof**: First we notice that neither of the invocations of Algorithm 1 in Algorithm 2 involves $e = [a,b]$ being an element of $R^e \cup B^e$. Furthermore, the recursive invocations




**Figures 8a-8d.** An example of the application of Algorithm 2

of Algorithm 1 from itself insure that the precondition of $e \notin R^e \cup B^e$ is always satisfied. Since $e \notin R$, it is not a boundary edge of the original triangulation. Some node (possibly on the boundary) must be strictly left of $\vec{ab}$ and unseparated from $m$, the midpoint of $a$ and $b$. We conclude that when Algorithm 1 is invoked by Algorithm 2 then the set $X$ in Algorithm 1 is never empty. Since each recursive invocation of Algorithm 1 reduces the size of *nodeset* by one, Algorithm 1 (and hence Algorithm 2) must terminate.

To prove the theorem, we must show that the invocations of Algorithm 1 in steps 3 and 4 of Algorithm 2 produce a modified Delaunay triangulation of the region $\Omega^e$ determined in step 1. First we recognize that Algorithm 1 can produce triangles only employing the nodes of the original *nodeset* and thus, of the original triangulation. Furthermore all triangles produced by the algorithm lie within the region $\Omega^e$, and it follows from Theorem 2 that all satisfy the modified circle test. To complete the proof we must show that after the inclusion of the new triangles produced by the algorithm:

2. No triangle contains a node other than its vertices.
3. The interiors of the triangles are pairwise disjoint.
4′. The union of the triangles is $\Omega$.
6. Each element of $R$ is an edge in some triangle.

Property 2 follows from the modified circle test since no non-vertex node could lie in a triangle without being interior to a circumcircle. Property 3 follows from Lemmas 3 and 4. For Property 4′, consider the portion of the region that would remain untriangulated if this property did not hold at the conclusion of Algorithm 2. Since Algorithm 1 fails to recur only when a required edge is encountered, we conclude that any untriangulated region must be bounded by edges in $R^e \cup B^e$. Yet no edges are added to this set during Algorithm 2, so such an untriangulated region could only be all of $\Omega^e$. Since at least two triangles are formed (one on either side of the original edge $e$), we cannot have any such untriangulated portion. Finally, Property 6 holds since the entire region is triangulated. Those elements of $B^e$ remain as boundary edges. The other required edges, $R^e$, are never intersected by the interiors of triangles and thus must be triangle edges. Lastly, the edge $e$ clearly is present in the triangulation. □

Finally, Algorithm 2 is employed in an iterative fashion to solve the generalized Delaunay triangulation problem. Recall that $S$ is the set of nodes, $l$ is the number of boundary curves, $B_1, \ldots, B_l$ are the boundary curves themselves, and $E$ is the set of required edges that are not on the boundary.

**Algorithm 3**: *generalizedDelaunaytriangulation* $(S, l, B_1, \cdots, B_l, E)$

1. Determine a standard Delaunay triangulation of S. (This is a triangulation over the convex hull). Let $R$ be the set of triangle edges on the convex hull.
2. For each $e \in B_1 \cup \cdots \cup B_l \cup E$, addedge $(e, T, R)$.
3. For $i = 1,...,l$, delete those triangles whose interiors are in the exterior of $B_i$.

An application of this algorithm to the original example, figure 1, is shown in Figures 9a-9d. Figure 9a displays the problem constraints. The standard Delaunay triangulation (step 1) is shown in Figure 9b. Figure 9c displays the result of imposing the edge constraints (step 2). Finally, following the removal of the exterior triangles (step 3, those removed are marked with circles in Figure 9c), we obtain the generalized Delaunay triangulation shown in Figure 9d.

