VERTEX DECOMPOSITIONS OF PLANAR GRAPHS*

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TR-87-34 September 1987

*This work was supported in part by the U.S.Office of Naval Research under contracts N00014-86-K-597 and N00014-86-K-0763.

Abstract

Vertex decompositions of graphs are useful in designing fast parallel algorithms for vertex coloring. Vertex arboricity of a graph is the minimum number of sets into which a vertex set can be partitioned such that the subgraph induced by each set is an acyclic graph (a forest). It is well known that the vertex arboricity of planar graphs is 3 and that of outerplanar graphs is 2, from a result due to Chatrand and Kronk [CK 69]. In this paper, it is shown that outerplanar graphs can be vertex partitioned into two sets, one inducing a forest and the other an independent set. As a result it can be concluded that all 4-connected planar graphs can be vertex partitioned into three sets such that two of them induce forests and the third is an independent set.

1 Introduction

Vertex decompositions are useful in coloring the vertices of a graph on a parallel computer [BJ 85]. Recently it was shown [D 86] that the vertices of a planar graph can be colored with six colors on a parallel computer efficiently. The idea is to partition the vertex set into two sets such that each of them induces an outerplanar graph. One way of obtaining such a partition is by picking any vertex v and finding the shortest distance, in terms of the number of edges, from v to all other vertices, and partitioning the vertex set depending on whether the distance is odd or even. Each partition can be colored with 3 colors, in parallel, thus giving a six-coloring of the whole graph. Since the partitioning algorithm, which is a single source shortest path computation, has a fast parallel solution six-coloring has a fast parallel algorithm.

In this paper we show that a large class of planar graphs, 4-connected planar graphs, can be vertex partitioned into three sets, where two of them induce forests and the third one is an independent set. As this result is obtained by appealing to Tutte's theorem [T 56] on planar graphs and since it is not known if Tutte's theorem parallelizes, it is not clear if our result yields a parallel algorithm for five-coloring of 4-connected planar graphs. However polynomial time algorithms to construct such decompositions on a sequential computer can be easily extracted from our proofs, given the result on the complexity of finding Hamiltonian circuit in 4-connected planar graphs due to Gouyou-Beauchamps[GB 82]. Also, our approach raises several interesting questions for further research.

2 Definitions

Definition 1 $V(G)$ and $E(G)$ stand for the vertex set and the edge set of a given graph G, respectively.

Definition 2 A graph G is said to be *planar* if G can be embedded in a plane without any edge crossings. A planar graph is called *outerplanar* if G can be embedded in a plane so that every vertex lies on the exterior region.

Definition 3 The subgraph G' *induced* by a subset $V'(G)$ of the vertex set $V(G)$ is a graph whose vertex set is $V'(G)$ and the edge set $E(G')$ is a subset of $E(G)$ such that the edges in $E(G')$ are those with both end nodes in $V'(G)$.

Definition 4 A *Hamiltonian circuit* of a graph G is a closed path in which each vertex of G appears exactly once.

Definition 5 A connected graph G is *r-connected* if at least r vertices must be removed to disconnect the graph.

Definition 6 A vertex set $V(G)$ is called an *independent set* if the edge set of the graph induced by $V(G)$ is an empty set. Similarly, a subset $V'(G)$ of $V(G)$ induces a forest if the subgraph induced by $V'(G)$ has no cycles.

Definition 7 A subset $V'(G)$ induces a *maximal forest (tree)* if $V'(G)$ induces a forest (tree) and for all $v \in V(G) - V'(G)$, the subgraph induced by $V'(G) \cup \{v\}$ has a cycle.

3 Results

We assume the given planar graph G is triangulated. If it is not, then apply any of the known sequential/parallel algorithms to triangulate it.

Triangulated graphs are also referred to as maximal planar graphs since no more edges can be added to a triangulated graph without destroying its planarity. As our main result claims that partitions induce forests and an independent set and if we prove this result for maximal planar graphs then it holds, clearly, for any subgraph induced by a smaller edge set.

