A COLLECTION OF 120 COMPUTER SOLVED GEOMETRY PROBLEMS IN MECHANICAL FORMULA DERIVATION*

Shang-Ching Chou and Xiao-Shan Gao†

Department of Computer Sciences
The University of Texas at Austin
Austin, Texas 78712-1188

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ABSTRACT

This is a collection of 120 geometric problems mechanically solved by a program based on the methods introduced by us. Researchers can use this collection to experiment with their methods/programs similar to ours. It consists of two parts: the exact specification of the input to our program and a collection of 120 examples. A typical example consists of an informal description of the geometric problem, the input to the program which is the exact specification of the problem, the result of the problem, and a diagram.

Keywords: Elementary geometry, formula derivation, Gröbner bases, Ritt–Wu's method, Heron's formula, Brahmagupta's Formula, locus equations, Peaucellier's linkage.

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†On leave from Institute of Systems Science, Academia Sinica, Beijing.

1. Introduction

We have implemented a program to derive geometric formulas mechanically based on the methods described in the paper [1]. This is a collection of the geometric problems mechanically solved by our program. It is a supplement to that paper [1]. Researchers can use this collection to experiment with their methods/programs similar to ours. It consists of two parts: the exact specification of the input to our program and a collection of 120 examples. A typical example consists of an informal description of the geometric problem, the input to the program which is the exact specification of the problem, the result of the problem, and a diagram. Most parts of the collection of 120 examples are produced by computers, including the text file in the TEX form and insertion of diagrams in the PostScript form.

The Specificaiton of Inputs

An input to our program is an algebraic specification of a given geometric problem. It is in LISP forms. Eventually, the program transforms it to a set of polynomial equations and a set of polynomial inequations. A typical example of the input can be as follows (see Example 9):

$((x_1\ x_3\ x_4\ u_1\ u_2)\ (u_3\ x_2\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10})$
$(D\ (0\ 0)\ B\ (x_1\ 0)\ C\ (x_2\ 0)\ A\ (x_3\ x_4)\ E\ (x_5\ x_6)\ F\ (x_7\ x_8)\ P\ (x_9\ x_{10}))$
(collinear $D\ P\ A$)
(collinear $B\ F\ A$)
(collinear $C\ E\ A$)
(collinear $C\ P\ F$)
(collinear $B\ P\ E$)
(x-ratio $u_1\ A\ F\ F\ B$)
(x-ratio $u_2\ A\ E\ E\ C$)
(x-ratio $u_3\ A\ P\ P\ D$)
non-deg u_3 (pp– $x_2\ x_3$)).

The first element is a list of parameters; the second is a list of dependent variables in their increasing order. The order of dependent variables is important for the methods based on Ritt-Wu's method and the Gröbner basis method. The program tries to find the relation set among the parameters and the first variable of the list of dependent variables (for the definition of the relation set see [1]). In this particular example, it is the relation set among x_1, x_3, x_4, u_1, u_2 and u_3. The third element of the input list is the assignment of coordinates to the points involved in the problem. The elements after the third and before the element "non-deg" (if it occurs) are geometric conditions or polynomials in the parameters and dependent variables. They are eventually transformed to the set of polynomial equations of the problem. Similarly, the elements after "non-deg" represent inequations. In this example, the inequation set of the problem is $\{u_3 \neq 0,\ x_2 - x_3 \neq 0\}$. Now we explain each other element in the input list and its corresponding polynomials.

Let P_i be points with the coordiantes (x_i, y_i). Other elements in the input are one of the following forms:

S1. (collinear $P_1\ P_2\ P_3$). Points P_1, P_2, and P_3 are collinear. Its corresponding polynomial equation is:

$$(y_3 - y_1)(x_2 - x_1) - (y_2 - y_1)(x_3 - x_1) = 0.$$

S2. (parallel P_1 P_2 P_3 P_4). Line P_1P_2 is parallel to line P_3P_4. Its corresponding polynomial equation is:

$$(y_4 - y_3)(x_2 - x_1) - (y_2 - y_1)(x_4 - x_3) = 0.$$

S3. (perpendicular P_1 P_2 P_3 P_4). Line P_1P_2 is perpendicular to line P_3P_4. Its corresponding polynomial equation is:

$$(y_4 - y_3)(y_2 - y_1) + (x_4 - x_3)(x_2 - x_1) = 0.$$

S4. (cocyclic P_1 P_2 P_3 P_4). Four points P_1, P_2, P_3, and P_4 are on the same circle. Its corresponding polynomial equation is:

$$\begin{vmatrix} x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \\ x_4^2 + y_4^2 & x_4 & y_4 & 1 \end{vmatrix} = 0$$

S5. (mid-x P_1 P_2 P_3). $2x_1 - x_2 - x_3 = 0$.

S6. (mid-y P_1 P_2 P_3). $2y_1 - y_2 - y_3 = 0$.

S7. (distance P_1 P_2 d). The distance between points P_1 and P_2 is equal to d. Its corresponding polynomial equation is $d^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2 = 0$.

S8 (sq-distance P_1 P_2) is the quantity of the square of the distance between P_1 and P_2. Its corresponding polynomial is $(x_1 - x_2)^2 + (y_1 - y_2)^2$.

S9. (eqdistant P_1 P_2 P_3 P_4). The segments P_1P_2 and P_3P_4 are congruent. Its corresponding polynomial equation is:

$$(y_4 - y_3)^2 + (x_4 - x_3)^2 - (y_2 - y_1)^2 - (x_2 - x_1)^2 = 0.$$

S10. (point-line-dis P_1 P_2 P_3 d). The distance from the point P_1 to the line P_2P_3 is d. Its corresponding polynomial equation is equivalent to the following equation:

$$d^2 - \frac{(\text{collinear } P_2\ P_3\ P_1)^2}{(\text{sq-distance } P_2\ P_3)} = 0.$$

S11. (eqangle P_1 P_2 P_3 P_4). $\tan(P_2P_1P_3\ ) - \tan(P_3P_1P_4) = 0$ or:

$$\begin{aligned}&(\text{collinear } P_1\ P_2\ P_3)(\text{perpendicular } P_1\ P_3\ P_1\ P_4)\\ &\quad -(\text{collinear } P_1\ P_3\ P_4)(\text{perpendicular } P_1\ P_2\ P_1\ P_3) = 0.\end{aligned}$$

S12. (eqtangent P_1 P_2 P_3 P_4 P_5 P_6). $\tan(P_1P_2P_3) = \tan(P_4P_5P_6)$ or:

$$\begin{aligned}&(\text{collinear } P_2\ P_1\ P_3)(\text{perpendicular } P_5\ P_4\ P_5\ P_6)\\ &\quad -(\text{collinear } P_5\ P_4\ P_6)(\text{perpendicular } P_2\ P_1\ P_2\ P_3) = 0.\end{aligned}$$

S13. (x-ratio r P_1 P_2 P_3 P_4). Its corresponding polynomial equation is equivalent to the equation $r = (x_1 - x_2)/(x_3 - x_4)$.

S14. (y-ratio r P_1 P_2 P_3 P_4). Its corresponding polynomial equation is equivalent to the equation $r = (y_1 - y_2)/(y_3 - y_4)$.

S15. (c–ratio r P_1 P_2 P_3 P_4). r is the cross ratio (P_1P_2, P_3P_4). Its corresponding polynomial equation is equivalent to the equation

$$r = \frac{(x_3 - x_1)(x_2 - x_4)}{(x_4 - x_1)(x_2 - x_3)}.$$

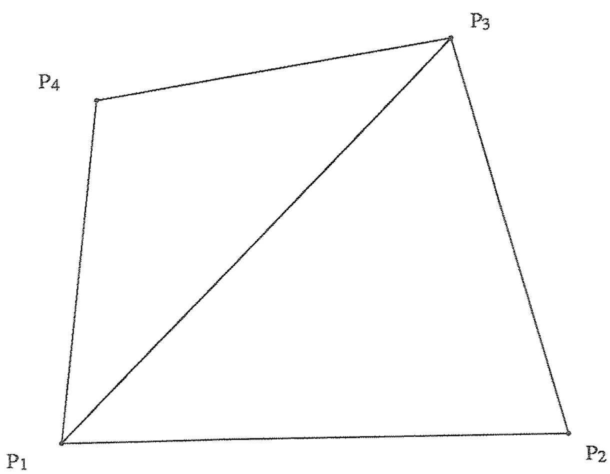

Fig. 1.1

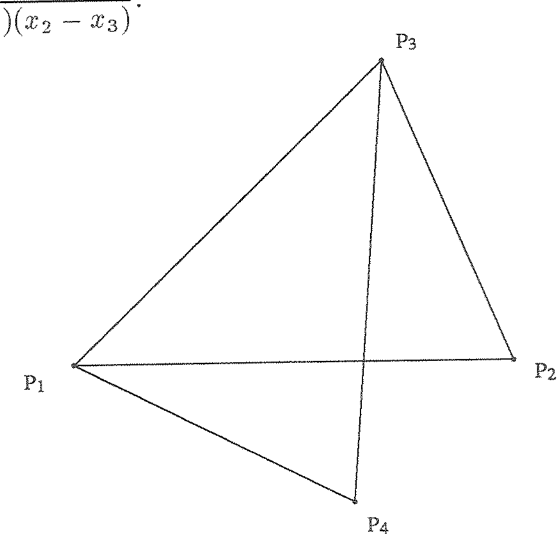

Fig. 1.2

S16. (area k P_1 ... P_n). k is the sum of the signed areas of oriented triangles $P_1P_2P_3$, $P_1P_3P_4$, ..., $P_1P_{n-1}P_n$ or:

$$2k = \sum_{i=2}^{n-1} \text{area}(P_1P_iP_{i+1}),$$

where $\text{area}(P_iP_jP_k) = (x_j - x_i)y_k + (-y_j + y_i)x_k + x_iy_j - y_ix_j$. The polygon $P_1P_2 \cdots P_n$ is not necessarily convex. Our definition of the area for a polygon is more general (see Example 5.2 in [1]: Brahmagupta's Formula).

S17. (pp+ p_1 p_2) means the sum of polynomials p_1 and p_2.

S18. (pp− p_1 p_2) means the difference of polynomials p_1 and p_2.

S19. (pp∗ p_1 p_2) means the product of polynomials p_1 and p_2.

S20. (pp∧ p_1 n) means the n-th power of polynomial p_1.

References

[1] S. C. Chou and X. S. Gao, "Mechanical Formula Derivation in Elementary Geometries", TR-89-21, Computer Sciences Department, The University of Texas at Austin, August 1989.

2. Examples Mechanically Solved

Example 1. If O is any point in the plane of triangle ABC, find the relation among the areas of triangles ABC, OBC, OCA, and OAB.

$((x_1\ x_2\ u_1\ u_2\ u_3)\ (u_4\ x_3\ x_4\ x_5)$
$(O\ (0\ 0)\ A\ (x_1\ 0)\ B\ (x_2\ x_3)\ C\ (x_4\ x_5))$
(area u_1 O B C)
(area u_2 O C A)
(area u_3 O A B)
(area u_4 A B C))

The result is: $u_4 - u_3 - u_2 - u_1 = 0$.

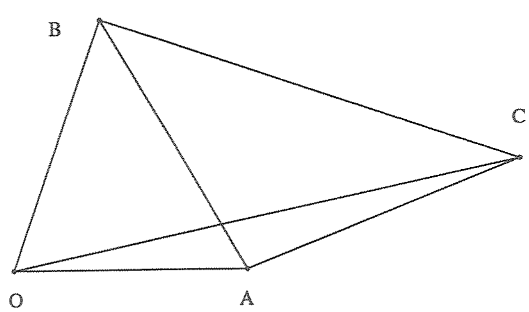

Fig. 1

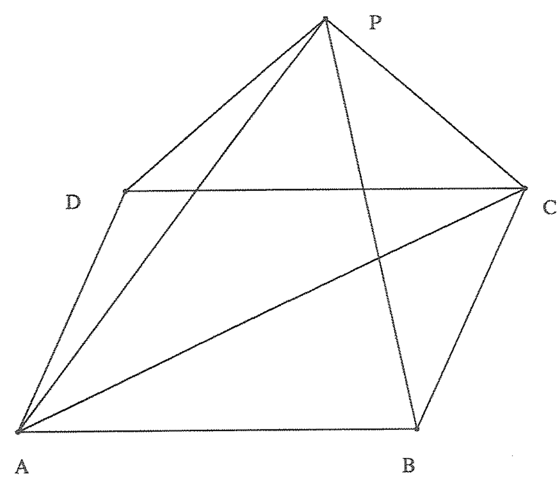

Fig. 2

Example 2. If P is any point in the plane of the parallelogram $ABCD$, find the relation among the areas of triangles PAB, PCD, and ABC.

$((x_1\ x_2\ x_5\ u_1\ u_2)\ (u_3\ x_3\ x_4\ x_6)$
$(A\ (0\ 0)\ B\ (x_1\ 0)\ C\ (x_2\ x_3)\ D\ (x_4\ x_3)\ P\ (x_5\ x_6))$
(parallel A D B C)
(area u_1 P A B)
(area u_2 P C D)
(area u_3 A B C)
non-deg x_3)

The result is: $u_3 - u_2 - u_1 = 0$.

Example 3. (Menelaus' Theorem) If the sides AB, BC, and CA of a triangle ABC are cut by a transversal in the points D, E, and F respectively, find the relation among AD, DB, BE, and EC, CF, and FA. *

((x_1 x_3 x_4 u_1 u_2) (u_3 x_2 x_5 x_6 x_7 x_8)
 (A (0 0) B (x_1 0) C (x_3 x_4) D (x_2 0) F (x_5 x_6) E (x_7 x_8))
 (collinear A F C)
 (collinear D E F)
 (collinear B E C)
 (x-ratio u_1 A D D B)
 (x-ratio u_2 B E E C)
 (x-ratio u_3 C F F A))

$$((x_1\ x_3\ x_4\ u_1\ u_2)\ (u_3\ x_2\ x_5\ x_6\ x_7\ x_8)$$

The result is: $u_1u_2u_3 + 1 = 0$.

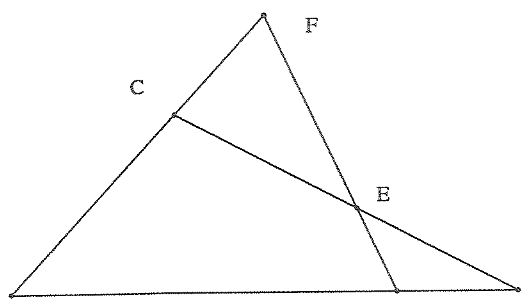

Fig. 3

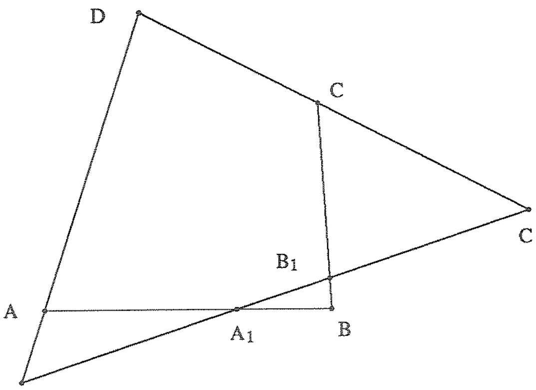

Fig. 4

Example 4. (Menelaus' Theorem for a Quadrilateral) If the sides AB, BC, CD, and DA of a quadrilateral $ABCD$ are cut by a transversal in the points A_1, B_1, C_1 and D_1 respectively, Find the relation among the ratios AA_1/A_1B, BB_1/B_1C, CC_1/C_1D, and DD_1/D_1A.

$((u_1\ u_2\ u_3\ u_4\ r_1\ r_2\ r_3)\ (r_4\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9)$
$(A\ (0\ 0)\ B\ (u_1\ 0)\ C\ (u_2\ u_3)\ D\ (u_4\ x_2)\ A_1\ (x_3\ 0)\ B_1\ (x_4\ x_5)\ C_1\ (x_6\ x_7)\ D_1\ (x_8\ x_9))$
(collinear $A\ D_1\ D$)
(collinear $B\ B_1\ C$)
(collinear $C\ D\ C_1$)
(collinear $A_1\ B_1\ C_1$)
(collinear $A_1\ B_1\ D_1$)
(x-ratio $r_1\ A\ A_1\ A_1\ B$)
(x-ratio $r_2\ B\ B_1\ B_1\ C$)
(x-ratio $r_3\ C\ C_1\ C_1\ D$)
(x-ratio $r_4\ D\ D_1\ D_1\ A$))

The result is: $r_1r_2r_3r_4 - 1 = 0$.

* Here, we shall find the relation among the ratios: AD/DB, BE/EC, and CF/FA instead of the relation among AD, DB, BE, EC, CF, and FA.

