AUTOMATED REASONING IN MECHANICS USING RITT-WU'S METHOD*

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ABSTRACT

Methods of automated reasoning in mechanics have been presented and implemented on computers. The paper consists of two parts. In part I, a mechanical method developed by W.T. Wu on the basis of the work of J. F. Ritt has been used to prove theorems in mechanics. In particular, a mechanical study of the complete logical relationship between Kepler's laws and Newton's gravitational laws has been given. Wu's work on the same topic has been extended in several ways. Many other examples from mechanics are also given. In part II, a method of mechanical derivation of formulas from a set of differential polynomials has been presented. The method has been used successfully to some problems in mechanics. In particular, a mechanical derivation of Newton's gravitational laws from Kepler's laws has been given without knowing Newton's Laws in advance.

Keywords: Mechanical theorem proving, mechanical derivation of formulas, Wu's method, differential polynomial, Ritt-Wu's decomposition algorithm, mechanics, Newton's gravitational laws, Kepler's laws.

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Part I. Mechanical Theorem Proving in Plane Mechanics

1. Introduction

In [WU2], Wu Wen-Tsün gave a mechanical proof of the theorem that Kepler's laws implied Newton's gravitational laws using a method developed by him on the basis of the work of J. F. Ritt [RI2]. In our opinion, this interesting paper marked a beginning of the automated reasoning in mechanics.

We follow the same direction, but using the complete decomposition method of Ritt-Wu, to explore mechanical reasoning in mechanics. The reason for using the complete decomposition algorithm instead of simple generation of the first characteristic set (as Wu did in [WU2]) is that there are some complications in proving theorems involving differential polynomials (see Section 2). In Part I of this paper, we concentrate on proving physics theorems of equation type, i.e., theorems whose hypotheses and conclusions can be represented by differential polynomial equations with universal quantifiers. Many examples from kinematics and dynamics are given. In particular, we extend Wu's work on the Kepler-Newton problem (i.e., K1, K2, N1, N2; see below), giving a complete, mechanical solution of the logical relationship between these laws. The results on computer show that

(1) K1 and K2 imply N1 and N2 (originally given by Wu [WU2].)

(2) K2 is equivalent to N2.

(3) N1 and N2 imply K1.

(4) K1 and N1 do not imply K2.

Statement (3) involves the existential quantifier. In order to get rid of the existential quantifier, in Section 3 we introduce a general technique which will be useful for study of mechanical theorem proving in differential geometry and mechanics. In Section 2, we briefly review Ritt-Wu's method for mechanical theorem proving involving differential polynomials, emphasizing our observation that there are some essential differences between proving theorems involving (ordinary) polynomials and proving theorems involving differential polynomials (e.g., theorems in differential geometry and mechanics). In section 4, a mechanical treatment of Newton-Kepler problem is presented. In section 5, more examples in kinematics as well as dynamics are given.

2. An Introduction To Ritt-Wu's Method

In this section, we will briefly introduce Ritt-Wu's Method, clarifying several key points. We find some essential differences between proving theorems involving differential polynomials (abbr. d-pols) and proving theorems in elementary geometry, in which only ordinary polynomials are involved.

First, in elementary geometry the use of general form of Ritt-Wu's zero decomposition theorem is not necessarily needed because subsidiary conditions produced by the method automatically are connected with nondegeneracy and can be converted to their geometric forms for a large class of geometric statements [CH1]. Alternatively, we can use the notion of "generally true" to justify the addition of subsidiary conditions to the set of hypothesis equations.

But in the case of proving theorems involving differential polynomials, the physics or geometry meaning of the subsidiary conditions obtained from the characteristic set of the hypothesis set is often not clear, and can change the meaning of the original statement to be proved completely.* Thus, the complete decomposition theorem is necessary to implement a prover which can be used to deal with complicated problems. Second, in the case of differential polynomials, the notion of "generally true" should be modified. Our preliminary investigation of this notion shows that the description of the nondegenerate components needs two numbers: the dimension and the differential order of the related differential algebraic sets.

Now we give a brief introduction to Ritt-Wu's method with our modifications in the d-pol case. A full description of the method can be found in [WU1] or in [CG1] which is our improvement and clarification of Wu's original method in [WU1].

Let K be a field of characteristic zero, all differential polynomials (abbr. d-pols) in this paper are considered in $K\{x_1, ..., x_n\}$, the differential polynomial ring in variables $x_1, ..., x_n$ [RI2]. Differentiations are considered w.r.t the variable t representing time in mechanics. For an ascending chain ASC [RI2], denote the set of the d-pols with zero pseudo remainders w.r.t ASC by $PD(ASC)$. $PD(ASC)$ is an important notion introduced by us [CG1]. The following Ritt-Wu's zero decomposition algorithm is the basis of the method.

Theorem 1. There is a mechanical procedure that can decide in a finite number of steps whether $Zero(PS/R)$** is empty (or equivalently R is in the radical ideal generated by PS) for a given finite d-pol set PS and a d-pol R, and in the non-empty case give a decomposition of the following forms:

$$Zero(PS/R) = \cup_{i=1}^{n} Zero(ASC_i/R_i)$$

$$Zero(PS/R) = \cup_{i=1}^{n} Zero(PD(ASC_i)/R),$$

where each ASC_i is an irreducible ascending chain and R_i is the non-zero pseudo remainder of $J_i R$ w.r.t ASC_i; here J_i is the product of the initials and separants of the d-pols in ASC_i.

For an irreducible ascending chain $ASC = \{A_1, ..., A_p\}$, we make a renaming of the variables. If A_i is of class m_i, we replace x_{m_i} by x_i, the other variables are denoted by $u_1, ..., u_q$, ($q = n-p$). The u and the x are called the parameters and the leading variables of ASC respectively. For each A_i, denote the maximal number of differentiation of x_i effectively occurred in A_i by $ORD(A_i)$. We define $DIM(ASC) = q = n - p$ as the dimension of ASC and $ORD(ASC) = \sum_{i=1}^{n} ORD(A_i)$ as the order of ASC w.r.t to the parameter set $u_1, ..., u_q$. For the legitimacy of the definitions, the reader is referred to [RI2].

We only consider the physics statements of equation type, i.e., statements whose hypotheses and conclusions can be represented by differential polynomial equations with universal quantifiers. Let $HYP = \{H_1 = 0, ..., H_r = 0\}$ and $G = 0$ be the hypotheses and conclusion of a physic statement respectively, where the H_i and G are d-pols. At first we shall divide the variables occurred in H_i and G into two groups: $u_1, ..., u_q$ and $x_1, ..., x_p$ in the sense that in the statement the u generally can take any values, and once the u are given, all the x are determined as functions of the u. We call the u and the x the parameters and the dependent variables of the statement respectively.

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* The similar phenomenon can also happen with inappropriate choices of parameters. See [CK1]

** Here the zeros are considered in all possible extension of K in the sense of Ritt [RI2].

To decide whether $G = 0$ follows from HYP generically w.r.t the u, we first apply Theorem 1 to HYP:

$$Zero(HYP) = \cup_{i=1}^{s} Zero(PD(ASC_i^*)/R) \cup \cup_{j=1}^{t} Zero(PD(ASC_j)/R) \tag{1}$$

where each ASC_i^* does not contain a d-pol with the u only and each ASC_j contains at least one d-pol with the u only. Let $r = max_{i=1}^{s} ORD(ASC_i^*)$. A component $Zero(PD(ASC_i^*)/R)$ is said to be the main (or non-degenerate) component of HYP w.r.t the parameters u, if $DIM(ASC_i^*) = q$ and $ORD(ASC_i^*) = r$. All others are said to be degenerate components. Note that the main components of HYP are uniquely determined by the parameters [CG1]. The following is our clarification of Wu's notion of a theorem to be generally true.

Definition 1. A physics statement with hypotheses HYP and conclusion G is said to be generally true if for all main components $C = Zero(PD(ASC_i^*)/R)$, we have $C \subset Zero(G)$, i.e., the pseudo remainder of G w.r.t the ascending chain of each main component of the statement is zero.

We have proved:

Theorem 2. For a physics statement of equation type, once its parameter and dependent variables are given, there is a mechanical procedure that can decide whether this statement is generally true in a finite number of steps.

Remark 1. To reduce the number of the components occurring in the decomposition, we can first give some nondegenerate conditions $B_1 \neq 0, ..., B_l \neq 0$ for the d-pols B_i, and then use the general form of theorem 1 for $R = \prod_{i=1}^{l} B_i$ to prove that the statement is true under the nondegenerate condition $R \neq 0$.*

Remark 2. Another method can also be used to reduce the number of components. For variable x, if only some power of x, say x^n, occurs in the equations, then we may use a new variable to represent x^n. To do so, we can avoid the occurrence of the separant of x^n.

Remark 3. In theorem proving in the d-pol case, if a variable c is an arbitrary constant, i.e., c only satisfies the condition $c' = 0$, then c may be treated as a parameter in the sense that when differentiating a d-pol we always assume $c' = 0$. This kind of parameter is called a constant parameter.

3. An Algebraic Lemma

As we will see later in Sections 4 and 5, in the case of d-pols, a geometric or physical statement is usually represented by more than one equations, i.e., a main d-pol equation and some simple equations which mean that certain variables in the main equation are constants. For this kind of statements, the existential quantifier is involved. K1 in Section 4 is such an example. We can represent K1 as $\forall x \forall y \forall r \exists p \exists e \exists f (r - p - ex - fy = 0 \wedge p' = 0 \wedge e' = 0 \wedge f' = 0)$. We must eliminate the existential quantifier, otherwise Wu's method cannot be used in this case, because Wu's method can only used to prove statements involving the universal quantifier.

We distinguish two cases. In the first case, the existential quantifier is in the hypothesis of a statement, but not in the conclusion. In the second case, the existential quantifier is in

* This is actually another approach of mechanical theorem proving. For details, see section 3 [CG1].

the conclusion. The first case is only a problem of formulation and can be solved easily using the simple logic rule $(\exists x(f(x)) \to g) \iff \forall x(f(x) \to g)$ where g is free of x. Thus we have a statement with the universal quantifier only. But in the second case, the logic rule $(f \to \exists x g(x)) \iff \exists x(f \to g(x))$ (here f is free of x) tells us that we cannot eliminate the existential quantifiers only by logic transformation. A general solution for this kind of problems is given below. At first we define a function DLR recursively as follows:

$$
\begin{aligned}
&DLR(y_1) = y_1 \\
&DLR(y_1, y_2) = y_1' y_2 - y_2' y_1 \\
&DLR(y_1, ..., y_r) = DLR(DLR(y_1, y_2), ..., DLR(y_1, y_r))
\end{aligned}
$$

We have:

Lemma 1. For variables $x_1, ..., x_n$, there exist $a_1, ..., a_n$ such that:

$$
\begin{aligned}
&a_1' = 0, ..., a_n' = 0 \\
&a_1 x_1 + ... + a_n x_n = 0
\end{aligned}
$$

if and only if $DLR(x_1, ..., x_n) = 0$.