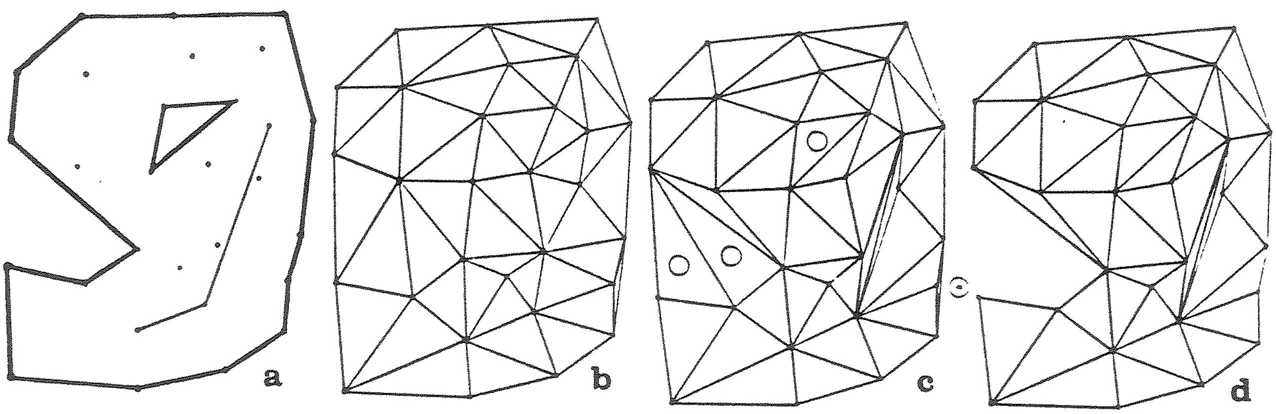



**Figure 9a-9d.** An example of the application of Algorithm 3

## 3. Shortest Paths and Distances

In this section, we present some theoretical results regarding paths and distances along paths within regions with polygonal boundaries. The set Ω is as before. We define the concept of visibility as follows: for any points $a, b \in \Omega$, $a$ is *visible from b* if the line segment from $a$ to $b$ is contained in Ω. The *visibility region of a point a* in Ω is defined to be the subset of points in Ω visible from $a$. This will be denoted by V(a). For any points $a, b \in \Omega$, a *path from a to b* is a sequence of points $<p_0, p_1, ..., p_k>$ so that $a = p_0$, $b = p_k$, and $p_i$ is visible from $p_{i-1}$ for $i = 1, \cdots, k$. We denote the Euclidean distance from $a$ to $b$ by $|a - b|$. The *length of a path* $p = <p_0, ..., p_k>$ is $l(p) = \sum_{i=1}^{k} |p_i - p_{i-1}|$. Let P(a,b) be the set of all paths from $a$ to $b$. The *distance from a to b* is $d(a,b) = \text{infimum}_{p \in P(a,b)}\, l(p)$. If for $p \in$ P(a,b) we have $d(a,b) = l(p)$, then $p$ is termed a *shortest path from a to b*.

By virtue of Ω being a connected subset of the plane, it is arc-wise connected as well (Simmons [10]), and thus paths between any pair of points exist. The distance function is, therefore, defined for any pair.

From the definitions, it immediately follows that:

1. $d(a,b) \geq 0$.
2. $d(a,a) = 0$.
3. $d(a,b) = d(b,a)$.

The following lemmas establish $d$ as a metric on $\Omega$.

**Lemma 5**: For any $a$, $b$, and $c \in \Omega$, $d(a,c) \leq d(a,b) + d(b,c)$.

**Lemma 6**: For any $a$ and $b \in \Omega$, if $a \neq b$ then $d(a,b) > 0$.

Theorem 4 guarantees the existence of shortest paths. Theorem 5 characterizes all shortest paths as consisting essentially of boundary nodes with non-consecutive elements that are mutually invisible. (A result similar to Theorem 5 is asserted without proof in Lee and Preparata [3], Lozano-Perez and Wesley [4], and Sharir and Schorr [9] among others.)