Example 5. (Menelaus' Theorem for a Pentagon) If the sides P_1P_2, P_2P_3, P_3P_4, P_4P_5, and P_5P_1 of a polygon $P_1P_2P_3P_4P_5$ are cut by a transversal in the points Q_1, Q_2, Q_3, Q_4, and Q_5 respectively, find the relation among the ratios P_1Q_1/Q_1P_2, P_2Q_2/Q_2P_3, P_3Q_3/Q_3P_4, P_4Q_4/Q_4P_5, and P_5Q_5/Q_5P_1.

$((x_2\ x_3\ y_3\ x_4\ x_5\ u_1\ u_2\ u_3\ u_4)\ (u_5\ y_4\ y_5\ z_1\ z_2\ w_2\ z_3\ w_3\ z_4\ w_4\ z_5\ w_5)$
$(P_1\ (0\ 0)\ P_2\ (x_2\ 0)\ P_3\ (x_3\ y_3)\ P_4\ (x_4\ y_4)\ P_5\ (x_5\ y_5)\ Q_1\ (z_1\ 0)\ Q_2\ (z_2\ w_2)\ Q_3\ (z_3\ w_3)$
$Q_4\ (z_4\ w_4)\ Q_5\ (z_5\ w_5))$
(collinear $Q_2\ P_2\ P_3$)
(collinear $Q_3\ P_3\ P_4$)
(collinear $Q_4\ P_4\ P_5$)
(collinear $Q_5\ P_5\ P_1$)
(collinear $Q_1\ Q_2\ Q_3$)
(collinear $Q_1\ Q_2\ Q_4$)
(collinear $Q_1\ Q_2\ Q_5$)
(x-ratio $u_1\ P_1\ Q_1\ Q_1\ P_2$)
(x-ratio $u_2\ P_2\ Q_2\ Q_2\ P_3$)
(x-ratio $u_3\ P_3\ Q_3\ Q_3\ P_4$)
(x-ratio $u_4\ P_4\ Q_4\ Q_4\ P_5$)
(x-ratio $u_5\ P_5\ Q_5\ Q_5\ P_1$))

The result is: $u_1u_2u_3u_4u_5 + 1 = 0$.

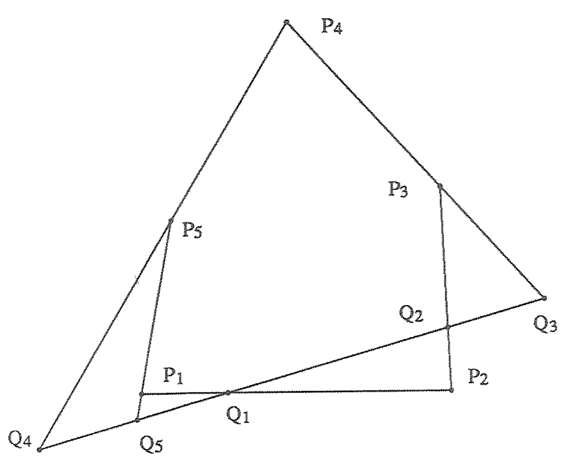

Fig. 5

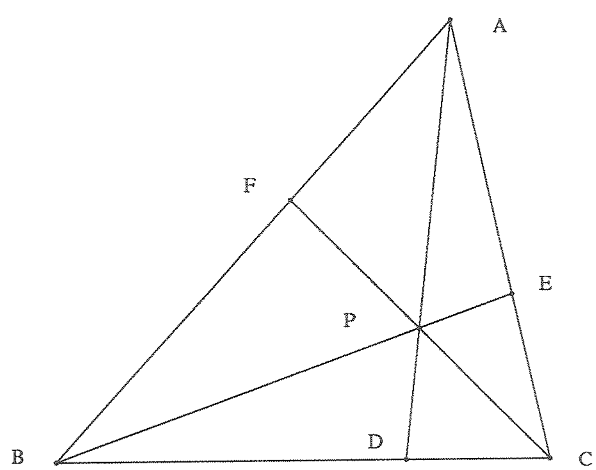

Fig. 6

Example 6. (Ceva's Theorem) Let D, E, and F be points on BC, AC, AB of a triangle ABC. If AD, BE, CF are concurrent in P find the relation among the ratios BD/DC, CE/EA, and AF/FB.

$((x_1\ x_3\ x_4\ u_1\ u_2)\ (u_3\ x_2\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10})$
$(D\ (0\ 0)\ B\ (x_1\ 0)\ C\ (x_2\ 0)\ A\ (x_3\ x_4)\ E\ (x_5\ x_6)\ F\ (x_7\ x_8)\ P\ (x_9\ x_{10}))$
(collinear $D\ P\ A$)
(collinear $B\ F\ A$)
(collinear $C\ E\ A$)
(collinear $C\ P\ F$)

(collinear $B\ P\ E$)
(x-ratio $u_1\ B\ D\ D\ C$)
(x-ratio $u_2\ C\ E\ E\ A$)
(x-ratio $u_3\ A\ F\ F\ B$))

The result is: $u_1 u_2 u_3 - 1 = 0$.

Example 7. In example 6, find the relations among PD/AD, PE/BE, and PF/CF.

(($x_1\ x_4\ x_3\ u_1\ u_2$) ($u_3\ x_2\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}$)
(D (0 0) B (x_1 0) C (x_2 0) A ($x_3\ x_4$) E ($x_5\ x_6$) F ($x_7\ x_8$) P ($x_9\ x_{10}$))
(collinear $D\ P\ A$)
(collinear $B\ F\ A$)
(collinear $C\ E\ A$)
(collinear $C\ P\ F$)
(collinear $B\ P\ E$)
(x-ratio $u_1\ P\ D\ A\ D$)
(x-ratio $u_2\ P\ E\ B\ E$)
(x-ratio $u_3\ P\ F\ C\ F$))

The result is: $u_3 + u_2 + u_1 - 1 = 0$.

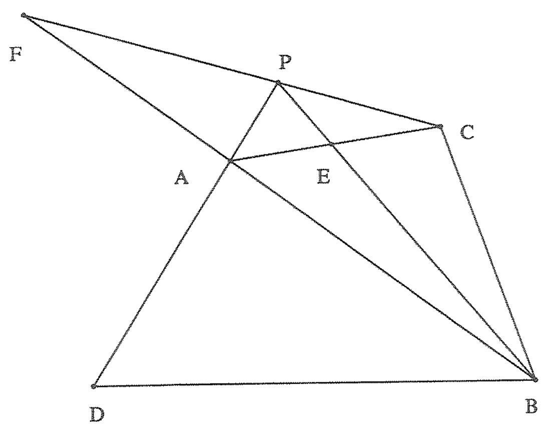

Fig. 7

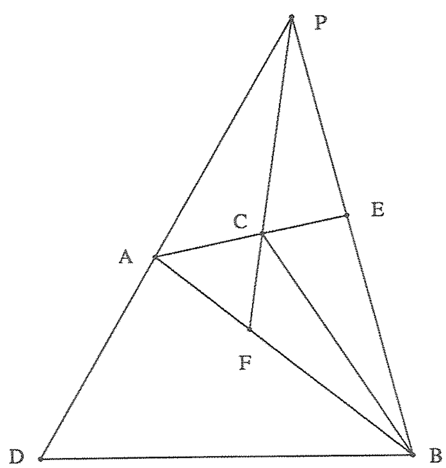

Fig. 8

Example 8. In example 6, find the relations among AP/AD, BP/BE, and CP/CF.

(($x_1\ x_3\ x_4\ u_1\ u_2$) ($u_3\ x_2\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}$)
(D (0 0) B (x_1 0) C (x_2 0) A ($x_3\ x_4$) E ($x_5\ x_6$) F ($x_7\ x_8$) P ($x_9\ x_{10}$))
(collinear $D\ P\ A$)
(collinear $B\ F\ A$)
(collinear $C\ E\ A$)
(collinear $C\ P\ F$)
(collinear $B\ P\ E$)
(x-ratio $u_1\ A\ P\ A\ D$)
(x-ratio $u_2\ B\ P\ B\ E$)

(x-ratio u_3 C P C F))

The result is: $u_3 + u_2 + u_1 - 2 = 0$.

Example 9. In example 6, the the relations among AF/FB, AE/EC, and AP/PD.