Proof. We prove it by induction. For $n = 1$, as a_1 is an arbitrary constant then we have $a_1 x_1 = 0$ if and only if $x_1 = 0$. For $n = 2$, $a_1 x_1 + a_2 x_2 = 0$ can be written as $x_1/x_2 = a_2/a_1$. This formula is true if and only if $(x_1/x_2)' = 0$ or equivalently, $DLR(x_1, x_2) = x_1' x_2 - x_1 x_2' = 0$ as $x_1 x_2 \neq 0$. Now assume the lemma for $r = k$. $DLR(x_1, ..., x_{k+1}) = 0$ means that $DLR(DLR(x_1, x_2), ..., DLR(x_1, x_{k+1})) = 0$ or equivalently, $b_1 DLR(x_1, x_2) + ... + b_k DLR(x_1, x_{k+1}) = 0$ for some arbitrary constants $b_1, ..., b_k$. The last equation is actually $(b_1 x_2/x_1 + ... + b_k x_{k+1}/x_1)' = 0$ or equivalently $b_1 x_2/x_1 + ... + b_k x_{k+1}/x_1 = b_0$ for a constant b_0. This proves our lemma.

Lemma 2. For variables $x_1, ..., x_n$, there exist $a_0, ..., a_n$ such that:

$$
\begin{aligned}
&a_0' = 0, ..., a_n' = 0 \\
&a_1 x_1 + ... + a_n x_n = a_0
\end{aligned}
$$

if and only if $DLR(1, x_1, ..., x_n) = 0$.

Proof. $a_1 x_1 + ... + a_n x_n = a_0$ for some constants $a_0, ..., a_n$ if and only of $a_1 x_1' + ... + a_n x_n' = 0$, which is equivalent to $DLR(x_1', ..., x_n') = DLR(1, x_1, ..., x_n) = 0$.

Remark 4. $DLR(x_1, ..., x_n)$ is actually the relation among $x_1, ..., x_n$ defined by

$$
\begin{aligned}
&a_1' = 0, ..., a_n' = 0 \\
&a_1 x_1 + ... + a_n x_n = 0
\end{aligned}
$$

In Part II of this paper, we give a general solution of this kind of problem. This method can be used to the following more general case: for a polynomial P find a d-pol $DNR(x_1, ..., x_n)$ such that $P(a_1, ..., a_r, x_1, ...x_n) = 0$ for constants $a_1, ..., a_r$ if and only if $DNR(x_1, ..., x_n) = 0$.

Summing up, if a statement involving the existential quantifier like K1 is in the hypotheses then we can use either (2) or $DLR(1, x, y, r)$, but if such statement is the conclusion then we must use $DLR(1, x, y, r)$.

4. Mechanical Treatment of the Newton-Kepler Problem

We formulate Kepler's first and second law and Newton's gravitational laws as follows.

K1. Each planet describes an ellipse with the sun in one focus.

K2. The radius vector drawn from the sun to a planet sweeps out equal areas in equal times.

N1. The acceleration of a planet is inversely proportional to the square of the distance from the sun to the planet.

N2. The acceleration vectors of planets are directed towards the sun.

For the convenience of representing the physical laws, we use a mixed coordinate system with the center of the force (i.e., the sun) as the origin point. The differential polynomial equations corresponding to K1, K2, N1, and N2 are (2), (3), (4), and (5), respectively:

$$\begin{aligned} &p' = 0 \\ &e' = 0 \\ &f' = 0 \\ &K_1 = r - p - ex - fy = 0 \end{aligned} \tag{2}$$

$$\begin{aligned} &h' = 0 \\ &y'x - yx' - h = 0 \end{aligned} \tag{3}$$

$$\begin{aligned} &c' = 0 \\ &ar^2 - c = 0 \end{aligned} \tag{4}$$

$$x''y - y''x = 0 \tag{5}$$

where a is the magnitude of the acceleration of the planet; r is the length of the radius vector drawn from the sun to the planet, so we have

$$\begin{aligned} &r^2 - x^2 - y^2 = 0 \\ &a^2 - x''^2 - y''^2 = 0 \end{aligned} \tag{6}$$

As a simple application of lemma 2, we can infer that K2 and N2 are equivalent, for by lemma 2, (3) is equivalent to $DLR(1, (x'y - y'x)) = x''y - y''x$, which is exactly the d-pol representing N2. So we need only to consider the relations among N1, N2 (or K2), and K1.

4.1. Mechanical Proof of N1 Under K1 and K2 (or N2)

By lemma 2, we only need to prove $DLR(1, ar^2) = a'r^2 + 2ar'r = 0$ under (2), (5), and (6). Here we shall adopt a simplification: considering a special coordinate system such that the center of the ellipse is on the x-axis. In this coordinate system, K_1 becomes $K_{11} = r - p - ex$. Using theorem 1 to $K_{11} = 0$, (5) and (6) under the variable order: $p < e < f < x < y < r < a$ with p, e, and f as constant parameters, our computer program found two components under the non-degenerate condition $a \neq 0$ meaning that the acceleration and hence the force is zero. In that case, the conclusion is obviously true.

$$
\begin{aligned}
&((e^3-1)x^3+(3e^2-1)px^2+3epx+p^3)x''+pxx'=0 \\
&y^2-(e^2-1)x^2-2pex-p^2=0 \\
&r-p-ex=0 \\
&a^2-y''^2-x''^2=0
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
&p=0 \\
&y^2+(-e^2+1)x^2=0 \\
&r-ex=0 \\
&a^2-y''^2-x''^2=0
\end{aligned} \tag{8}
$$

The pseudo remainder of $DLR(1, ar^2)$ w.r.t (7) is zero. But the remainder of $DLR(1, ar^2)$ w.r.t (8) is not zero. According to section 2, (7) is the main component with dimension 3 and order 2, and (8) is a degenerate component. The physical meaning of (8) is that the ellipse becomes two lines passing the origin point. For a particle moving along such lines condition (5) is always true. Thus there are no restrictions for the velocity and acceleration and hence the conclusion cannot be true. We have proved mechanically that N1 can be deduced from K1 and K2 under the non-degenerate condition $ap \neq 0$. The CPU time for calculating the remainder of $DLR(1, ar^2)$ w.r.t (7) is 7.9 secs and the largest d-pol occurred in the process has 204 terms.

4.2. Mechanical Proof of K1 Under N1 and N2 (or K2)

Note that we cannot adopt the simplification used in section 4.1 in this case, because from N1 and N2 we can only know that the planet is on an ellipse with its focus on the origin point but its center does not necessaryly lie on the x-axis. By lemma 2, K1 is equivalent to

$$
DLR(1,x,y,r)=r'''(y''x'-y'x'')+r''(-y'''x'+y'x''')+r'(y'''x''-y''x''') \tag{9}
$$

and the hypotheses are $DLR(1, ar^2)$ (N1), (5) and (6). The variables are $x < y < r < a$ with no parameters. Using theorem 1, our program found three components under the non-degenerate condition $a \neq 0$.

$$
\begin{aligned}
&9x'''''x''^2x^3+x''''(-45x'''x''x^3+18x''^2x'x^2)+40x'''^3x^3 \\
&\quad -30x'''^2x''x'x^2-6x'''x''^2x'^2x+18x''^4x'x18-4x''^3x'^3=0 \\
&y^2(3x''''x''x^2-4x'''^2x^2+2x'''x''x'x+6x''^3x+2x''^2x'^2) \\
&\quad +x'''^2x^4+4x'''x''x'x^3+4x''^2x'^2x^2=0 \\
&r^2-x^2-y^2=0 \\
&a^2-x''^2-y''^2=0
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
&xx'''+2x'x''=0 \\
&xy'-x'y=0 \\
&r^2-x^2-y^2=0
\end{aligned} \tag{11}
$$

$$a^2 - x''^2 - y''^2 = 0$$

$$\begin{aligned} & x = 0 \\ & yy''' + 2y'y'' = 0 \\ & r^2 - x^2 - y^2 = 0 \\ & a^2 - x''^2 - y''^2 = 0 \end{aligned} \tag{12}$$

The pseudo remainders of (9) w.r.t to (10), (11) and (12) are all zero which means K1 can be deduced from N1 and N2 under the degenerate condition $a \neq 0$. The CPU time used to compute the remainder of (9) w.r.t (10) is 1630.85 secs and the largest d-pol occurred in the process has 13708 terms. For (11), the CPU time is 6.06 secs and the largest d-pol occurred has 36 terms. For (12), CPU time is 0.217 secs and the largest d-pol occurred has 6 terms.

In this problem, (10) is the main component with the dimension zero and the order five which representing an ellipse orbit. The physical meaning of the order is the number of the initial values to determine the motion of the system. In case (10), we need five such values: the initial position (two), velocity (two), and the initial value of the force. (11) and (12), with the dimension zero and the orders four and three respectively, are degenerate components. (11) means the planet moves on a straight line, in this case we need four initial values: the direction of the line, the position (one), the velocity (one), and the value of the force. (12) is a special case of (11).

4.3. Mechanical Proof That N1 and K1 Do Not Imply K2 (or N2)

Naturally we will ask whether K2 or N2 can be deduced from N1 and K1? In physics this means that if a particle moves in an ellipse orbit and is under a force whose magnitude is inversely proportional to the square of the distance from the focus of the ellipse to the particle, we ask whether the force must toward the focus. The answer is negative. Now the conclusion is (5) and the hypotheses are K_{11} (here again this simplification can be adopted), (4), and (6). The variables are under the following order $p < e < h < x < y < r < a$ with p, e, and h as constant parameters. The d-pol ascending chain representing the main component of the hypotheses is

$$\begin{aligned} & c_1 x''^2 + c_2 x'' + c_3 = 0 \\ & y^2 - (e^2 - 1)x^2 - 2pex - p^2 = 0 \\ & r - p - ex = 0 \\ & r^2 a - h = 0 \end{aligned} \tag{13}$$

where c_1, c_2, and c_3 are polynomials of p, e, h, x, and x'. The pseudo remainder of (5) w.r.t to (13) is not zero. Note that the degree of the leading variables of the d-pols in (13) are not greater than two, for which Chou's prover can be used to prove that (13) is irreducible [CH1]. We still need to prove that the differential equation system (13) (here we need only consider the last equation) exists real solutions provided the initial values $x(0), x'(0)$ are given. This is guaranteed by the existence theorem for differential equation, e.g p166, [BC1]. Now we have proved that N2 or K2 is not a logic consequence of K1 and N1.