**Theorem 4**: For any $a$ and $b \in \Omega$, a shortest path $<p_0,...,p_k>$ exists with the properties that

i. $p_i$ is a boundary node for $i = 1,...,k-1$.
ii. $p_i$ is not visible from $p_j$ for $|i-j| \geq 2$.

**Proof**: We may identify a path $<p_0,...,p_k>$ in $\Omega$ between $a$ and $b$ with a point in $\Omega^{k+1}$, the $k$+1-fold Cartesian product of $\Omega$ with itself. Since $\Omega$ is a compact subset of two dimensional Euclidean space, $\Omega^{k+1}$ is a compact subset of $2k$+2 dimensional Euclidean space. The set $\{ (p_0, \ldots, p_k) : p_i \in V(p_{i-1}), \text{ for } i = 1,...,k \}$ of images of all paths from $a$ to $b$ is compact in $\Omega^{k+1}$. The distance measure is continuous on this compact set and thus assumes its minimum on it. We conclude that a minimum length path of $k$+1 points exists between any points of $\Omega$.

It remains to be shown that a minimum length path of an arbitrary number of points exists. We shall show that beyond a certain value of $k$, the lengths of shortest paths of $k$+1 points does not decrease as $k$ increases. Thus minimizing path length over all values of $k$ is in fact equivalent to minimizing path length over a finite set of $k$'s. To show this we shall employ the generalized Delaunay triangulation. By forming a generalized Delaunay triangulation of $\Omega$ into some number $m$ of triangles, we notice that any path of $2m$+1 or more points must have at least three points in a common triangle. Thus the path must be of the form $<p_0,...,p_i,...,p_j,...,p_l,...,p_k>$, where $p_i$, $p_j$, and $p_l$ are common to a single triangle. Since the segment $[p_i,p_l]$ is also in that triangle (by convexity), the path $<p_0,...,p_i,p_l,...,p_k>$, is a path from $p_0$ to $p_k$ with strictly fewer points and no greater length. We conclude that when searching for shortest paths, we need not consider paths with more than $2m$ points: a shortest path of $k$+1 points for $k+1 \leq 2m$, is a shortest path independent of the number of points.[5]

To prove the first of the two properties, notice that if $p_i$ is collinear with $p_{i-1}$ and $p_{i+1}$, then deleting $p_i$ from the path has no effect on the path length. Now assume $p_{i-1}$, $p_i$, and $p_{i+1}$ are not collinear and that $p_i$ is not a boundary node. Since $p_i$ is interior, a small circular neighborhood of it is also interior. We may replace $p_i$ in the path by the two

[5] The essential elements of the proof of the existence of a shortest path are that $\Omega$ is a closed, path-wise connected set which can be represented as the union of a finite number of convex sets. The generalized Delaunay triangulation yields such a representation.

points $p'_i$ and $p''_i$ located on the respective intersections of $[p_{i-1},p_i]$ and $[p_i,p_{i+1}]$ with the circle boundary (Figure 10). The new path $<p_0,...,p_{i-1},p'_i,p''_i,p_{i+1},...,p_k>$, is strictly shorter than the original path with $p_i$. Thus there exists a shortest path in which $p_i$ is a boundary node for $i = 1,...,k-1$.