((x_1 x_3 x_4 u_1 u_2) (u_3 x_2 x_5 x_6 x_7 x_8 x_9 x_{10})
(D (0 0) B (x_1 0) C (x_2 0) A (x_3 x_4) E (x_5 x_6) F (x_7 x_8) P (x_9 x_{10}))
(collinear D P A)
(collinear B F A)
(collinear C E A)
(collinear C P F)
(collinear B P E)
(x-ratio u_1 A F F B)
(x-ratio u_2 A E E C)
(x-ratio u_3 A P P D)
non-deg u_3 (pp- x_2 x_3))

The result is: $u_3 - u_2 - u_1 = 0$.

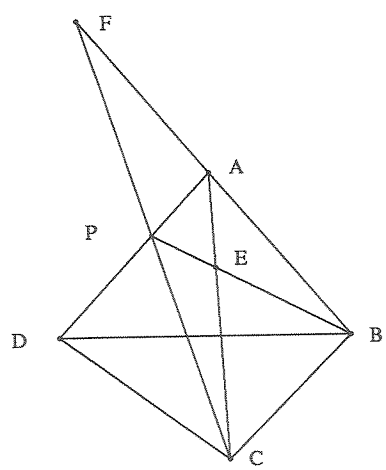

Fig. 9

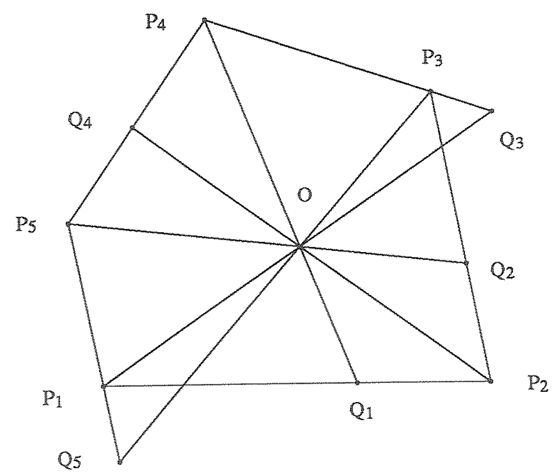

Fig. 10

Example 10. (Ceva's Theorem for a Pentagon) As Figure 10. Find the relation among the ratios: P_1Q_1/Q_1P_2, P_2Q_2/Q_2P_3, P_3Q_3/Q_3P_4, P_4Q_4/Q_4P_5, and P_5Q_5/Q_5P_1.

((x_1 x_2 y_2 x_3 x_4 u_1 u_2 u_3 u_4) (u_5 y_3 y_4 x_5 y_5 z_1 w_1 z_2 w_2 z_3 z_4 w_4 z_5 w_5)
(O (0 0) P_1 (x_1 0) P_2 (x_2 y_2) P_3 (x_3 y_3) P_4 (x_4 y_4) P_5 (x_5 y_5) Q_1 (z_1 w_1) Q_2 (z_2 w_2)
Q_3 (z_3 0) Q_4 (z_4 w_4) Q_5 (z_5 w_5))
(collinear P_1 Q_1 P_2)
(collinear P_2 Q_2 P_3)
(collinear P_3 Q_3 P_4)
(collinear P_4 Q_4 P_5)
(collinear P_5 Q_5 P_1)
(collinear O P_2 Q_4)
(collinear O P_3 Q_5)

(collinear O P_4 Q_1)
(collinear O P_5 Q_2)
(x-ratio u_1 P_1 Q_1 Q_1 P_2)
(x-ratio u_2 P_2 Q_2 Q_2 P_3)
(x-ratio u_3 P_3 Q_3 Q_3 P_4)
(x-ratio u_4 P_4 Q_4 Q_4 P_5)
(x-ratio u_5 P_5 Q_5 Q_5 P_1)
non-deg (pp− x_5 x_4))

The result is: $u_1u_2u_3u_4u_5 - 1 = 0$.

Example 11. If AD, BE, and CF are any three concurrent ceva lines of a triangle ABC, and if D_1 denotes the point of intersection of BC and FE, find the relation among BD, DC, BD_1 and D_1C.

((x_1 x_2 x_3 x_6 u_1) (u_2 x_4 x_5 x_7 x_8 x_9 x_{10} x_{11})
(P (0 x_1) B (x_2 0) C (x_3 0) D (x_4 0) D_1 (x_5 0) A (x_6 x_7) F (x_8 x_9) E (x_{10} x_{11}))
(collinear A B F)
(collinear A E C)
(collinear A P D)
(collinear B P E)
(collinear C P F)
(collinear E F D_1)
(x-ratio u_1 B D D C)
(x-ratio u_2 B D_1 D_1 C))

The result is: $u_2 + u_1 = 0$.

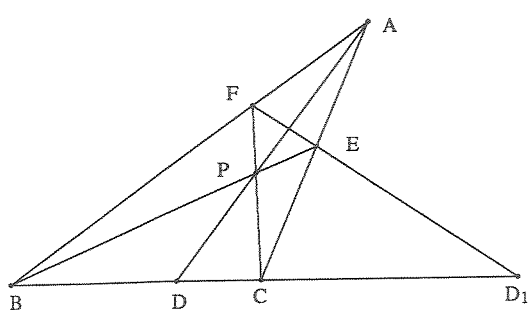

Fig. 11

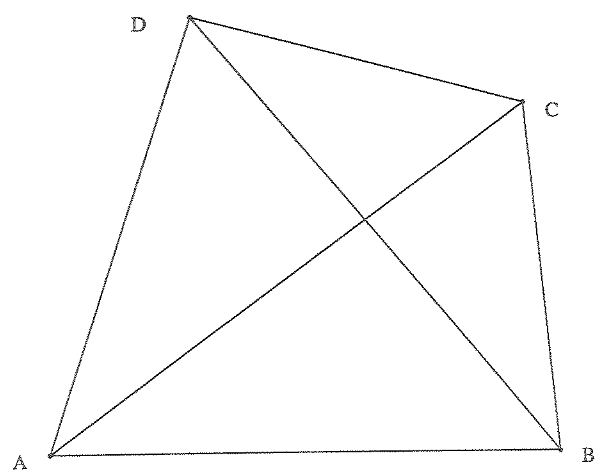

Fig. 12

Example 12. Find the relation among the six distances of the four points A, B, C, and D in a plane.

((u_1 u_2 u_3 u_4 u_5) (u_6 x_1 x_2 x_3 x_4)
(A (0 0) B (u_1 0) C (x_1 x_2) D (x_3 x_4))

(distance A C u_2)
(distance D A u_3)
(distance B C u_4)
(distance B D u_5)
(distance C D u_6))

The result is: $u_1^2u_6^4 + ((u_4^2 - u_2^2 - u_1^2)u_5^2 + (-u_3^2 - u_1^2)u_4^2 + (u_2^2 - u_1^2)u_3^2 - u_1^2u_2^2 + u_1^4)u_6^2 + u_2^2u_5^4 + ((-u_3^2 - u_2^2)u_4^2 + (-u_2^2 + u_1^2)u_3^2 + u_2^4 - u_1^2u_2^2)u_5^2 + u_3^2u_4^4 + (u_3^4 + (-u_2^2 - u_1^2)u_3^2 + u_1^2u_2^2)u_4^2 = 0.$

Example 13. Determine the area of a planar pentagon $A_0A_1A_2A_3A_4$ in terms of the area of five triangles with vertices taken from A_0, A_1, A_2, A_3, and A_4.