4.4. The Equivalence of N1 and K1 Under Condition K2 (or N2)

We know from section 4.1 and 4.2 that N1 and K1 are equivalent under the condition of K2 (or N2) and some non-degenerate conditions. In this section, we give a direct proof for this using a simple technique presented in p17 [GA1] which can be stated briefly as below. To prove the equivalence of H_1 and H_2 under an ascending chain ASC, let R_1 and R_2 be the pseudo remainders of H_1 and H_2 w.r.t ASC respectively. If R_1 and R_2 satisfy $D_1 R_1 = D_2 R_2$ then we may say that H_1 and H_2 are equivalent under the non-degenerate conditions: $D_1 \neq 0, D_2 \neq 0$, $I_i \neq 0$, and $S_i \neq 0$, where I_i and S_i are the initials and separants of the d-pols in ASC respectively.

Here the ascending chain which represents K2 (or N2) is

$$\begin{aligned} &x''y - y''x = 0 \\ &r^2 - x^2 - y^2 = 0 \\ &a^2 - x''^2 - y''2 = 0 \end{aligned} \tag{14}$$

Let R_1 and R_2 be the pseudo remainders of $DLR(1, ar^2)$ (N1) and $DLR(1, x, y, r)$ (K1) w.r.t (14) respectively, then we have:

$$\begin{aligned} R_1 &= rx''(x^2 + y^2)(3xx''yy' + xx'''y^2 - x'x''y^2 + x^3x''' + 2x^2x'x'') \\ R_2 &= xx'(y'x - x'y)(3xx''yy' + xx'''y^2 - x'x''y^2 + x^3x''' + 2x^2x'x'') \end{aligned}$$

Hence N1 and N2 are equivalent under the non-degenerate conditions: $a \neq 0$, $r \neq 0$, $x'' \neq 0$, $x \neq 0$, $y \neq 0$, and $y'x - x'y \neq 0$. The physical meaning of these conditions are clear: $a = 0$ means the acceleration and hence the force is zero; $r = 0$ means the orbit of the planet becomes a point; $x'' = 0$ means the planet has no x-acceleration, by (5) and $x \neq 0$ this implies $y'' = 0$ or the acceleration is zero; at last $x = 0$ and $y = 0$ are special cases of $x'y - y'x = 0$ which means that the planet sweeps zero area per secs and hence must be in a line passing the center of the force.

The total CPU time to calculate the pseudo remainders and to factorize the remainders is 7.817 secs which is even a little faster than the proof of N1 under condition K1 and K2. The reason is that to prove the equivalence we need only to eliminate a, r, and y, but to prove N1 or K1 separately we need to eliminate more variables: a, r, y, and x. So it is more complicated. But the second method is more general.

5. More Examples Mechanically Proved in Plane Mechanics

Lots of theorems in mechanics can be proved mechanically in this way especially those whose conclusions are about certain properties of an object such as the velocity, the acceleration, or the orbit of the object. Those whose conclusions are about the time or the length of orbits in which integration is needed, are beyond the scope of the tools developed by us. Here are some of the examples mechanically proved in plane mechanics.

Using rectangular coordinate system, To describe a motion we always have:

$$\begin{aligned} v^2 - x'^2 - y'^2 = 0 \\ a^2 - x''^2 - y''^2 = 0 \end{aligned} \tag{15}$$

where v and a stand for velocity and acceleration of the motion respectively.

5.1. Problems in Plane Kinematics

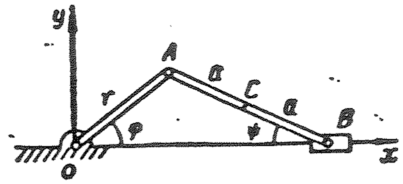

Fig. 1

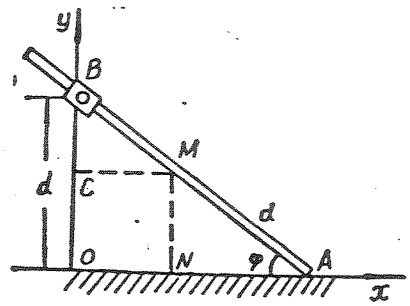

Fig. 2

Example 1. As in figure 1, a bar AB slips on a wall OB. The point A moves with a constant velocity c. Show that the velocity and the acceleration of point M are

$$v = c\sqrt{1 - 2\sin^3(\varphi) + \sin^4(\varphi)}, \quad a = \frac{c^2}{d}\sin^3(\varphi)\sqrt{1 + 3\cos^3(\varphi)}$$

respectively.

Let $A = (z, 0), M = (x, y)$, then we have:

$$\begin{aligned} & z'' = 0 \\ & \cos(\varphi)d - \sin(\varphi)z = 0 \\ & \sin^2(\varphi) + \cos^2(\varphi) - 1 = 0 \\ & x - (z - \cos(\varphi)d) = 0 \\ & y - \sin(\varphi)d = 0 \end{aligned} \tag{16}$$

The conclusions are

$$\begin{aligned} C_1 &= v^2 - z'^2(1 - 2\sin^3(\varphi) + \sin^4(\varphi)) = 0 \\ C_2 &= a^2d^2 - \sin^6(\varphi)z'^4(1 + 3\cos^3(\varphi)) = 0 \end{aligned}$$

By remark 2, we can treat v^2 and a^2 as new variables. Using theorem 1 to (15) and(16) under the following variable order: $d < \sin(\varphi) < \cos(\varphi) < z < x < y < v < a$ with d as constant parameter, our program found three components under the non-degenerate condition $a \neq 0$. The main component is

$$\begin{aligned} & (\sin^3(\varphi) - \sin(\varphi))\sin(\varphi)'' + (-3\sin^2(\varphi) + 2)\sin(\varphi)'^2 = 0 \\ & \cos^2(\varphi) + \sin^2(\varphi) - 1 = 0 \\ & \sin(\varphi)z - \cos(\varphi)d = 0 \\ & x - (z - \cos(\varphi)d) = 0 \\ & y - \sin(\varphi)d = 0 \\ & v^2 - x'^2 - y'^2 = 0 \\ & a^2 - x''^2 - y''^2 = 0 \end{aligned}$$

The pseudo remainders of C_1 and C_2 w.r.t the three ascending chains are all zero. Thus, the results are true under the non-degenerate condition $a \neq 0$.

Example 2. As in figure 2, in a central slider-crank mechanism, the bar OA rotates around point O with angular velocity w. Show that the velocity of point C is

$$v = \frac{rw}{2\cos(\psi)}\sqrt{\cos^2(\varphi) + 4\sin(\varphi)\cos(\psi)\sin(\psi+\varphi)}$$

Let $C = (x, y)$. We have:

$$\begin{aligned}
&w^2 - \sin(\varphi)'^2 - \cos(\varphi)'^2 = 0\\
&\sin^2(\varphi) + \cos^2(\varphi) - 1 = 0\\
&\sin^2(\psi) + \cos^2(\psi) - 1 = 0\\
&r\sin(\varphi) - 2a\sin(\psi) = 0\\
&x - r\cos(\varphi) - a\cos(\psi) = 0\\
&y - a\sin(\psi) = 0
\end{aligned} \tag{17}$$

The conclusion is

$$C_3 = 4v^2\cos^2(\psi) - r^2w^2(\cos^2(\varphi) + 4\sin(\varphi)\cos(\psi)\sin(\varphi+\psi)) = 0$$

By remark 2, we can treat v^2, a^2 and w^2 as new variables. Using theorem 1 to (17) and (15) under the following variable order: $r < a < \sin(\varphi) < \cos(\varphi) < \sin(\psi) < \cos(\psi) < x < y < w < v$ with r, a, and $\sin(w)$ as parameters in which r and a are constant, our program found ten components under the non-degenerate condition $ra \neq 0$. The main component is just (17). The pseudo remainders of C_3 w.r.t the ten ascending chains are all zero.

Example 3. A bar rotates in a plane around the focus of an ellipse in the same plane. The equation of the ellipse is $r = \frac{a_1(1-e^2)}{1+e\cos(w)}$. Show that the velocity of the intersection point of the bar and the ellipse can be represented as $v = \frac{rw'}{b}\cdot\sqrt{r(2a_1 - r)}$, where b is the short radius of the ellipse and w is the inclination of the radius to the x-axis.

We have:

$$\begin{aligned}
&r^2w' - xy' + x'y = 0\\
&e^2a_1^2 - a_1^2 + b^2 = 0\\
&\sin^2(w) + \cos^2(w) - 1 = 0\\
&r(1 + e\cos(w)) - a_1(1 - e^2) = 0\\
&x - r\cos(w) = 0\\
&y - r\sin(w) = 0
\end{aligned} \tag{18}$$

The conclusion is

$$C_4 = v^2b^2 - r^3w'^2(2a_1 - r) = 0$$

The first d-pol of (18) is the relation between the velocity and angular velocity a mechanical derivation of which can be found in Part II of this paper. The variables are under the following order: $e < a_1 < b < \sin(w) < \cos(w) < r < x < y < w < v$ with e, a_1 as constant parameters and $\sin(w)$ as ordinary parameter. Treat v^2 as a new variable. Using theorem 1 to (18) and (15), our program found three components under the non-degenerate condition $a_1 b \neq 0$. The main component is

$$\begin{aligned}
&b^2 + a_1^2 e^2 - a_1^2 = 0\\
&\cos^2(w) + \sin^2(w) - 1 = 0\\
&(e\cos(w) + 1)r + a_1 e^2 - a_1 = 0\\
&x - \cos(w) r = 0\\
&y - \sin(w) r = 0\\
&w' + \sin(w)\cos(w)' - \sin(w)'\cos(w) = 0\\
&v^2 - x'^2 - y'^2 = 0
\end{aligned}$$

The pseudo remainders of C_4 w.r.t the three ascending chains are all zero.

Example 4. A particle moves on a circle with radius r. The angle between its velocity and acceleration is a constant.

(a). Show that $1/v = 1/v_0 - \frac{t\cot(\theta)}{r}$ where v_0 is the initial value of the velocity and θ is the angle between the velocity and acceleration of the particle.

(b). Show that $v = v_0 e^{(w - w_0)\cot(\theta)}$ where w is the inclination of the radius to the x-axis.