Theorem 1 *The vertex set of any maximal outerplanar graph can be partitioned into two sets, one of which induces a tree and the other is an independent set.*

Proof: The theorem clearly holds for any graph with the size of the vertex set less than 3. So assume that the graph has more than two vertices. Let v be a vertex of degree 2. Note that the existence of such a vertex is guaranteed in any maximal outerplanar graph with more than two vertices. Assume, inductively, that there is partition V_1 and V_2 of the vertex set $V(G)-\{v\}$, where V_1 induces a tree and V_2 induces an independent set. To extend the partition to $V(G)$, include v in V_2 if both neighbors are in V_1. Clearly, $V_2 \cup v$ is an independent set of G. If both neighbors are not in V_1

then include v in V_1. The following observation proves that $V_1 \cup v$ induces a tree. Notice that there is an edge between the two neighboring vertices of v, as the graph is maximal. Hence both neighbours of v cannot be in V_2 and one of them has to be in V_1. Therefore we have the desired partition of $V(G)$.

Observe that if the graph is not maximal then, by a similar proof, we would have a set inducing a forest (not a tree) and the other an independent set. Also, it can be verified, by strengthening the inductive hypothesis, that the tree or the forest (when the outerplanar graph is not maximal) induced by $V_1 \cup v$ is *maximal.* □

Theorem 2 *For a planar graph G, if there exists a subset $V'(G)$ of the vertex set of G that induces a* maximal tree *then there exists a partition of $V(G)$ consisting of three sets where two sets induce forests and the third is an independent set.*

Proof: Outerplanar graphs can also be characterized as planar graphs without any subgraph homeomorphic to K_4 or $K_{2,3}$ [BCL 79]. Let the subgraph induced by $V(G) - V'(G)$ be G''. We will show that G'' is outerplanar. From application of the previous theorem we would get the desired decomposition.

Assume G'' has a subgraph G_{K_4} that is homeomorphic to K_4. Let a, b, c, d be the vertices that correspond to the four vertices of K_4. Consider any embedding of G_{K_4} on a sphere. Notice that any three *distinct* vertices x_1, x_2, x_3 in $\{a, b, c, d\}$ form a closed curve in the embedding and let the fourth vertex be $x_4 \in \{a, b, c, d\}$. All the the four closed curves induced by some distinct x_1, x_2, x_3 have one region empty and the other contains x_4. The tree induced by $V'(G)$ is connected and therefore is enclosed by one of the four closed curves. But since $V'(G)$ induces a maximal tree, inclusion of any vertex from the complementary set would create a cycle. Therefore there are atleast two edges from x_4 to two nodes of the tree. By Jodan-curve theorem [M 75] this is impossible unless the G is nonplanar. Similarly, by considering the three closed surves of a graph homeomorphic to $K_{2,3}$ we can conclude that G'' can not have a subgraph homeomorphic to $K_{2,3}$ unless G is nonplanar. Therefore G'' is outerplanar. □

Corollary 1 *The vertex set of any 4-connected planar graph can be partitioned into three sets such that two sets induce forests and the third is an independent set.*

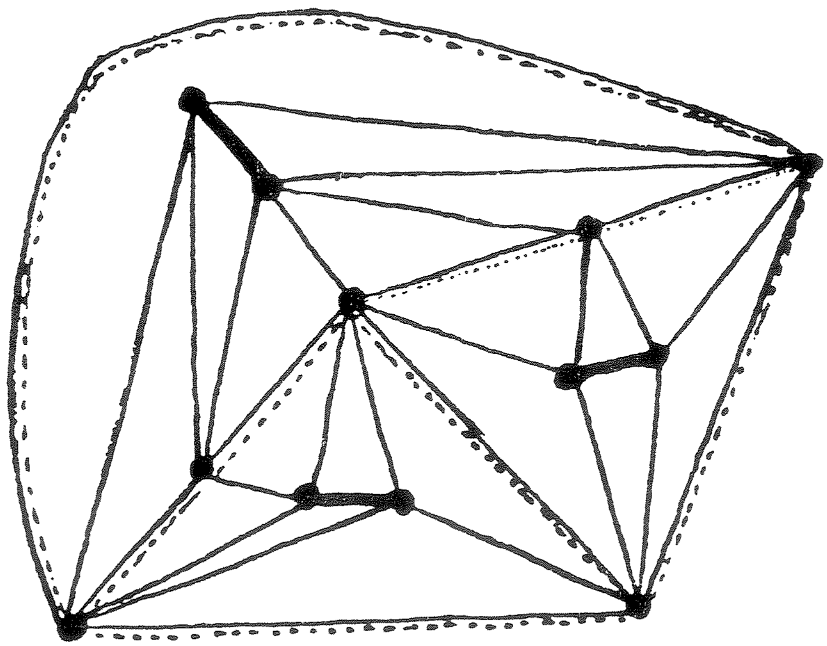
Proof: We know from Tutte's theorem that any 4-connected planar graph is Hamiltonian. For a given embedding in a plane, any Hamiltonian circuit divides the remaining edges of the graph into two categories: *internal* chords and *external* chords. Removal of all internal chords or external chords leaves a maximal outerplanar graph, for which we know there is a maximal tree from Theorem 1. Clearly the same tree is also a maximal tree for the whole graph. □