((x_1 x_3 u_1 u_2 u_3 u_4 u_5) (u_6 x_4 x_5 x_6 x_7 x_8)
(A_0 (0 0) A_1 (x_1 0) A_2 (x_3 x_4) A_3 (x_5 x_6) A_4 (x_7 x_8))
(area u_1 A_0 A_1 A_2)
(area u_2 A_1 A_2 A_3)
(area u_3 A_2 A_3 A_4)
(area u_4 A_3 A_4 A_0)
(area u_5 A_4 A_0 A_1)
(area u_6 A_0 A_1 A_2 A_3 A_4))

The result is: $u_6^2 + (-u_5 - u_4 - u_3 - u_2 - u_1)u_6 + (u_4 + u_1)u_5 + u_3u_4 + u_2u_3 + u_1u_2 = 0.$

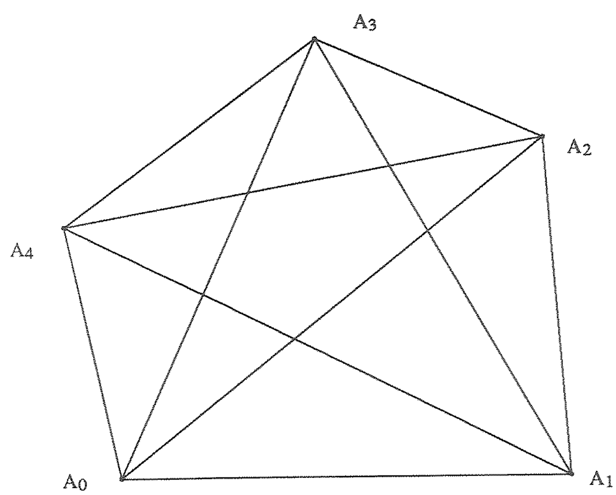

Fig. 13

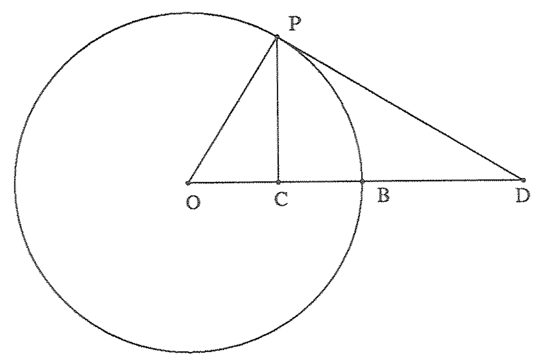

Fig. 14

Example 14. Point D is outside the circle and point C is the intersection of the chord of contact of the tangents from D to the circle of inversion and the diametrical line OD. Find the relation among OC, OD, and the radius of the circle.

((u_1 u_2) (u_3 x_1 x_2 x_3 x_4)
(O (0 0) B (x_1 0) P (x_2 x_3) D (x_4 0) C (x_2 0))
(eqdistant P O B O)
(perpendicular O P P D)
(pp– u_1 x_2)
(pp– u_2 x_4)

(pp– u_3 x_1))

The result is: $u_3^2 - u_1 u_2 = 0$.

Example 15. The same as example 14. Find the cross ratio (AB, CD).

((x_1 x_2) (u_1 x_3 x_4 x_5)
(O (0 0) B (x_1 0) P (x_2 x_3) D (x_4 0) C (x_2 0) A (x_5 0))
(eqdistant P O B O)
(perpendicular O P P D)
(pp+ x_5 x_1)
(c–ratio u_1 A B C D))

The result is: $u_1 + 1 = 0$.

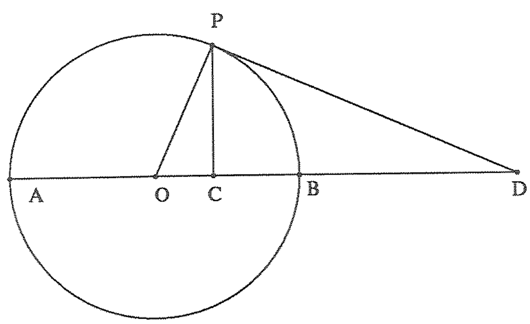

Fig. 15

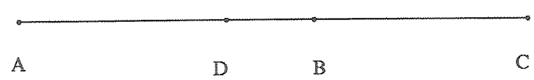

Fig. 16

Example 16. A, B, C, and D are four collinear points. If $(AB, CD) = -1$, find the relation among AB, AC, and AD.

((x_1 u_1 u_2) (u_3 x_2 x_3 x_4)
(A (x_1 0) B (x_2 0) C (x_3 0) D (x_4 0))
(c–ratio –1 A B C D)
(pp– u_1 (pp– x_2 x_1))
(pp– u_2 (pp– x_3 x_1))
(pp– u_3 (pp– x_4 x_1)))

The result is: $(2u_2 - u_1)u_3 - u_1 u_2 = 0$.

Example 17. (The Inverse of Example 16) A, B, C, and D are four collinear points. If $2/AB = 1/AC + 1/AD$, find cross ratio (AB, CD).

((x_1 u_1 u_2) (r u_3 x_2 x_3 x_4)
(A (x_1 0) B (x_2 0) C (x_3 0) D (x_4 0))
(pp– u_1 (pp– x_2 x_1))

(pp− u_2 (pp− x_3 x_1))
(pp− u_3 (pp− x_4 x_1))
(pp− (pp* (pp− (pp* 2 u_2) u_1) u_3) (pp* u_1 u_2))
(c–ratio r A B C D))

The result is: $r + 1 = 0$.

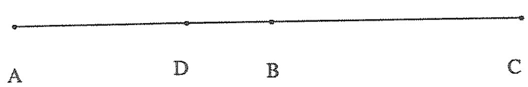

Fig. 17

Fig. 18

Example 18. A, B, C,and D are four collinear points. If $(AB, CD) = -1$, find the relation among OB, OC, and OD where O is the midpoint of AB.

((x_1 u_1 u_2) (u_3 x_2 x_3 x_4 x_5)
(A (x_1 0) B (x_2 0) C (x_3 0) D (x_4 0) O (x_5 0))
(c–ratio −1 A B C D)
(mid-x O A B)
(pp− u_1 (pp− x_2 x_5))
(pp− u_2 (pp− x_3 x_5))
(pp− u_3 (pp− x_4 x_5)))

The result is: $u_2 u_3 - u_1^2 = 0$.

Example 19. As figure 19. Find the cross ratio $(T_1 S, UV)$.

((x_1 x_2 y_2 x_3 y_3) (u_1 x_4 y_4 x_5 x_6 y_6 x_7 y_7)
(A (0 0) B (x_1 0) C (x_2 y_2) D (x_3 y_3) T_1 (x_4 y_4) S (x_5 0) U (x_6 y_6) V (x_7 y_7))
(collinear T_1 D A)
(collinear T_1 C B)
(collinear C D S)
(collinear U A C)
(collinear U T_1 S)
(collinear V D B)
(collinear V T_1 S)
(c–ratio u_1 U V T_1 S)
non-deg (pp− x_5 x_4) (pp− x_6 x_4) (pp− x_7 x_4) u_1)

The result is: $u_1 + 1 = 0$.

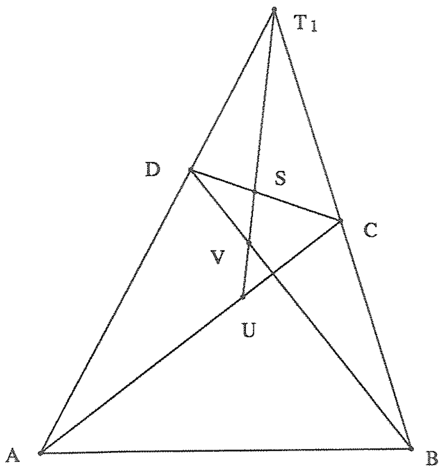

Fig. 19

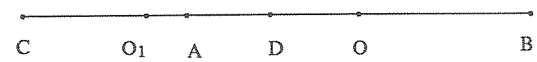

Fig. 20

Example 20. If $(AB, CD) = -1$, find the relation among OB, O_1C, and OO_1 where O and O_1 are the midpoints of AB and CD respectively.

((x1 u1 u2) (u3 x2 x3 x4 x5 x6)
(A (x1 0) B (x2 0) C (x3 0) D (x4 0) O (x5 0) O1 (x6 0))
(c–ratio −1 A B C D)
(mid-x O A B)
(mid-x O1 C D)
(pp− u1 (pp− x2 x5))
(pp− u2 (pp− x3 x6))
(pp− u3 (pp− x6 x5)))

The result is: $u_3^2 - u_2^2 - u_1^2 = 0$.

Example 21. Find the cross-ratio of the lines joining any point on a circle to the vertices of an inscribed square.

((x1 x3) (u1 x2 x4 x5 x6)
(O (0 0) A (x1 0) B (0 x1) C (x2 0) D (0 x2) P (x3 x4) B1 (x5 0) D1 (x6 0))
(pp+ x2 x1)
(eqdistant P O A O)
(collinear P B B1)
(collinear P D D1)
(c–ratio u1 A C B1 D1))

The result is: $u_1 + 1 = 0$.

Example 22. Triangle ABC is inscribed in a circle of which DE is the diameter perpendicular to side AB. If lines DC and EC intersect AB in P and Q, find the cross ratio (AB, PQ).