We have:

$$\begin{aligned}
&w'^2 - \sin(w)'^2 - \cos(w)'^2 = 0\\
&\sin^2(w) + \cos^2(w) - 1 = 0\\
&x - r\cos(w) = 0\\
&y - r\sin(w) = 0\\
&\cos^2(\theta) v^2 a^2 - (x'x'' + y'y'')^2 = 0
\end{aligned} \tag{19}$$

The conclusions are

$$\begin{aligned}
&C_5 = v'^2 r^2 (1 - \cos^2(\theta)) - v^4 \cos^2(\theta) = 0\\
&C_6 = v'^2 (1 - \cos^2(\theta)) - v^2 \cos^2(\theta) w'^2 = 0
\end{aligned}$$

The equivalence of C_5 and C_6 to (a) and (b) comes from lemma 1. The variables are under the following order: $r < \cos(\theta) < \sin(w) < \cos(w) < x < y < w < v$ with r and $\cos(\theta)$ as constant parameters. Using theorem 1, our program found two components under the non-degenerate condition $v \neq 0$:

$$\begin{aligned}
&((\cos^2(\theta) - 1)\sin^4(w) + (-2\cos^2(\theta) + 2)\sin^2(w) + \cos^2(\theta) - 1)\sin(w)''^2\\
&+ (((-2\cos^2(\theta) + 2)\sin^3(w) + (2\cos^2(\theta) - 2)\sin(w))\sin(w)'^2)\sin(w)''\\
&+ (-\sin^2(w) + \cos^2(\theta))\sin(w)'^4 = 0\\
&\cos^2(w) + \sin^2(w) - 1 = 0\\
&x - r\cos(w) = 0\\
&y - r\sin(w) = 0
\end{aligned}$$

$$w^2 - \cos(w)'^2 - \sin(w)'^2 = 0$$
$$v^2 - y'^2 - x'^2 = 0$$

$$\cos(\theta) = 0$$
$$(\sin^2(w) - 1)\sin(w)'' - \sin(w)\sin(w)'^2 = 0$$
$$\cos^2(w) + \sin^2(w) - 1 = 0$$
$$x - r\cos(w) = 0$$
$$y - r\sin(w) = 0$$
$$w^2 - \cos(w)'^2 - \sin(w)'^2 = 0$$
$$v^2 - y'^2 - x'^2 = 0$$

The pseudo remainders of C_5 and C_6 w.r.t the two ascending chains are zero.

Example 5. The y-axis projection of the velocity of a particle moving within a plane is a constant c. Show that $a = \frac{v^3}{cp}$ where p is the curvature radius of the orbit of the particle.

By a formula for curvature in differential geometry, we have:

$$\begin{aligned} &a - x'' = 0 \\ &v^2 - x'^2 - c^2 = 0 \\ &p^2c^2x''^2 - (y'^2 + x'^2)^3 = 0 \end{aligned} \tag{20}$$

The conclusion is

$$C_7 = a^2c^2p^2 - v^6 = 0$$

Fix an order for the variables: $c < x < v < a < p$ with c as constant parameter. Using theorem 1 to (20), our program found three components of which the main component is just (20). The pseudo remainders of C_7 w.r.t the three ascending chains are zero which means that the result is unconditionally true.

5.2. Problems About the Motion Under a Central Force

We use the same coordinate system as section 4. The equations describing a motion under a central force can be formulated as below

$$\begin{aligned} &x''y - y''x = 0 \\ &r^2 - x^2 - y^2 = 0 \\ &h - xy' + x'y = 0 \\ &r^2w' - h = 0 \end{aligned} \tag{21}$$

where h and w stands for the angular momentum per unit mass and the inclination of the radius to the x-axis respectively. In example 6, 7, and 8, the variables are under the following order $x < y < r < h < w < v < a$ with x as parameter.

Example 6. The orbit described by a particle under a central attractive force is an ellipse with its center at the center of the force if and only if the force is varying directly as the distance.

The two conclusions are

$$C_8 = DLR(a, r) = 0$$
$$C_9 = DLR(1, x^2, xy, y^2) = 0$$

By lemma 1, C_9 is equivalent to $c_1x^2 + c_2xy + c_3y^2 + 1 = 0$ for arbitrary constants c_1, c_2, and c_3. The pseudo remainders of C_8 and C_9 w.r.t (15) and (21) are

$$(x'''x - x''x')(y'x - x'y)y'x'' = 0$$
$$(x'''x - x''x')(x^2 + y^2)x'' = 0$$

Hence C_8 and C_9 are equivalent under the non-degenerate conditions: $y'x - x'y \neq 0, x'' \neq 0, y' \neq 0$, and $r \neq 0$ the physical meaning of which have been given in section 4.4.

Example 7. (Binet's formula) The differential equation of the orbit of a particle moving under a central force P per unit mass is $h^2u^2(d^2u/dw^2 + u) = P$, where $u = 1/r$.

By changing the differential w.r.t w to the differential w.r.t time, Binet's formula becomes

$$C_{10} = a^2w'^6 - h^4u^4(u''w' - w''u' + uw'^3)^2$$

Suppose $u > y$. Using theorem 1 to (15), (21), and $ur - 1 = 0$ under the on-degenerate condition $a \neq 0$, our program found two components of which (15), (21), and $ur - 1 = 0$ consist of the main component. The pseudo remainders of C_{10} w.r.t the two ascending chains are zero.

Example 8. For a particle under a central force, the magnitude of the force per unit mass is

$$P = 1/2 \cdot h^2\, dq^{-2}/dr$$

where q is the distance from the force center to the tangent line of the orbit.

By simple calculation, we have:

$$H_1 = g(xy' - yx')^2 - (x'^2 + y'^2) = 0$$

where $g = q^{-2}$. The conclusion is

$$C_{11} = (2ar')^2 - (h^2g')^2 = 0$$

Suppose $g > y$, then (15), (21), and H_1 form a triangular form already. Using theorem 1, we know this triangular form is the only component of the decomposition under the nondegenerate condition $a \neq 0$. The pseudo remainder of C_{11} w.r.t (15), (21), and $H_1 = 0$ is zero.

Example 9. If the orbit of a particle moving under a central force is $r^2 = c^2\cos(2w)$, show that the force per unit mass is $p = \frac{-3c^4h^2}{r^7}$

$r^2 = c^2\cos(2w)$ is equivalent to

$$H_2 = r^4 - c^2(2x^2 - r^2)$$

The conclusion is

$C_{12} = (ar^7)^2 - (3c^4h^2)^2$

The variables are under the following order: $c < x < y < r < w < h < a$ with c as constant parameter. Under the non-degenerate condition $a \neq 0$, our program found one component for (15), (21), and $H_2 = 0$:

$$\begin{aligned}
&(64x^7 - 48c^2x^5 - 15c^4x^3 - c^6x)x''^2 + ((-48c^2x^4 - 30c^4x^2 - 3c^6)x'^2)x'' + 9c^4xx'^4 = 0\\
&((24x^4 + 11c^2x^2 + c^4)x'' - 3c^2xx'^2)y^2 + (8x^6 - 7c^2x^4 - c^4x^2)x'' + 3c^2x^3x'^2 = 0\\
&r^2 - y^2 - x^2 = 0\\
&h - xy' + x'y = 0\\
&r^2w' - h = 0\\
&v^2 - y'^2 - x'^2 = 0\\
&a^2 - y''^2 - x''^2 = 0
\end{aligned} \tag{22}$$

The pseudo remainder of C_{12} w.r.t (22) is zero.

Example 10. If the magnitude of the velocity of a particle under a central force is inversely proportional to the distance from the force center to the particle, show that the orbit of the particle is

$$\ln(r/r_0) = \pm\frac{\sqrt{c^2 - h^2}}{h} \cdot w$$

where $c = vr$ is the proportional constant and r_0 is the initial value of r.

The conclusion is equivalent to

$$C_{13} = h^2r'^2 - (c^2 - h^2)r^2w'^2 = 0$$

Fix an order for the variables: $c < x < y < r < w < h < a$ with c as constant parameter. Using theorem 1 to (15), (21), and $vr - c = 0$, our program found 12 components under the non-degenerate condition $a \neq 0$, of which the main component is

$$\begin{aligned}
&c^2x^4x'''^4 - 4c^2x^3x'x''x'''^3 + ((64x^5x'^2 + 32c^2x^3)x''^3 + 6c^2x^2x'^2x''^2)x'''^2\\
&+ (-256x^5x'x''^5 + (128x^4x'^3 - 64c^2x^2x')x''^4 - 4c^2xx'^3x''^3)x''' + 256x^5x''^7\\
&+ (-256x^4x'^2 + 256c^2x^2)x''^6 + (64x^3x'^4 + 32c^2xx'^2)x''^5 + c^2x'^4x''^4 = 0\\
&(x^2x'''^2 - 2xx'x''x''' + 16xx''^3 + x'^2x''^2)y^2 + x^4x'''^2 + 6x^3x'x''x''' + 9x^2x'^2x''^2 = 0\\
&r^2 - y^2 - x^2 = 0\\
&h - xy' + x'y = 0\\
&r^2w' - h = 0\\
&rv - c = 0\\
&a^2 - y''^2 - x''^2 = 0
\end{aligned}$$

The pseudo remainders of C_{13} all the ascending chains are zero.

The most difficult one of the ten examples is example 9. To calculate the pseudo remainder of C_{12} w.r.t (22), the CPU time used is 669.4 secs, and the largest d-pol occurred in the process has 5597 terms.

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Part II. Mechanical Formula Derivation in Plane Mechanics

1. Introduction

In [WU3], Wu Wen-Tsün introduced a method of mechanical derivation of formulas in elementary geometries which was successfully used to solve quite a few difficult problems in Euclidean and Non-Euclidean geometries. In our opinion, this kind of method is more important in mechanics, for unlike in elementary geometries where we usually need to give proofs for some known results and conjectures, in mechanics most problems are about how to find some unknown formulas. But unfortunately, there are some difficulties when extending the method to mechanics, or more generally, to the problems involving differential polynomials. As shown in the appendix of [WU2], the relations we want do not generally occur in the final ascending chain obtained by Ritt-Wu's well ordering algorithm. Instead, they occur in the process. Therefore, more human assistance has to be involved. The problem is that we often want to find a relatively simple formula instead of a complicated one, though the two are equivalent. In this paper, we present a method which can be used to find such a simple relation mechanically. The method is used for plane mechanics and is quite successful for certain problems. As a particular result, we give a mechanical derivation of Newton's square inverse gravitational law from Kepler's laws in the sense that the formula representing Newton's law occurs as the first element of the ascending chain of the non-degenerate component.