4 Discussion

Even though we can deduce that there is a maximal tree in every 4-connected planar graphs from Tutte's theorem, it appears to be difficult to find one efficiently without finding the Hamiltonian circuit. The example in Fig.1 shows that if we try to build a maximal tree starting from a vertex and adding vertices that do not create a cycle together with the subset collected so far, then we will be forced to pick vertices, because of the maximality condition, that have no edges in common with the vertices collected so far and as a result the induced subgraph would be a forest instead of a tree.

See figure 1.

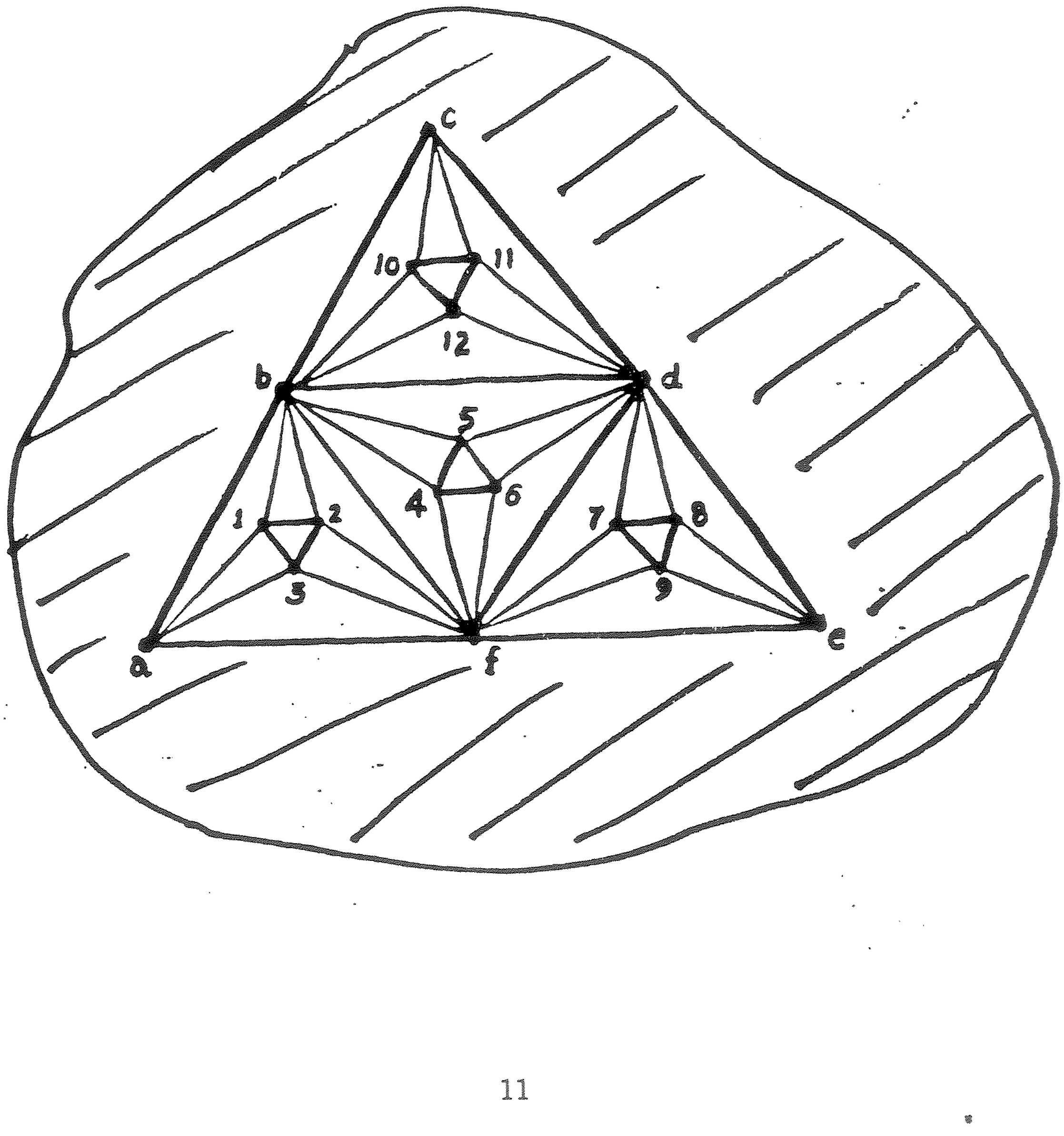
Figure 1. A 4-connected, triangulated graph in which removal of a *maximal* forest (the subgraph with dark solid edges) leaves a subgraph (the subgraph with dotted edges) that is homeomorphic to K_4.