The second property follows immediately from the triangle inequality. □

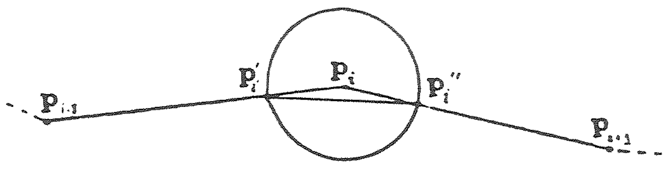



**Figure 10.**

It is obvious that extraneous points may occur in paths. These points have no effect on length or "footprint" (i.e., the union of line segments $[p_i,p_{i-1}]$ for $i = 1,...,k$). For example, for any path $p = <p_0,...,p_k>$, the path $p' = <p_0,...,p_i,p'_i,p_{i+1},...,p_k>$, where $p'_i = \frac{1}{2}(p_i+p_{i+1})$, has exactly the same length and footprint as $p$. We shall term a path essential (i.e., capturing the essence) if no such extraneous points occur. Precisely, a path $p = <p_0,...,p_k>$ is *essential* if for $i = 1,...,k-1$, $p_{i-1}$, $p_i$, and $p_{i+1}$ are not collinear. It is clear that an essential path can be extracted from any path by deleting the middle point of such collinear triples (in general, by deleting the middle $n$-2-tuple from collinear $n$-tuples). In particular, this operation can be applied to shortest paths. The following theorem characterizes all shortest essential paths.

**Theorem 5**: If $p = <p_0,...,p_k>$ is a shortest essential path then

i. $p_i$ is a boundary node for $i = 1,...,k-1$.

ii. $p_i$ is not visible from $p_j$ for $|i-j| \geq 2$.

iii. $<p_i, p_{i+1},...,p_{j-1},p_j>$ is a shortest path from $p_i$ to $p_j$ for $0 \leq i \leq j \leq k$.

**Proof**: The proof of Theorem 4 shows that if for $i = 1,...,k-1$ $p_i$ is not a boundary node then either $p_{i-1}$, $p_i$, and $p_{i+1}$ are collinear (in which case the path is not essential) or the path can be shortened. Property i thus holds. Similarly for property ii, if $p_i$ is visible from $p_j$ with $|i-j| \geq 2$ then either $p_i,...,p_j$ are collinear or a shorter path could be constructed. Finally, for property iii, $<p_i,...,p_j>$ is clearly a path from $p_i$ to $p_j$. If a shorter path between them existed then a shorter path would exist between $a$ and $b$. □

## 4. Closest Nodes

In this section, we show that the generalized Delaunay triangulation is the natural structure for solving closest node problems. This is a result of the fact that the the closest node to any given node must share a triangle with it. The second closest node either shares a triangle with it or shares a triangle with the first closest node. The generalization of this is presented as the corollary to Theorem 6.

The generalized Delaunay triangulation to be employed in this theory is exactly as presented in section 2 except that the interior required edges have no role (and thus $E$ is assumed to be empty). To emphasize the fact that the paths are contained in $\Omega$, we refer to the boundary curves as *barriers*. The set of nodes on the boundary is denoted by $B$.

For $a, b$, and $c \in \Omega$, we say $c$ is *closer to a than is b* if $d(a,c) < d(a,b)$. We term $b$ a *k-th closest node to a* if there are $k-1$ nodes $c_1,...,c_{k-1}$ (distinct from $a$ and $b$) so that $d(a,c_i) \le d(a,b)$ and no other node (except $a$) closer to $a$ than $b$ is. For completeness, we refer to $a$ itself as the *0-th closest node to a.*

There are obvious ambiguities associated with this definition since there may not be a unique k-th closest node. Furthermore, a given node may be both a k-th closest node and a j-th closest node for $k \ne j$. Both of these situations correspond to points that are equidistant from $a$. If $S = \{a, w, x, y, z\}$ and $d(a,w) = 3$, $d(a,x) = 1$, $d(a,y) = 2$, and $d(a,z) = 2$, then $x$ is the unique closest node to $a$. However, $y$ is both a second and third closest node, and so is $z$. The unique fourth closest node to $a$ is $w$. A more traditional usage of "closest" would term both $y$ and $z$ second closest nodes and nothing would be third closest. The definition here simplifies the following presentation significantly.

Theorem 6 and its corollary will be employed to develop algorithms for solving various closest node problems. Prior to presenting them two lemmas are proved.

**Lemma 7**: Let $\Delta abc$ be a triangle of a generalized Delaunay triangulation, $x$ be a point in its interior, and $U$ be the set of points interior to $\bigcirc\Delta abc$ unseparated from $x$. The closure of $U$ is convex.