In formula derivation, another difference between problems involving differential polynomials and problems involving ordinary polynomials is that the relations we find are generally some differential equations, but what we really want is the solution of these equations. So, to make the algorithm complete, we need a method to solve the multivariate algebraic differential equation, which is still an open problem as far as we know. In this paper, we give a partial solution to decide whether the solution of a multivariate algebraic differential equation is a polynomial equation.

In section 2, we give a precise description of the differential version of Wu's method of mechanical formula derivation. A uniqueness theorem is proved. In section 3, we shall show the defect of the method when we try to derive Newton's laws from Kepler's laws and give a solution to this problem. In section 4, we give a refined method of mechanical formula derivation based on the idea of section 3. In section 5, more examples are given.

We assume the reader is familiar with Ritt-Wu's zero decomposition theorem for d-pols, a brief description of which may be found in Part I of this paper. For the full theory, the reader is referred to [WU1] or [CG2] with our improvement.

2. An Algorithm of Mechanical Formula Derivation (I)

Suppose a geometry-mechanics problem can be given by a set of d-pol equations DPS in variables $u_1, ..., u_q, x_1, ..., x_p$ with the u as parameters, and for a particular variables among the x, say x_1, we want to find the relations between x_1 and the u determined by DPS.

The problem can be divided into two subproblems:

Problem 1. Find the formulas (in the form of differential equations) between x_1 and the u.

Problem 2. Find the general form of the solution of these differential equations.

Rit-Wu's zero decomposition algorithm can be used to give a solution to the first problem. The second is still an open problem as far as we know. We give a partial solution to this problem, i.e., to decide whether the solution of the differential equations is a polynomial equation.

As an instance, let us consider the relations between the u and x_1 defined by DPS. Applying theorem 1 of Part I to DPS under the variable order $u_1 < ... < u_q < x_1 ... < x_p$, we have

$$\text{Zero(DPS)} = \cup_{i=1}^{s} \text{Zero}(PD(ASC_i^*)) \cup_{j=1}^{t} \text{Zero}(PD(ASC_j))$$

where the Zero$(PD(ASC_i^*))$ are the main components. Then the first d-pol of each ASC_i^*, $A_{i,1}$, must be a d-pol of the u and x_1 and the set of the prime ideals $RS = \{PD(A_{1,1}), .., PD(A_{s,1})\}$ is defined to be the relation between x_1 and the u determined by DPS. Note that the set RS is not uniquely determined. But we have the following property.

Theorem 1. Let RS_1 and RS_2 be two relation sets derived from the same d-pol set DPS, then we have

$$\cup_{I \in RS_1} \text{Zero}(I) = \cup_{J \in RS_2} \text{Zero}(J)$$

Proof. Let $ASC_i, i = 1, ..., s$ and $ASC_j', j = 1..., t$ be the asc chains which give the main components in the two decompositions respectively. By the uniqueness of the manifold decomposition

$$\cap_{i=1}^{s} PD(ASC_i) = \cap_{j=1}^{t} PD(ASC_j')$$

Restricting the ideals in the above formula to $K\{u_1, ..., u_q, x_1\}$, we have

$$\cap_{i=1}^{s} PD(A_{i,1}) = \cap_{j=1}^{t} PD(A_{j,1}')$$

which proves the theorem. Q.E.D

The following result provides a method to delete some of the redundant d-pols in the relation.

Lemma 1. Let ASC_1 and ASC_2 be two ascending chains of which ASC_2 is irreducible. If the pseudo remainders of the d-pols in ASC_1 w.r.t ASC_2 are zero and the pseudo remainder of the product of the initials and separants of the d-pols in ASC_1 w.r.t ASC_2 is not zero, then $PD(ASC_1) \subset PD(ASC_2)$.

Proof. As ASC_2 is irreducible, $PD(ASC_2)$ is a prime ideal. According to the definition of $PD(ASC_2)$, we have $ASC_1 \subset PD(ASC_2)$ and $J \not\subset PD(ASC_2)$ where J is any product of the separants and initials of the d-pols in ASC_1. Let $P \in PD(ASC_1)$, then there exists a product J_1 of the separants and initials of the d-pols in ASC_1 such that

$$J_1 P \equiv 0 \quad [ASC_1]$$

From which we have $J_1 P \in PD(ASC_2)$. Hence $P \in PD(ASC_2)$ as J_1 is not in $PD(ASC_2)$.

Here is our algorithm.

Algorithm 1. For a d-pol set $DPS \in K\{u, x_1, ..., x_p\}$ find the relations between the u and x_1 defined by DPS.

Step 1. Apply theorem 1 in Part I to DPS, we have:

$$\text{Zero(DPS)} = \cup_{i=1}^{s}\text{Zero}(PD(ASC_i^*)) \cup_{j=1}^{t} \text{Zero}(PD(ASC_j))$$

where the ASC_i^* are the main components.

Step 2. Let $RP = \{R_1, ..., R_s\}$ be the set of the first d-pols of the ASC_i^*.

Step 3. For each pair R_i and R_j in RP, if the pseudo remainder of R_i w.r.t R_j is zero and the pseudo remainder of the product of the initial and separant of R_i w.r.t R_j is not zero, then delete R_i from RP.

Step 4. At last, we have $RP = \{R_1', ..., R_t'\}$. The relations are $\{PD(R_1'), ..., PD(R_t')\}$.

We consider a special form of problem 2:

Problem 3. If $u_1, ..., u_q$ satisfy an algebraic differential equation: $P(u_1, ..., u_q) = 0$ for a d-pol P, decide whether the u satisfy $R(a_1, ..., a_m, u_1, ..., u_q) = 0$ for a polynomial R and some constants $a_1, ..., a_m$.

Here we only have a partial solution to this problem: If such a relation exists, our method can find it in a finite number of steps, but we cannot prove there are no such relations.

Algorithm 2. A partial solution to problem 3.

Step 1. For $n = 1$ to infinite, decide whether $u_1, ..., u_q$ satisfy a polynomial equation of degree n with some constants. This is equivalent to whether the pseudo remainder of P w.r.t DLR_n $= DLR(1, u_1, u_2, ..., u_q^n)$ is zero.

Step 2. Let n_0 be the first number such that the pseudo remainder of P w.r.t DLR_{n_0} is zero, then the u satisfy a polynomial equation of degree n_0 with some constants.

Step 3. By lemma 1 in Part I, we can go further to decide the particular form of polynomial equation of degree n_0, for which $u_1, ..., u_n$ are satisfied.

3. Mechanical Derivation of N1 From K1 and K2 (or N2)

We use the same coordinate system and differential equations for Newton's laws and Kepler's laws as the first part of this paper. Our task is to derive N1 from K1 and K2 (or N2) using algorithm 1 without knowing N1 in advance. The hypotheses are

$$\begin{aligned} N_2 &= x''y - y''x = 0 \\ K_{11} &= DLR(1, x, r) = x''r' - r''x' = 0 \\ h_1 &= r^2 - x^2 - y^2 = 0 \\ h_2 &= a^2 - x''^2 - y''^2 = 0 \end{aligned}$$

The ascending chain representing the main component under the variable order $r < a < x < y$ is

$$\begin{aligned} & r^2r'r'''' + (-r^2r'' + 6rr'^2)r''' + 6r'^3r'' = 0 \\ & r'a - rr''' - 3r'r'' = 0 \end{aligned} \qquad (1)$$

$$(r'^2a - rr''^2 - r'^2r'')x^2 + r^3r''^2 = 0$$
$$y^2 + x^2 - r^2 = 0$$

But from (1), we cannot obtain any relation between a and r easily. The problem is that the simplest relation between a and r is $DLR(1, ar^2) = r(a'r + 2r'a) = 0$ in which the highest order for a and r are one. But in (1) the highest order for r is four. This suggest that if only a, a', r and r' are allowed during the computation, we may hopefully get the relation we want.

Note that r'' occurs in K_{11}. We may eliminate it by taking the pseudo remainder of K_{11} w.r.t h_1. The remainder is

$$K_{12} = (x'y^3 + x^2x'y)y'' + x^2x'y'^2 + (-x''y^3 + (-x^2x'' - 2xx'^2)y)y' + x'^3y^2 = 0$$

which involves x and y alone.

We adopt a slightly different viewpoint from the method described in section 2 for the problem: N_2 and K_{12} decide the motion completely by giving the differential equations for x and y, and our problem is actually find the relation between two d-pols of x and y: $r^2 = x^2 + y^2$ and $a^2 = x''^2 + y''^2$ under the condition of N_2 and K_{12}. To do this, our first step is to determine the relation among x and y determined by N_2 and K_{12}. This can be done by applying algorithm 1 to N_2 and K_{12}. The ascending chain representing the main component is

$$\begin{aligned} &3x^2x'x''x'''' - 4x^2x'x'''^2 + (-3x^2x''^2 + 2xx'^2x'')x''' + 2x'^3x''^2 = 0 \\ &3x''^2y^2 + x^2x'x''' + 2xx'^2x'' = 0 \end{aligned} \tag{2}$$

The pseudo remainders of h_1, h_1', h_2, and h_2' w.r.t (2) are

$$\begin{aligned} &x^2x'x''' + (-3x^2 + 3r^2)x''^2 + 2xx'^2x'' = 0 \\ &x^2x'x'''^2 + (-3x^2x''^2 + xx'^2x'')x''' + (3xx' - 9rr')x''^3 - 2x'^3x''^2 = 0 \\ &xx'x''' - 3xx''^2 + 2x'^2x'' + 3a^2x = 0 \\ &2x^2x'x'''^2 + (-6x^2x''^2 + 2xx'^2x'')x''' + 6xx'x''^3 - 4x'^3x''^2 + 9aa'x^2x'' = 0 \end{aligned} \tag{3}$$

Applying algorithm 1 to (3) in the sense of ordinary polynomials under the variable order: $r < r' < a < a' < x < x' < x'' < x'''$, our computer program produced two ascending chains representing the main components. They are

$$\begin{aligned} &ra' + 2r'a = 0 \\ &2rax'^2 + ra'xx' - 2a^2x^2 + 2r^2a^2 = 0 \\ &rx'' - ax = 0 \\ &x^2x'x''' + (-3x^2 + 3r^2)x''^2 + 2xx'^2x'' = 0 \end{aligned} \tag{4}$$

$$\begin{aligned} &ra' + 2r'a = 0 \\ &rx'^2 - r'xx' + ax^2 - r^2a = 0 \\ &rx'' + ax = 0 \\ &x^2x'x''' + (-3x^2 + 3r^2)x''^2 + 2xx'^2x'' = 0 \end{aligned} \tag{5}$$

The first equation of (4) and (5) actually means that a is reversely proportional to r^2. Hence we have derived N1 from K1 and K2 mechanically. The crucial point is that we only allow

r', a' occurring in the process and all higher orders are forbidden. We do this by treating the elements in (3) as ordinary polynomials for the new variables: $a, a', r, r'x, x', x''$, and x'''.