Another natural question to ask is, do all planar graphs have such decompositions? Notice that triangulating an arbitrary planar graph makes it 3-connected. The triangulated graphs would be 4-connected if there are no separating triangles. One possible approach would be to find a forest in the interior (in an embedding) of every separating triangle whose removal leaves a graph in which the vertices from the interior do not form a cycle with two of the three vertices of the separating triangle. Then we can, inductively, "empty" the interiors of the separating triangles starting with the innermost triangles. Unfortunately this approach does not work, as is shown in the following. Consider the graph shown in Fig.2. The triangles $\{a, b, f\}$, $\{b, c, d\}$, $\{d, e, f\}$ and $\{b, d, f\}$ are separation triangles. The interior consists of the subgraph induced by the vertex set $V_i = \{1, 2, ..., 12\}$. There are several subsets V_i' of V_i that induce forests. One such subset is $V_i' = \{1, 2, 4, 5, 7, 8, 10, 11\}$. It is not difficult to verify that for every such V_i' the complementary graph induced by $\{a, b, ..., f\} \cup (V_i - V_i')$ has a subgraph homeomorphic to $K_{2,3}$.

See figure 2.

Figure 2. A triangulated planar graph. The vertex sets $\{a, b, f\}$, $\{b, c, d\}$, $\{d, e, f\}$ and $\{b, d, f\}$ form separation triangles.



To summarize, we gave polynomial time computable algorithms to partition the vertex set of a 4-connected planar graph where two sets of the partition are 2-colorable and the third set is 1-colorable. Unfortunately, the sets are constructed using a Hamiltonian circuit, and also the sets are maximal in some sense. As it is not clear if a Hamiltonian circuit can be constructed efficiently on a parallel computer and since maximal objects are hard to construct in parallel [KW 85] the following questions are relevant:

1. Is it possible to a find a partition of planar graphs where each set in the partition is easily colorable on a parallel computer?

2. Given a planar graph does there exist a subset of the vertex set that induces a connected tree which is maximal? If so, can it found efficiently, i.e. in polynomial time?

3. Can the vertex sets of all planar graphs be partitioned into three sets where two of them induce forests and third is an independent set? (Note that this is a weaker question than the above question)

4. Is it possible to find the desired three partitions of the vertex set in

4-connected graphs without finding the Hamiltonian circuit?

References

[BCL 79] M. Behzad, G. Chatrand and L. Lesniak-Foster, *Graphs and Digraphs*, (Prindle, Weber and Schmidth, Boston, 1979).

[BJ 85] F. Bauernoppel and H. Jung, *Fast Parallel Vertex Coloring*, Proc. of 5th conference on Fundamentals of Computation (LNCS 199, Springer-Verlag, 1985) 28-35.

[CK 69] G. Chatrand and H.V. Kronk, *The Point-Arboricity of Planar Graphs*, J. London Math. Soc., 44(1969) 612-616.

[D 86] K. Diks, *A Fast Parallel Algorithm for Six-Coloring of Planar Graphs*, Proc. on Math. Foundations of Computer Science (1986) 273-282.

[GB 82] D. Gouyou-Beauchamps, *The Hamiltonian Circuit Problem is Polynomial for 4-connected Planar Graphs*, SIAM Journal of Computing, Vol. 11, No.3 (1982) 529-539.

[KW 85] R.M. Karp and A. Wigderson, *A Fast Parallel Algorithm for the Maximal Independent Set Problem*, JACM, Vol. 32, No.4 (1985) 762-773.

[M 75] J.R. Munkres, *Topology: A First Course* (Prentice-Hall, NJ, 1975).

[T 56] Tutte, W., *A Theorem on Planar Graphs*, Trans. Amer. Math. Soc., 82(1956) 99-116.