**Proof**: Recall that since all edges on the boundary of $\Omega$ are required and $U$ consists of points unseparated from $x$, $U$ is a buset of $\Omega$. Since $\Delta abc$ satisfies the modified circle test, no node is within $U$. Thus $U$ is the intersection of the interior of $\bigcirc\Delta abc$ and some set of half-planes supported by those required edges that intersect the circumcircle. This intersection is convex as is its closure. □

**Lemma 8**: If $(b, c, q, d)$ is an ordered quadruple of points on a common circle, then either $|b - q| > |b - d|$ and $|b - q| > |d - q|$ or $|b - q| > |b - c|$ and $|b - q| > |c - q|$ (or both).

**Proof**: The shorter of the circular arcs subtended by $[b,q]$ is strictly greater than the circular arcs subtended by $[b,c]$ and $[c,q]$ or those subtended by $[b,d]$ and $[d,q]$. The distances between the vertices are monotonically related to the lengths of the subtended arcs. □

With respect to a fixed generalized Delaunay triangulation we define node $a$ to be a *neighbor* of node $b$ if there is a triangle of which both $a$ and $b$ are vertices. The set of all neighbors of a node $a$ is denoted by $N(a)$.

**Theorem 6**: For any $a$ and $b \in S$, if $a \ne b$ then some neighbor of $b$ (possibly $a$ itself) is closer to $a$ than $b$ is and the same neighbor (if not equal to $a$) is closer to $b$ than $a$ is.

**Proof**: The proof is by contradiction: what is assumed to be a shortest essential path from $a$ to $b$ is reduced in length if no neighbor of $b$ is closer to $a$ than $b$. Let $\langle p_0, p_1, ..., p_k \rangle$ be a shortest essential path from $a$ to $b$. Consider the segment $[p_{k-1}, p_k] = [p_{k-1}, b]$. If $p_{k-1}$ is a neighbor of $b$ the proof is complete. Otherwise, the segment intersects the interior of some edge $[c,d]$ of a triangle $\Delta bcd$ (Figure 11). The edge $[p_{k-1}, b]$ also intersects the interior of the arc from $c$ to $d$ of $\bigcirc\Delta bcd$ subtended by $\angle cbd$. Let this point of intersection be $q$ and let $w$ be a point on $(q,b)$ and interior to $\Delta bcd$. Obviously $c$, $d$, and $q$ are in the closure of the region interior to $\bigcirc\Delta bcd$ and unseparated from $w$. But from Lemma 7, this closure is convex and thus no point on $[q,c]$ or $[q,d]$ is exterior to $\Omega$.

We conclude that both $c$ and $d$ are visible from $q$. However, by virtue of Lemma 8, either $c$ or $d$ must be closer to $b$ and $q$ than $|b-q|$. Without loss of generality, supposing this closer point is $c$, then the path $<p_0, p_1, ..., q, c>$ is strictly shorter than $<p_0, p_1, ..., q, b>$ which has the same length as $<p_0, p_1, ..., p_{k-1}, b>$. Thus $c$ is closer to $a$ than $b$ is. Also, since $d(a,b) \geq d(q,b) > d(c,b)$, $c$ is closer to $b$ than $a$ is. □

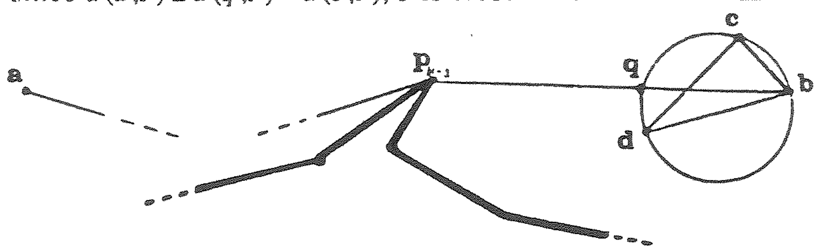



**Figure 11**

**Corollary**: Let $c_j$ be a j-th closest node to $a$ for $j = 0,...,k$ and $c_i \neq c_j$ for $i \neq j$. Then for some $i$ and $j$ with $i \leq j \leq k-1$, $c_k \in N(c_j) \cap V(c_i)$.