4. An Algorithm of Mechanical Formula Derivation (II)

Based on the idea of section 3, we formulate our problem as below.

Problem 4. Given two groups of d-pols.

$$\begin{aligned} &P_1(x_1, ..., x_p) = 0 \\ &\dots \\ &P_r(x_1, ..., x_p) = 0 \end{aligned} \tag{6}$$

$$\begin{aligned} &Q_1(u_1, x_1, ..., x_p) = 0 \\ &\dots \\ &Q_q(u_q, x_1, ..., x_p) = 0 \end{aligned} \tag{7}$$

find the differential equations satisfied by $u_1, ..., u_q$ with the lowest order for the u.

Algorithm 3. A solution to problem 4.

Let $VS = \{u_1, ..., u_q\}, PS_0 = \{Q_1, ..., Q_q\}$

Step 1. Apply algorithm 1 to (6) and (7) in the sense of d-pols under the following variable order: $u_1 < \cdots < u_q < x_1 < \cdots < x_p$ with $u_1, ..., u_{q-1}$ as parameters. If there exist no nondegenerate components containing d-pols involving the u alone, then there exists no relation among the u at all. If there exists one d-pol of the u in some non-degenerate components, then these relations are the relations we want. For the non-degenerate components involving more than one d-pols of the u, do step 2.

Step 2. Apply Ritt-Wu's decomposition algorithm to (6) in the sense of d-pols.

$$Zero((6)) = \cup_{i=1}^{n} Zero(PD(ASC_i))$$

Step 3. For each nondegenerate component $Zero(PD(ASC_{i_0}))$, let PS_1 be the set of the pseudo remainders of the d-pols in PS_0 w.r.t ASC_{i_0}.

Step 4. Use Ritt-Wu's decomposition algorithm to PS_1, ASC_{i_0} and (6) in the sense of ordinary polynomial under the following variable order: $u_1 < \cdots < u_q < x_1 < x_1' < \cdots x_p < x_p' \cdots$ for the variables occurred in PS, ASC_{i_0}, and (6).

Step 5. If there exists a polynomial involving variables in VS alone in some non-degenerate components, then this relation is what we want.

Step 6. Otherwise, for $i = 1$ to q, $j = 1$ to infinite, do as follows: adding $u_{i,j}$ (the j-th derivation of u_i w.r.t t) to VS, adding $Q_{i,j}$ (the j-th derivation of Q_i w.r.t t) to PS_0, and repeating step 3, 4, and 5 for the new VS and PS_0. As there are actually some relations among $u_1, ..., u_q$, the process must terminate at a finite number of steps.

In algorithm 3, by treating the d-pols as ordinary polynomials, we can get the relation among the u with lowest order. This algorithm only fits to those problem in which such relation is exactly the one we want. Fortunately, most of the problems we encountered belong to this kind of problems as shown by the examples in section 5 below.

5. More Examples in Plane Mechanics

In Part I of this paper, ten formulas in mechanics have been mechanically proved. Actually, most of these formulas can be mechanically derived using algorithm 3. For convenience, we use the same example number with a prime here to show its corresponding example in Part I.

5.1. Problems in Plane Kinematics

Example 1, 2, and 3 in Part I of this paper belong to the same class of problems: in a mechanism, the velocity and acceleration of one point are given, find the velocity and acceleration of another point. This kind of problems is a class of typical examples which can be solved by algorithm 3.

Example 1'. As in figure 1, a bar AB slips on a wall OB. The point A moves with a constant velocity c. Find the velocity and acceleration of point M.

Let $A = (z, 0), M = (x, y)$. The two groups of hypotheses are

$$\begin{aligned} & z'' = 0 \\ & \cos(\varphi)d - \sin(\varphi)z = 0 \\ & \sin^2(\varphi) + \cos^2(\varphi) - 1 = 0 \\ & x - z + \cos(\varphi)d = 0 \\ & y - \sin(\varphi)d = 0 \end{aligned} \tag{8}$$

$$\begin{aligned} & c - z' = 0 \\ & v^2 - x'^2 - y'^2 = 0 \end{aligned} \tag{9}$$

Applying theorem 1 in Part I to (8) under the following variable order: $d < \sin(\varphi) < \cos(\varphi) < z < x < y$ with d as the constant parameter, our program found the following ascending chain which represents the main component.

$$\begin{aligned} & (\sin^3(\varphi) - \sin(\varphi))\sin(\varphi)'' + (-3\sin^2(\varphi) + 2)\sin(\varphi)'^2 = 0 \\ & \cos^2(\varphi) + \sin^2(\varphi) - 1 = 0 \\ & \sin(\varphi)z - d\cos(\varphi) = 0 \\ & x - z + \cos(\varphi)d = 0 \\ & y - \sin(\varphi)d = 0 \end{aligned} \tag{10}$$

The pseudo remainders of the d-pols in (9) w.r.t (10) are

$$\begin{aligned} & \sin(\varphi)'^2(\sin^3(\varphi)d^2 - \sin^2(\varphi)d^2 - \sin(\varphi)d^2 - d^2) + \sin^5(\varphi)v^2 + \sin(\varphi)4v^2 = 0 \\ & \cos(\varphi)\sin^2(\varphi)c + \sin(\varphi)'d = 0 \end{aligned} \tag{11}$$

Applying theorem 1 in Part I to (8), (10) and (11) in the sense of ordinary polynomials under the variable order $d < c < \sin(\varphi) < v < \sin(\varphi)' < \cos(\varphi) < z < x < y < z''$, we have:

$$v^2 + \sin^4(\varphi)c^2 - 2\sin^3(\varphi)c^2 - c^2 = 0$$
$$d^2 \sin(\varphi)'^2 + \sin^6(\varphi)c^2 - \sin^4(\varphi)c^2 = 0$$
$$c\sin^2(\varphi)\cos(\varphi) + \sin(\varphi)'d = 0$$
$$\sin(\varphi)z - \cos(\varphi)d = 0$$
$$x - (z - \cos(\varphi)d) = 0$$
$$y - \sin(\varphi)d = 0$$
$$z'' = 0$$

The first polynomial is just the velocity formula of example 1. To find the formula for acceleration, we only need to replace (9) by:

$$c - z' = 0$$
$$a^2 - x''^2 - y''^2 = 0$$

and repeat the above process. The relation obtained by algorithm 3 is $a^2d^2 + 3\sin^8(\varphi)c^4 - 4\sin^6(\varphi)c^4 = 0$ which is exactly the acceleration formula of example 1.

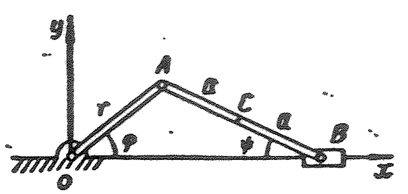

Fig. 1

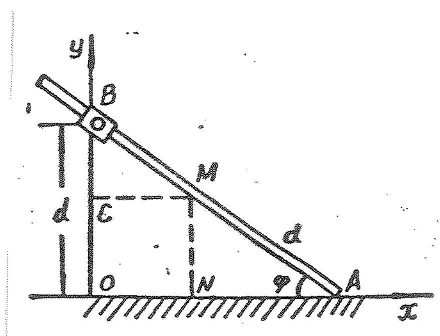

Fig. 2

Example 2'. As in figure 2, in a central slider-crank mechanism, the bar OA rotates around O with angular velocity φ'. Find the velocity of C.

Let $C = (x, y)$. The two groups of hypotheses are

$$\begin{aligned} &\sin^2(\varphi) + \cos^2(\varphi) - 1 = 0 \\ &\sin^2(\psi) + \cos^2(\psi) - 1 = 0 \\ &r\sin(\varphi) - 2a\sin(\psi) = 0 \\ &x - r\cos(\varphi) - a\cos(\psi) = 0 \\ &y - a\sin(\psi) = 0 \end{aligned} \tag{12}$$

$$v^2 - x'^2 - y'^2 = 0 \tag{13}$$

$$\sin(\varphi)' - \cos(\varphi)\varphi' = 0$$

Applying theorem 1 in Part I to (12) under the following variable order: $r < a < \varphi < v < \cos(\psi) < \sin(\psi) < \cos(\varphi) < \sin(\varphi) < x < y$ with r and a as the constant parameters, our program found the following ascending chain which represents the main component.

$$\begin{aligned}
&\sin^2(\psi) + \cos^2(\psi) - 1 = 0 \\
&r^2\cos^2(\varphi) + 4a^2\sin^2(\psi) - r^2 = 0 \\
&r\sin(\varphi) - 2a\sin(\psi) = 0 \\
&x - r\cos(\varphi) - a\cos(\psi) = 0 \\
&y - a\sin(\psi) = 0
\end{aligned} \tag{14}$$

The pseudo remainders of the d-pols in (13) w.r.t (14) are

$$\begin{aligned}
&r\sin(\psi)\cos(\varphi)\varphi' - 2a\cos(\psi)\cos(\psi)' = 0 \\
&(4a^2\cos^4(\psi) + (r^2 - 8a^2)\cos^2(\psi) - r^2 + 4a^2)v^2 \\
&+ ((-8a^3r\cos^3(\psi) + 8a^3r\cos(\psi))\cos(\psi)'^2)\cos(\varphi) \\
&+ (-16a^4\cos^4(\psi) + 20a^4\cos^2(\psi) + a^2r^2 - 4a^4)\cos(\psi)'^2 = 0
\end{aligned} \tag{15}$$

Applying theorem 1 in Part I to (12) and (15) in the sense of ordinary polynomial under the variable order: $r < \cos(\psi) < \sin(\psi) < \cos(\varphi) < \sin(\varphi) < a < x < y < \varphi' < v < \cos(\psi)'$, our program produces the ascending chain representing the main component

$$\begin{aligned}
&\ldots \\
&4\cos^2(\psi)v^2 + ((-4r^2\cos^2(\psi) + r^2)\sin^2(\varphi) - 4r^2\cos(\psi)\sin(\psi)\cos(\varphi)\sin(\varphi) - r^2)\varphi'^2 = 0 \\
&2a\cos(\psi)\cos(\psi)' - r\sin(\psi)\cos(\varphi)\varphi' = 0
\end{aligned}$$

The polynomial involving v is equivalent to the result of example 2. Use the above method, we can also find a formula for the acceleration under the condition $\varphi'' = 0$.

$$\begin{aligned}
&16a^2\cos^6(\psi)a^2 + ((-32a^3r\cos^7(\psi) + (-8ar^3 + 32a^3r)\cos^3(\psi))\cos(\varphi) - 64a^4\cos^8(\psi) \\
&+ (-16a^2r^2 + 48a^4)\cos^6(\psi) + (-8a^2r^2 + 32a^4)\cos^4(\psi) - r^4 + 8a^2r^2 - 16a^4)\varphi'^4 = 0
\end{aligned}$$

This formula is not given in example 2.