((x0 x1 x4) (u1 x2 x3 x5 x6 x7 x8)

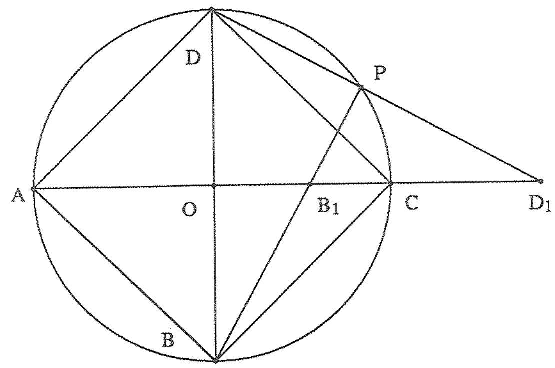

Fig. 21

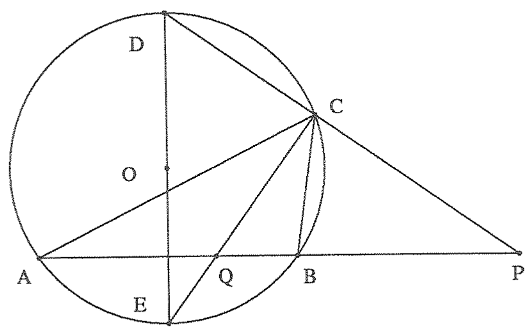

Fig. 22

(O (0 x_0) A (x_1 0) B (x_2 0) C (x_3 x_4) D (0 x_5) E (0 x_6) P (x_7 0) Q (x_8 0))
(pp+ x_2 x_1)
(eqdistant C O A O)
(eqdistant D O A O)
(mid-y O D E)
(collinear P D C)
(collinear Q E C)
(c–ratio u_1 A B P Q))

The result is: $u_1 + 1 = 0$.

Example 23. If L, M, and N are the midpoints of the sides BC, CA, and AB of a triangle ABC respectively. Find the cross ratio $L(MN, AB)$.

((x_1 x_2 x_3) (u_1 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11})
(A (0 0) B (x_1 0) C (x_2 x_3) L (x_4 x_5) M (x_6 x_5) N (x_7 0) P (x_8 x_9) Q (x_{10} x_{11}))
(mid-x L B C)
(mid-y L B C)
(mid-x M A C)
(mid-x N A B)
(collinear P L A)
(collinear P M B)
(collinear Q L N)
(collinear Q M B)
(c–ratio u_1 M Q P B))

The result is: $u_1 + 1 = 0$.

Example 24. If P, Q, and R are the feet of the altitudes on sides BC, CA, and AB of a triangle ABC, find the cross ratio $P(PR, AB)$.

((x_1 x_2 x_3) (u_1 x_4 x_5 x_6 x_7 x_8)
(A (0 0) B (x_1 0) C (x_2 x_3) P (x_4 x_5) Q (x_6 x_7) R (x_2 0) O (x_8 0))

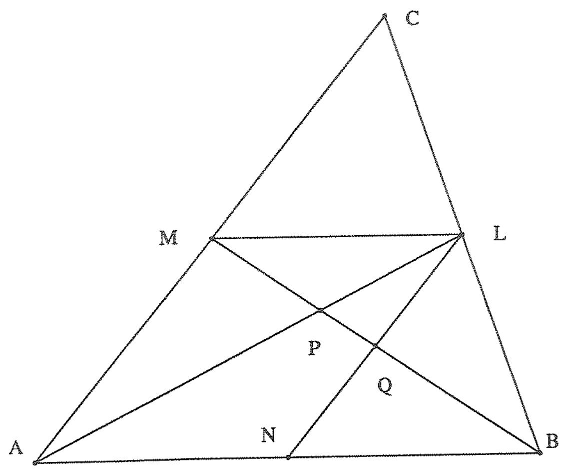

Fig. 23

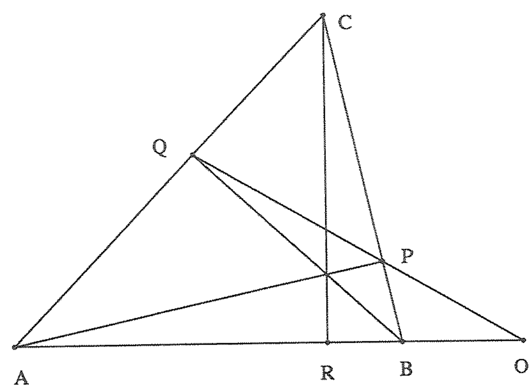

Fig. 24

(collinear P B C)
(collinear Q C A)
(perpendicular A P B C)
(perpendicular B Q C A)
(collinear O P Q)
(c–ratio u_1 O R A B))

The result is: $u_1 + 1 = 0$.

Example 25. Find the relation between the cross ratios of four lines passing through the same point cut by two different transversals.

((o_1 x_1 x_2 x_3 x_5 x_6 u_1) (u_2 x_4 y_5 y_6 x_7 y_7 x_8 y_8)
(O (0 o_1) A (x_1 0) B (x_2 0) C (x_3 0) D (x_4 0) A_1 (x_5 y_5) B_1 (x_6 y_6) C_1 (x_7 y_7)
D_1 (x_8 y_8))
(collinear O A A_1)
(collinear O B B_1)
(collinear O C C_1)
(collinear O D D_1)
(collinear A_1 B_1 C_1)
(collinear A_1 B_1 D_1)
(c–ratio u_1 A B C D)
(c–ratio u_2 A_1 B_1 C_1 D_1))

The result is: $u_2 - u_1 = 0$.

Example 26. If $(AB, CD) = -1$ and O is the midpoint of CD, find the relation among AC, AD, AB, and AO.

((x_1 x_2 u_1) (u_2 x_3 x_4 x_5)
(A (x_1 0) B (x_2 0) C (x_3 0) D (x_4 0) O (x_5 0))
(c–ratio -1 A B C D)
(mid-x O C D)

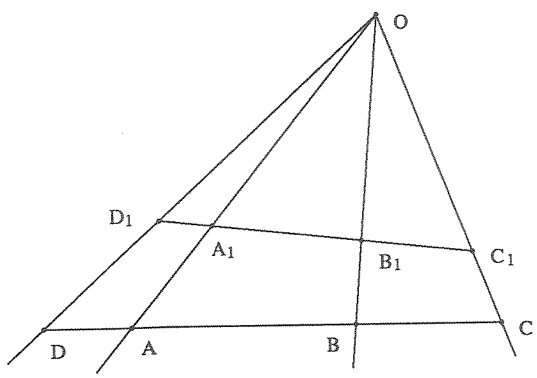

Fig. 25

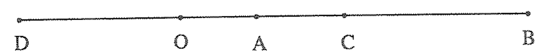

Fig. 26

(x-ratio u_1 A C A B)
(x-ratio u_2 A D A O))

The result is: $u_1 u_2 - 1 = 0$.

Example 27. If $(AB, CD) = -1$ and O is the midpoint of AB, find the relation among AC, AD, OC and OD.

((x_1 x_2 u_1) (u_2 x_3 x_4 x_5)
(A (x_1 0) B (x_2 0) C (x_3 0) D (x_4 0) O (x_5 0))
(c–ratio -1 A B C D)
(mid-x O A B)
(x-ratio u_1 O D O C)
(x-ratio u_2 A D A C))

The result is: $u_2^2 - u_1 = 0$.

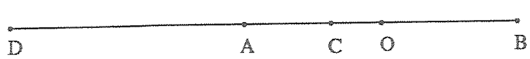

Fig. 27

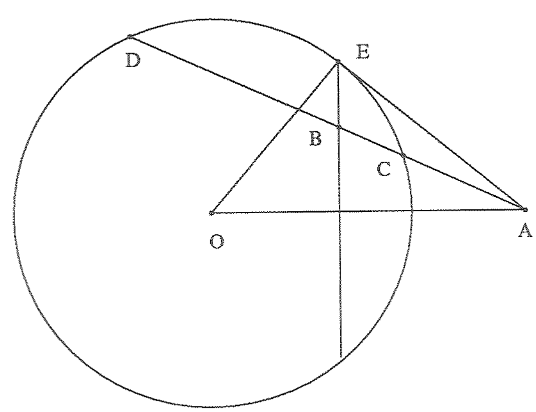

Fig. 28

Example 28. A secant from an external point A cuts a circle in C and D and cuts the chord of contact of the tangents to the circle from A in B. Find the cross ratio (AB, CD).

((x1 x3 x4) (u1 x5 x6 x7 x8)
(O (0 0) A (x1 0) P (x2 0) B (x3 x4) C (x5 x6) D (x7 x8))
(pp– (pp* x3 x1) (sq-distance O C))
(eqdistant C O D O)
(collinear A C D)
(collinear A B C)
(c–ratio u1 A B C D)
non-deg (pp– x7 x5))

The result is: $u_1 + 1 = 0$.