**Proof**: Suppose a shortest path from $c_0$ to $c_k$ is $<c_0, p_1, ..., p_{k-1}, c_k>$. The node $p_{k-1}$ is clearly closer to $c_0$ than $c_k$ is. Thus $p_{k-1}$ must be $c_i$ for some $i \leq k-1$. The theorem guarantees that some neighbor of $c_k$ is closer to $c_i$ than $c_k$ is (and therefore closer to $c_0$ than $c_k$ is), and this same neighbor (if it is not $c_i$) is closer to $c_k$ than $c_i$ is. Thus, this neighbor must be some $c_j$. The corollary will be proved if we can show $i \leq j$. But this is clear, since if $j < i$, then $d(c_0, c_j) \leq d(c_0, c_i)$. But $d(c_j, c_k) < d(c_i, c_k)$, and combining these inequalities contradicts the assumption that a shortest path from $c_0$ to $c_k$ passed through $c_i$. □

Both the theorem and corollary were presented for the barrier-free situation in Renka [6]. An algorithm for finding k closest nodes that exploited these results was stated in Cline and Renka [1]. This algorithm was used for estimating partial derivatives of bivariate functions for smooth surface fitting in Renka and Cline [8] and Renka [7]. In light of this, we find quite perplexing the statement in Preparata and Shamos [5,page 235]

> The Voronoi diagram[6], while very powerful, has no means of dealing with farthest points, k closest points, and other distance relationships.

Theorem 7 summarizes the theory we have developed for finding closest nodes. Its proof follows immediately from the corollary to Theorem 6.

**Theorem 7**: For $j = 0,...,k-1$, let $c_j$ be a j-th closest node to $a$ and assume $c_j \neq c_i$ for $j \neq i$. A k-th closest node $c_k$ is a minimizer of $|x - c_i| + d(a, c_i)$ over $x \in (N(c_j) - \{c_0, ..., c_{k-1}\}) \cap V(c_i)$ for $j = 0,...,k-1$, $i \leq j$, and $c_i \in B$. If $i^*$ denotes the value of i associated with a minimum and $p^*$ is a shortest path from $a$ to $c_{i^*}$, then $p^*$ concatenated with $<c_k>$ is a shortest path from $a$ to $c_k$.

Figure 12 shows an implementation of this method for finding a k-th closest node. With a slight modification, pointers could be added to maintain the shortest paths as well. The algorithm could be applied to find the closest nodes among some subset of $S$ (e.g. the

[6] The *Voronoi diagram* is dual to the Delaunay triangulation: obtaining either from the other is a triviality.

non-boundary nodes). An application of this is a "greedy algorithm" for solving the "constrained traveling salesman problem":

> Given a finite set of nodes $S$, a collection of polygonal boundary curves, $B_1,...,B_l$, defining $\Omega$ (a superset of $S$), and a subset $S^v$ of $S$, determine a shortest path (within $\Omega$) from any given point to itself that includes every element of $S^v$.

```
begin
d_k := -1 ;                                          { initialize }
for j := 0 to k-1 do                                 { loop over k closest points }
  for x ∈ N(c_j) - {c_0, . . . , c_{k-1}} do         { loop over other neighbors of c_j }
  begin
    if |x - c_0| < d_k or d_k = -1 then              { |x - c_0| is minimum possible for d(x,c_0) }
    begin
      if x ∈ V(c_0) then                             { |x - c_0| is d(x,c_0) if x is visible from c_0 }
      begin
        c_k := x ;                                   { update c_k and d_k }
        d_k := |x - c_0|
      end
      else                                           { x is not visible from c_0 }
      begin
        dx := d_j + |x - c_j| ;                      { this is the maximum possible for d(x,c_0) }
        for i := 1 to j-1 do                         { loop over boundary nodes which are closer }
          if dx > d_i + |x - c_i| and c_i∈B and x∈N(c_i) {   than c_j to c_0 and visible from }
              and x ∈ V(c_i) then dx := d_i + |x - c_i| ; {   x. Find sortest path using these }
        if dx < d_k or d_k := -1 then
        begin
          c_k := x ;                                 { update if shorter path found }
          d_k := dx
        end
      end
    end
  end
end.
```