Example 3'. A bar rotates in a plane around one focus of an ellipse in the same plane. The equation of the ellipse is $r = \frac{a_1(1-e^2)}{1+e\cos(w)}$. Find the relation between the velocity of the intersection point of the bar and the ellipse and the distance from the intersection point to the focus.

Let w be the inclination of the bar to the x-axis, and let b be the short axis of the ellipse. We have

$$\begin{aligned}
&e^2a_1^2 - a_1^2 + b^2 = 0 \\
&\sin^2(w) + \cos^2(w) - 1 = 0 \\
&r(1 + e\cos(w)) - a_1(1 - e^2) = 0 \\
&x - r\cos(w) = 0
\end{aligned} \tag{16}$$

$$y - r\sin(w) = 0$$

$$\begin{aligned} &v^2 - x'^2 - y'^2 = 0 \\ &\sin(w)' - \cos(w)w' = 0 \end{aligned} \tag{17}$$

Applying theorem 1 in Part I to (16) under the following variable order: $a_1 < b < e < r < \sin(w) < \cos(w) < x < y$ with a_1, b as the constant parameters, our program found the following ascending chain which represents the main component.

$$\begin{aligned} &a_1^2e^2 + b^2 - a_1^2 = 0 \\ &e^2r^2\sin^2(w) + (-e^2+1)r^2 + (2a_1e^2 - 2a_1)r + a_1^2e^4 - 2a_1^2e^2 + a_1^2 = 0 \\ &er\cos(w) + r + a_1e^2 - a_1 = 0 \\ &x - r\cos(w) = 0 \\ &y - r\sin(w) = 0 \end{aligned} \tag{18}$$

The pseudo remainders of the d-pols in (17) w.r.t (18) are

$$\begin{aligned} &(((b^2 - a_1^2)r^2)w')\sin(w) + a_1b^2er' = 0 \\ &(r^2 - 2a_1r + b^2)v^2 + (-r^2 + 2a_1r)r'^2 = 0 \end{aligned} \tag{19}$$

Applying theorem 1 in Part I to (16), (18) and (19) in the sense of ordinary polynomial under the variable order: $a_1 < b < r < w' < v < e < r' < \sin(w) < \cos(w) < x < y$, our program produces the ascending chain representing the main component

$$\begin{aligned} &b^2v^2 + (r^4 - 2a_1r^3)w'^2 = 0 \\ &a_1^2e^2 + b^2 - a_1^2 = 0 \\ &(-r^2 + 2a_1r)r'^2 + (r^2 - 2a_1r + b^2)v^2 = 0 \\ &(((b^2 - a_1^2)r^2)w')\sin(w) + a_1b^2er' = 0 \\ &er\cos(w) + r + a_1e^2 - a_1 = 0 \\ &x - r\cos(w) = 0 \\ &y - r\sin(w) = 0 \end{aligned}$$

The first polynomial is the result of example 3. Use the above method, we can also find a formula for the acceleration under condition $w'' = 0$.

$$(b^4r - 2a_1b^4)a^2 + (-r^7 + 6a_1r^6 + (-b^2 - 12a_1^2)r^5 + (4a_1b^2 + 8a_1^3)r^4 - 3a_1^2b^2r^3)w'^4 = 0$$

This formula is not given in example 3.

Example 4'. A particle moves on a circle with radius r. The angle between its velocity and acceleration is a constant.

(a) Find the velocity of the particle.

(b) Find the relation between the velocity and the inclination of the bar to the x-axis.

For (a), the hypotheses are

$$\cos^2(q) - \cot^2(q)(1 - \cos^2(q)) = 0$$

$$\begin{aligned}
&\cos^2(q)(x'^2+y'^2)(x''^2+y''^2)-(x'x''+y'y'')^2=0 \\
&x^2+y^2-r^2=0 \\
&v^2-x'^2-y'^2=0
\end{aligned} \tag{20}$$

To find the velocity actually means to find a formula of the velocity as a function of the time or the differential equation of the velocity. For this kind of problem algorithm 1 or the first step of algorithm 3 is enough. The non-degenerate components of (20) under the variable order $r < \cot(q) < v < \cos(q) < x < y$ with r and $\cot(q)$ as the constant parameter are

$$\begin{aligned}
&rv' \pm cot(q)v^2=0 \\
&(\cot^2(q)+1)\cos^2(q)-cot^2(q)=0 \\
&r^2x'^2+v^2x^2-r^2v^2=0 \\
&y^2+x^2-r^2=0
\end{aligned}$$

We have two solutions: $1/v-1/v_0=\pm\frac{t\cot(w)}{r}$. We cannot decide the sign by our method. For (b), let the inclination of the bar to the x-axis is w. The two groups of hypotheses are

$$\begin{aligned}
&\sin^2(w)+\cos^2(w)-1=0 \\
&x-r\cos(w)=0 \\
&y-r\sin(w)=0 \\
&\cos^2(q)-\cot^2(q)(1-\cos^2(q))=0 \\
&x^2+y^2-r^2=0 \\
&\cos^2(q)(x'^2+y'^2)(x''^2+y''^2)-(x'x''+y'y'')^2=0
\end{aligned} \tag{21}$$

$$\begin{aligned}
&v^2-x'^2-y'^2=0 \\
&\sin(w)'-\cos(w)w'=0
\end{aligned} \tag{22}$$

Applying theorem 1 in Part I to (21) under the following variable order: $r < \cot(q) < \cos(q) < \sin(w) < \cos(w) < x < y$ in the sense of d-pols, our program found the following ascending chain which represents the main component.

$$\begin{aligned}
&(cot^2(q)+1)\cos^2(q)-cot^2(q)=0 \\
&((\cos^2(q)-1)\sin^4(w)+(-2\cos^2(q)+2)\sin^2(w)+\cos^2(q)-1)\sin(w)''^2 \\
&+(((-2\cos^2(q)+2)\sin^3(w)+(2\cos^2(q)-2)\sin(w))\sin(w)'^2)\sin(w)'' \\
&+(-\sin^2(w)+\cos^2(q))\sin(w)'^4=0 \\
&\cos^2(w)+\sin^2(w)-1=0 \\
&x-r\cos(w)=0 \\
&y-r\sin(w)=0
\end{aligned} \tag{23}$$

The pseudo remainders of the d-pols in (22) and $(v^2-x'^2-y'^2)'$ w.r.t (23) are

$$\begin{aligned}
&w'\cos(w)-\sin(w)'=0 \\
&r^2\sin(w)'^2+v^2\sin^2(w)-v^2=0 \\
&((r^2\sin^2(w)-r^2)\sin(w)')\sin(w)''-r^2\sin(w)\sin(w)'^3
\end{aligned} \tag{24}$$

$$+ vv' \sin^4(w) - 2vv' \sin^2(w) + vv' = 0$$

Applying theorem 1 in Part I to (21), (23) and (24) in the sense of ordinary polynomial, our program produces four non-degenerate components:

$$\begin{aligned}
&v' \pm \cot(q) w' v = 0 \\
&w' r \pm v = 0 \\
&(\cot^2(q) + 1)\cos^2(q) - \cot^2(q) = 0 \\
&\sin(w)'^2 + w'^2 \sin^2(w) - w'^2 = 0 \\
&((r^2 \sin^2(w) - r^2)\sin(w)')\sin(w)'' - r^2 \sin(w)\sin(w)'^3 \\
&+ vv' \sin^4(w) - 2vv' \sin^2(w) + vv' = 0 \\
&w' \cos(w) - \sin(w)' = 0 \\
&y - r\sin(w) = 0 \\
&x - r\cos(w) = 0
\end{aligned}$$

The first element gives $v = \pm ce^{q\cot(w)}$ for a constants c, which is the second result of example 4.

Example 5'. The y projection of the velocity of a particle moving in a plane is a constant c. Find the relation among the acceleration, the velocity and the curvature radius of the orbit.

The hypotheses are

$$\begin{aligned}
&c' = 0 \\
&y' - c = 0
\end{aligned} \tag{25}$$

$$\begin{aligned}
&a^2 - x''^2 - y''^2 = 0 \\
&v^2 - x'^2 - y'^2 = 0 \\
&p^2 c^2 x''^2 - (y'^2 + x'^2)^3 = 0
\end{aligned} \tag{26}$$

(25) is an ascending chain obviously. The pseudo remainders of the d-pols in (26) w.r.t (25) are

$$\begin{aligned}
&c^2 p^2 x''^2 - x'^6 - 3c^2 x'^4 - 3c^4 x'^2 - c^6 = 0 \\
&x'^2 - v^2 + c^2 = 0 \\
&x'' - a = 0
\end{aligned} \tag{27}$$

Applying theorem 1 in Part I to (27) in the sense of ordinary polynomial under the variable order $c < p < v < a < x < x' < x''$, our program produces two main components:

$$\begin{aligned}
&cap \pm v^3 = 0 \\
&x'^2 - v^2 + c^2 = 0 \\
&x'' - a = 0
\end{aligned}$$

We have two solutions $cap = v^3$ and $cap = -v^3$ of which the later one is impossible.

5.2. Problems About the Motion Under a Central Force

Similarly as section 5.2 of Part I of this paper, we use a mixed coordinate system with the center of the force as the origin point. The following equations fit for all examples in this section.

$$N_2 = x''y - y''x = 0$$

$$\begin{aligned} h_1 &= r^2 - x^2 - y^2 = 0 \\ h_2 &= a^2 - x''^2 - y''^2 = 0 \\ h_3 &= h - y'x + x'y = 0 \\ h_4 &= yrw' - r'x + rx' = 0 \\ h_5 &= v^2 - x'^2 - y'^2 = 0 \end{aligned}$$

Example 6'. If a particle under a central attractive force describes an ellipse having its center at the center of the force. Find the relation between the force and the radius drawn from the particle to the force center.