Example 29. If $(AB, CD) = -1$ and O is collinear with A, B, C, D, find the relation among OB/AB, OC/AC, and OD/AD.

((x1 x2 u1 u2) (u3 x3 x4 x5)
(A (x1 0) B (x2 0) C (x3 0) D (x4 0) O (x5 0))
(c–ratio –1 A B C D)
(x-ratio u1 O B A B)
(x-ratio u2 O C A C)
(x-ratio u3 O D A D))

The result is: $u_3 + u_2 - 2u_1 = 0$.

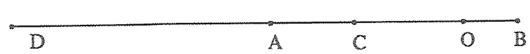

Fig. 29

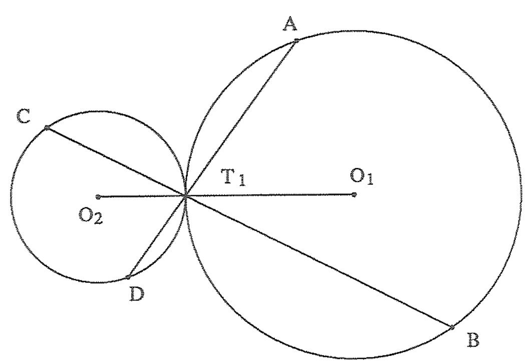

Fig. 30

Example 30. From the tangent point T_1 of two circles O_1 and O_2 two lines are drawn which cut the two circle in A, B, C, and D respectively. Find the relation among T_1A, T_1B, T_1C, and T_1D.

((x1 x3 x7 u1) (u2 x2 x4 x5 x6 x8 x9 x10)
(T1 (0 0) O1 (x1 0) O2 (x2 0) A (x3 x4) D (x5 x6) B (x7 x8) C (x9 x10))
(eqdistant A O1 T1 O1)

(eqdistant B O_1 T_1 O_1)
(eqdistant C O_2 T_1 O_2)
(eqdistant D O_2 T_1 O_2)
(collinear A T_1 D)
(collinear B T_1 C)
(x-ratio u_1 A T_1 T_1 D)
(x-ratio u_2 C T_1 T_1 B)
non-deg x_9)

The result is: $u_1 u_2 - 1 = 0$.

Example 31. Determine the area of a trapezoid in terms of its two parallel sides and its altitude.

((u_1 u_2 u_3 u_4 x_1) (k x_2)
(A (0 0) B (u_1 0) D (x_1 u_2) C (x_2 u_2))
(pp– u_3 (pp– x_2 x_1))
(area k A B C D))

The result is: $2k - u_2 u_3 - u_1 u_2 = 0$.

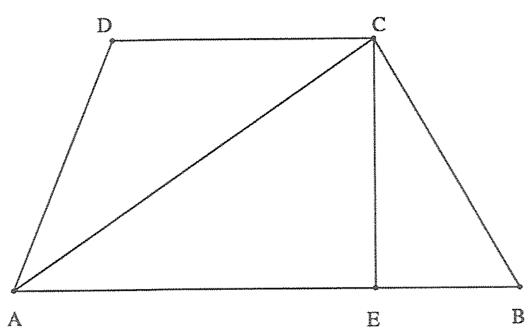

Fig. 31

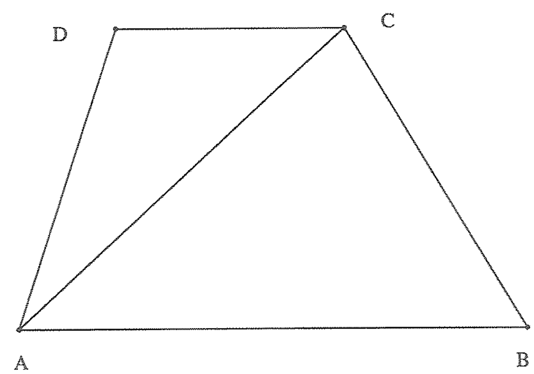

Fig. 32

Example 32. Determine the area of a trapezoid in terms of its four sides.

((u_1 u_2 u_3 u_4) (k x_1 x_2 x_3)
(A (0 0) B (u_1 0) D (x_1 x_3) C (x_2 x_3))
(distance A D u_3)
(distance B C u_4)
(pp– u_2 (pp– x_2 x_1))
(area k A B C D))

The result is: $(16u_2^2 - 32u_1u_2 + 16u_1^2)k^2 + (u_2^2 + 2u_1u_2 + u_1^2)u_4^4 + ((-2u_2^2 - 4u_1u_2 - 2u_1^2)u_3^2 - 2u_2^4 + 4u_1^2u_2^2 - 2u_1^4)u_4^2 + (u_2^2 + 2u_1u_2 + u_1^2)u_3^4 + (-2u_2^4 + 4u_1^2u_2^2 - 2u_1^4)u_3^2 + u_2^6 - 2u_1u_2^5 - u_1^2u_2^4 + 4u_1^3u_2^3 - u_1^4u_2^2 - 2u_1^5u_2 + u_1^6 = 0$; or $k^2 = -\frac{(u_2+u_1)^2(u_4-u_3+u_2-u_1)(u_4-u_3-u_2+u_1)(u_4+u_3+u_2-u_1)(u_4+u_3-u_2+u_1)}{16(u_2-u_1)^2}$.

Example 33. (The Secant Theorem) Two secants AB and CD intersect in P, find the relation among PA, PB, PC, and PD.

$((x_1\ u_1\ u_2\ u_3)\ (u_4\ x_2\ x_3\ x_4)$
$(P\ (0\ 0)\ A\ (u_1\ 0)\ B\ (u_2\ 0)\ C\ (x_1\ x_2)\ D\ (x_3\ x_4))$
(cocyclic $A\ B\ C\ D$)
(collinear $C\ D\ P$)
(distance $P\ C\ u_3$)
(distance $P\ D\ u_4$)
non-deg (pp– $x_4\ x_2$) (pp– $x_3\ x_1$))

The result is: $u_3u_4 + u_1u_2 = 0$ and $u_3u_4 - u_1u_2 = 0$.

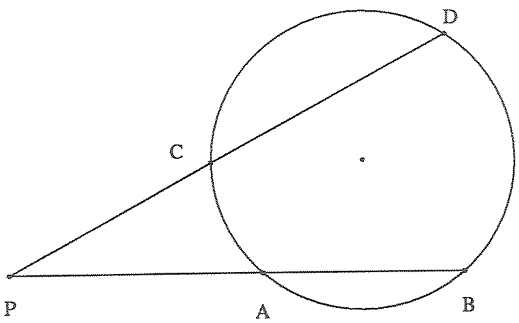

Fig. 33

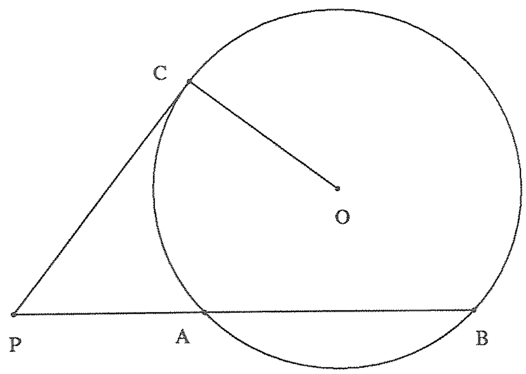

Fig. 34

Example 34. (The Tangent Theorem) A tangent CP and a secant of a circle intersect in P. Find the relation among PC, PA, and PB.

$((x_1\ u_1\ u_2)\ (u_3\ x_2\ x_3\ x_4)$
$(A\ (u_1\ 0)\ B\ (u_2\ 0)\ P\ (0\ 0)\ C\ (x_1\ x_2)\ O\ (x_3\ x_4))$
(mid-x $O\ A\ B$)
(eqdistant $O\ A\ O\ C$)
(perpendicular $O\ C\ P\ C$)
(distance $P\ C\ u_3$))

The result is: $u_3^2 - u_1u_2 = 0$.

Example 35. If a quadrilateral with sides a, b, c, and x is inscribed in a semicircle of diameter x, find the relation among a, b, c, and x.

$((u_1\ u_2\ u_3)\ (u_4\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6)$
$(O\ (0\ 0)\ A\ (x_1\ 0)\ B\ (x_2\ 0)\ C\ (x_3\ x_4)\ D\ (x_5\ x_6))$
(pp+ $x_1\ x_2$)
(eqdistant $C\ O\ A\ O$)
(eqdistant $D\ O\ A\ O$)

(pp− u_4 (pp− x_2 x_1))
(distance B C u_2)
(distance C D u_3)
(distance D A u_1))

The result is: $u_4^3+(-u_3^2-u_2^2-u_1^2)u_4+2u_1u_2u_3=0$ and $u_4^3+(-u_3^2-u_2^2-u_1^2)u_4-2u_1u_2u_3=0$.