**Figure 12.** The k-th closest node algorithm

Another interesting application of the closest node algorithm is that of finding optimal two-dimensional robot movement. Figure 13a shows the floor plan of a maze with initial position and exit position indicated. The problem is to determine a shortest path from one to the other. There is the additional complication that the robot covers some positive amount of area and as a result the center of the robot's projection onto the plane must maintain some minimal distance (its diameter) from the walls. For a given diameter, we can create a set of boundaries within which the robot could move without violating this proximity constraint. Such a set is shown in Figure 13b[7]. Application of the algorithm to find the closest nodes from the initial position, modified to yield the path as well as the distance, and applied until the exit position has been labeled a k-th closest

[7] By adjusting the polygonal boundaries at the corners, the region of allowed movement could be slightly increased but this will not be pursued here.

node, results in the path shown in Figure 13c.

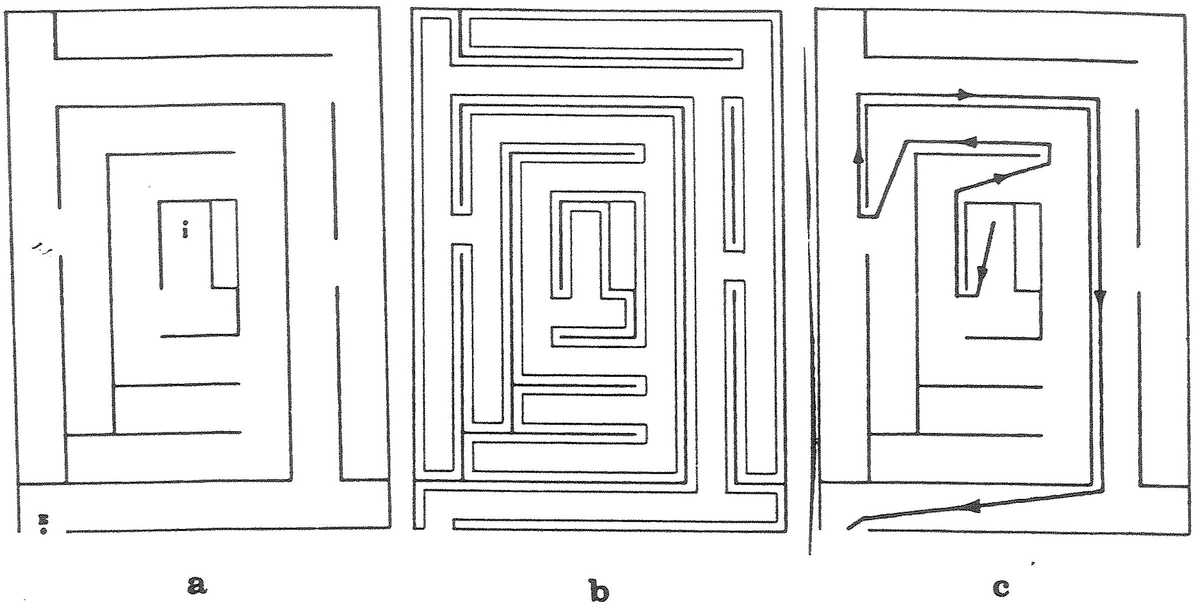



**Figure 13a-13c.** A robot motion problem

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