We adopt a simplification, i.e., assuming the equation of the ellipse is $bx^2 + cy^2 - 1 = 0$ or $DLR(1, x^2, y^2) = 0$. Applying theorem 1 in Part I to N_2 and $DLR(1, x^2, y^2) = 0$, our program found the following ascending chain which represents the main component

$$\begin{aligned} &x'''x - x''x' = 0 \\ &y'x' - yx'' = 0 \end{aligned} \tag{28}$$

The pseudo remainders of h_1, h_1', h_2, and h_2' w.r.t (28) are

$$\begin{aligned} &y^2x''^2 + x''^2x^2 - x^2a^2 = 0 \\ &y^2x'' + x'^2x - r'r = 0 \\ &y^2x''^3 + x''^2x'^2x - x'x^2 - a'a = 0 \\ &y^2 + x^2 - r^2 = 0 \end{aligned} \tag{29}$$

The variables are under the following order: $a < a' < r < r' < x < x' < x'' < x''' < y < y' < y''$. The ascending chains representing the non-degenerate components of (29), (28), N_2 and $DLR(1, x^2, y^2)$ in the sense of ordinary polynomials are

$$\begin{aligned} &r'a - ra' = 0 \\ &x'^2 + xr - x'r'r^2 - x^3a + xr^2a = 0 \\ &x''r - xa = 0 \\ &x'''x - x''x' = 0 \\ &y^2 + x^2 - r^2 = 0 \\ &y'x' - yx'' = 0 \\ &xy'' - x''y = 0 \end{aligned}$$

$$\begin{aligned} &r'a - ra' = 0 \\ &x'^2 + xr - x'r'r^2 + x^3a - xr^2a = 0 \\ &x''r + xa = 0 \end{aligned}$$

$$x'''x - x''x' = 0$$
$$y^2 + x^2 - r^2 = 0$$
$$y'x' - yx'' = 0$$
$$xy'' - x''y = 0$$

The first d-pol of both ascending chains are $DLR(a, r) = a'r - r'a$, i.e., a is proportional to r by algorithm 2.

Example 7'. (Binet's formula) For a motion under a central force, Find the relation among a, h, w', and u, where $ur - 1 = 0$.

In this example, we cannot find a unique solution which is exactly the formula we want, for there exist relations among every three of the four quantities.

(a). The relation among h, w', and u. Applying theorem 1 in Part I to h_1, h_3, h_4, and $ur - 1$ under the variable order $h < u < h < r < x < y$, our program found the following ascending chain which represents the main component

$$u^2h - w' = 0$$
$$ur - 1 = 0$$
$$r^2x'^2 - 2rr'xx' + (r'^2 + w'^2r^2)x^2 - w'^2r^4 = 0$$
$$w'ry + rx' - r'x = 0$$

The first d-pol gives the well known formula among u, h, and w. Note that we do not need the fact of the central force to get this relation. So this relation is generally true.

(b). The relation among a, w', and h. Applying theorem 1 in Part I to h_1, h_2 h_3, and h_4 under the variable order $a < w < h < x < y < r$ with h as the constant parameter, our program found the following ascending chain which represents the main component

$$(4w'^2w'''^2 + (-12w'w''^2 + 16w'^5)w''' + 9w''^4 - 24w'^4w''^2 + 16w'^8)h - 16a^2w'^5 = 0$$
$$4w'^2x'^2 + 4w'w''xx' + (w''^2 + 4w'^4)x^2 - 4w'^3h = 0$$
$$2w'^2y + 2w'x' + w''x = 0$$
$$r^2 - x^2 - y^2 = 0$$

The first d-pol gives the relation among w', h, and a.

(c). The relation among a, u, and h. The ascending chains representing the non-degenerate components for h_1, h_2 h_3, h_4, and $ur - 1$ under the variable order $h < u < h < r < x < y$ with h as the constant parameter are

$$u^6h^2 + uu'' - 2u'^2 \pm au^3 = 0$$
$$ur - 1 = 0$$
$$r^4x'^2 - 2r^3r'xx' + (r^2r'^2 + h^2)x^2 - h^2r^2 = 0$$
$$hy + r^2x' - rr'x = 0$$

The first d-pols give the relations among u, h, and a.

(d). The relation among a, u, and w'. In this case the general form of algorithm 3 must be used. The condition is N_2. The pseudo remainders of h_1, h_1', h_1'', h_2, $ur-1$, $(ur-1)'$,$(ur-1)''$, h_4, and h_4' w.r.t N_2 are

$$\begin{aligned}
&ru - 1 = 0\\
&r'u + u'r = 0\\
&r''u + 2r'u' + ru'' = 0\\
&y^2 + x^2 - r^2 = 0\\
&y'y + x'x - r'r = 0\\
&y'^2x + y^2x'' + x''x^2 + x'^2x - x(r''r + r'^2) = 0\\
&y^2x''^2 + x''^2x^2 - x^2a^2 = 0\\
&yrw' - r'x + rx' = 0\\
&yrw'' + (y'r + r'y)w' - r''x + rx'' = 0
\end{aligned} \tag{30}$$

Now apply theorem 1 in Part I to (30) in the sense of ordinary polynomials under the variable order $a < a' < u < u' < u'' < r < r' < r'' < x < x' < x'' < y < y'$. Our program produces two ascending chains which represent the main components

$$\begin{aligned}
&u^2w'^2 + uu'' - 2u'^2 \pm au^3 = 0\\
&ru - 1 = 0\\
&r'u + ru' = 0\\
&r''u + 2r'u' + ru'' = 0\\
&c_0x^2 + c_1 = 0\\
&c_2x' + c_3 = 0\\
&x''r - xa = 0\\
&yrw' - r'x + rx' = 0\\
&yy' + xx' - rr' = 0
\end{aligned} \tag{31}$$

where c_0, c_1, c_2, and c_3 are some d-pols. The first polynomial of (31) is equivalent to Binet's formula $h^2u^2(d^2u/dw^2 + u) = a$ under the relation $w' = hu^2$.

Example 8'. For a motion under a central force, find the relation among f, h, and the distance from the force center to the tangent line of the orbit.

We have

$$h_6 = (xy' - yx')^2 - p^2(x'^2 + y'^2) = 0$$

where p stands for the distance from the force center to the tangent of the orbit. The pseudo remainders of h_1, h_1', h_2, h_3, h_6, and h_6' w.r.t N_2 are

$$\begin{aligned}
&x^2 + y^2 - r^2 = 0\\
&xx' + yy' - rr' = 0\\
&x''^2y^2 + x^2x''^2 - f^2x^2 = 0\\
&xy' - yx' - h = 0
\end{aligned} \tag{32}$$

$$p^2y'^2 + p^2x'^2 - h^2 = 0$$
$$pp'xy'^2 + p^2x''yy' + p^2xx'x'' + pp'xx'^2 = 0$$

Applying theorem 1 in Part I to (32) in the sense of ordinary polynomials of under the variable order $a < p < p' < r < r' < x < x' < x'' < y < y'$, our program produces the following ascending chain which represents the main component

$$\begin{aligned}
&(p^4a^2 - p'^2h^2)r^2 - p^6a^2 = 0\\
&p^3ar' + p'h^2 = 0\\
&p^2r^2x'^2 - 2p^2rr'xx' + h^2x^2 - p^2h^2 = 0\\
&rx'' - ax = 0\\
&hy + r^2x' - rr'x = 0\\
&xy' - x'y - h = 0
\end{aligned} \tag{33}$$

We have two relations for a and p, the second is equivalent to the original result:$a = 1/2 \cdot h^2\, dq^{-2}/\, dr$.

Example 9'. If the orbit of a particle moving under a central force is $r^2 = c^2\cos(2w)$. Find the relation among a, r, and h.

$r^2 = c^2\cos(2w)$ is equivalent to

$$h_7 = y^4 + (2x^2 + c^2)y^2 + x^4 - c^2x^2 = 0$$

Applying theorem 1 in Part I to N_2 and h_7 in the sense of d-pol, our program produces the ascending chain which represents the main component

$$\begin{aligned}
&(64x^7 - 48c^2x^5 - 15c^4x^3 - c^6x)x''^2 + ((-48c^2x^4 - 30c^4x^2 - 3c^6)x'^2)x'' + 9c^4xx'^4 = 0\\
&((24x^4 + 11c^2x^2 + c^4)x'' - 3c^2xx'^2)y^2 + (8x^6 - 7c^2x^4 - c^4x^2)x'' + 3c^2x^3x'^2 = 0
\end{aligned} \tag{34}$$

The pseudo remainders of h_1, h_2, and h_3 w.r.t (34) are

$$\begin{aligned}
&(16x^6 + (-24r^2 + 18c^2)x^4 + (-11c^2r^2 + 2c^4)x^2 - c^4r^2)x'' + (-6c^2x^3 + 3c^2r^2x)x'^2 = 0\\
&c_1y + c_2 = 0\\
&c_3x'' + c_4 = 0
\end{aligned} \tag{35}$$

Applying theorem 1 in Part I to (35), h_1, and h_7 in the sense of ordinary polynomial under the variable order $c < a < r < x < x' < x'' < y$, our program produces two ascending chains which represent the main components

$$\begin{aligned}
&ar^7 \pm 3c^4h^2 = 0\\
&2c^2x^2 - r^4 - c^2r^2 = 0\\
&c_5x'^2 + c_6 = 0\\
&(16x^6 + (-24r^2 + 18c^2)x^4 + (-11c^2r^2 + 2c^4)x^2 - c^4r^2)x'' + (-6c^2x^3 + 3c^2r^2x)x'^2 = 0\\
&c_1y + c_2 = 0
\end{aligned} \tag{36}$$

where c_i ($i = 1, ..., 6$) are some d-pols. We have two results: $ar^7 - 3c^4h^2 = 0$ and $ar^7 + 3c^4h^2 = 0$ of which the later is impossible.

Example 10'. The magnitude of the velocity of a particle under a central force is inversely proportional to the distance from the force center to the particle. Find the orbit of the particle.

Let $c = vr$ be the proportional coefficient. We have equivalently:

$$h_8 = c^2 - (x^2 + y^2)(x'^2 + y'^2) = 0$$

This example can be solved by algorithm 1. The ascending chain representing the main component of N_2, h_1, h_3, and h_6 under the variable order: $r < x < y$ is

$$\begin{aligned} &r^2 r'^2 + h^2 - c^2 = 0 \\ &r^4 x'^2 - 2r^3 r' x x' + c^2 x^2 + r^4 r'^2 - c^2 r^2 = 0 \\ &hy + r^2 x' - r r' x = 0 \end{aligned} \qquad (37)$$

The first d-pol of (37) gives:

$$r^2 = r_0^2 + 2\sqrt{c^2 - h^2} \cdot t$$

where r_0 is the initial value of r. Use $h = r^2 w'$, we can get the original formula of example 10:

$$h^2 r'^2 - (c^2 - h^2) r^2 w'^2 = 0$$

or equivalently:

$$\ln(r/r_0) = \pm \frac{\sqrt{c^2 - h^2}}{h} \cdot w$$

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