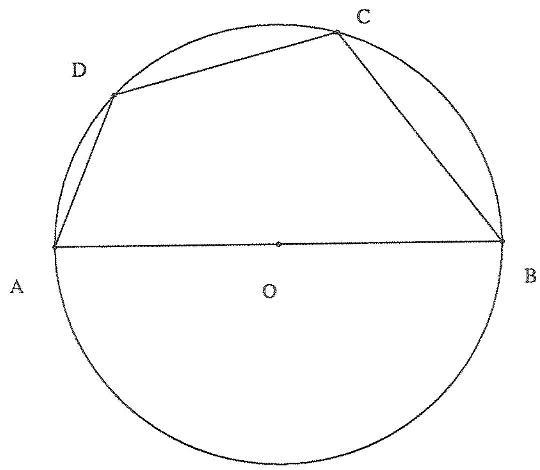

Fig. 35

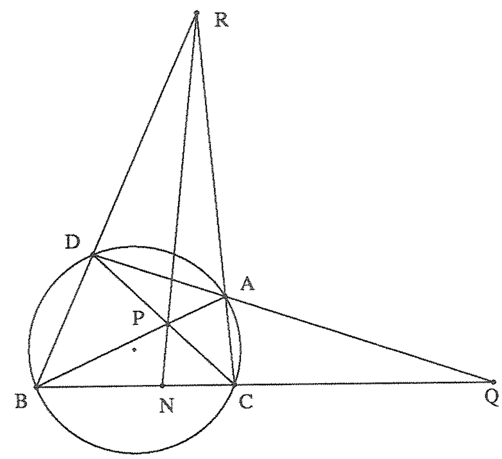

Fig. 36

Example 36. Let $ABCD$ be a complete quadrangle inscribed in a circle; $R = BD \cap AC$, $Q = BC \cap AD$, and $P = AB \cap CD$. Find the cross ratio (BC, NQ).

((x_1 x_2 x_3 x_4) (u_1 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11})
(B (0 0) C (x_1 0) D (x_2 x_3) A (x_4 x_5) Q (x_6 0) P (x_7 x_8) R (x_9 x_{10}) N (x_{11} 0))
(cocyclic B C D A)
(collinear Q A D)
(collinear P C D)
(collinear P B A)
(collinear R A C)
(collinear R B D)
(collinear R N P)
(c–ratio u_1 B C N Q)
non-deg (pp− x_2 x_1) (pp− x_5 x_3) (sq-distance D B) (pp− x_4 x_1) (pp− x_6 x_1)
(pp− x_6 x_1) (pp− x_{11} x_1) (pp− x_4 x_2))

The result is: $u_1+1=0$.

Example 37. The same as example 36. Find the cross ratio (AD, MQ).

((x_1 x_2 x_3 x_4) (u_1 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11} x_{12})
(B (0 0) C (x_1 0) D (x_2 x_3) A (x_4 x_5) Q (x_6 0) P (x_7 x_8) R (x_9 x_{10}) M (x_{11} x_{12}))
(cocyclic B C D A)
(collinear Q A D)
(collinear P C D)
(collinear P B A)

(collinear R A C)
(collinear R B D)
(collinear R P M)
(collinear A D M)
(c–ratio u_1 A D M Q)
non-deg (pp– x_4 x_2) (pp– x_{11} x_2) (pp– x_6 x_2) (pp– x_2 x_1) (pp– x_5 x_3)
(pp– x_{11} x_1) (pp– x_4 x_1))

The result is: $u_1 + 1 = 0$.

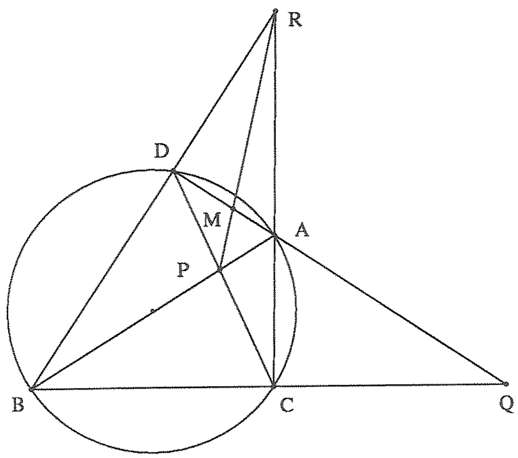

Fig. 37

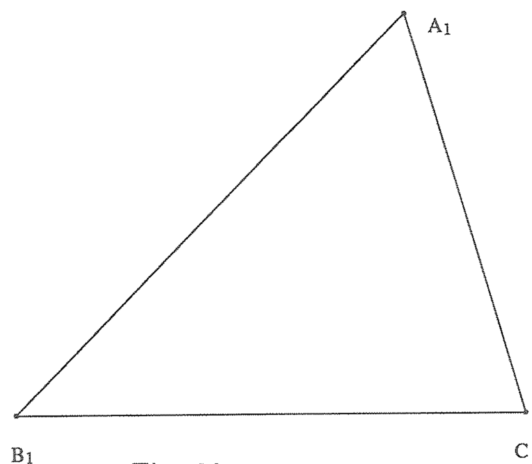

Fig. 38

Example 38. Determine the area of a triangle in terms of its three sides.

((a b c) (k x_1 x_2)
(B_1 (0 0) C_1 (a 0) A_1 (x_1 x_2))
(distance A_1 C_1 b)
(distance A_1 B_1 c)
(area k A_1 B_1 C_1))

The result is: $16k^2 + c^4 + (-2b^2 - 2a^2)c^2 + b^4 - 2a^2b^2 + a^4 = 0$; or $k^2 = (-c + b + a)(c - b + a)(c + b - a)(c + b + a)/16$.

Example 39. Determine the altitudes of a triangle in terms of its three sides.

((a b c) (h_a x_1 x_2)
(B_1 (0 0) C_1 (a 0) A_1 (x_1 x_2))
(distance A_1 C_1 b)
(distance A_1 B_1 c)
(pp– h_a x_2))

The result is: $4a^2h_a^2 + c^4 - (2b^2 + 2a^2)c^2 + b^4 - 2a^2b^2 + a^4 = 0$; or

$$h_a^2 = \frac{(c + b + a)(c + b - a)(c - b + a)(-c + b + a)}{4a^2}.$$

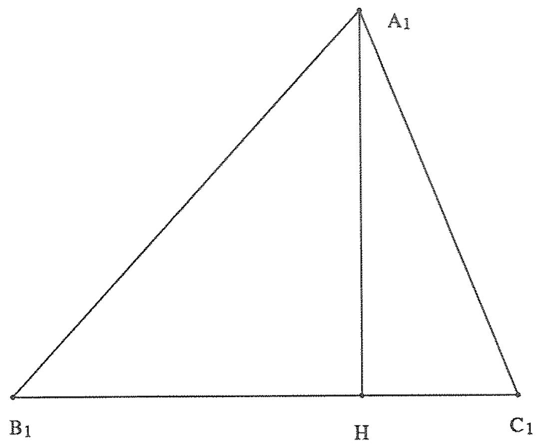

Fig. 39

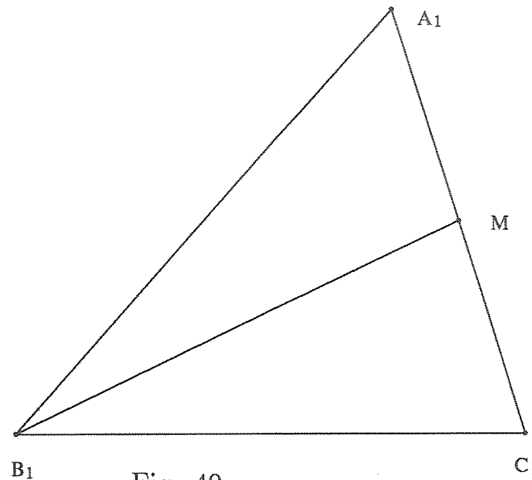

Fig. 40

Example 40. Determine the medians of a triangle in terms of its three sides.

((a b c) (m_a x_1 x_2 x_3 x_4)
(B_1 (0 0) C_1 (a 0) A_1 (x_1 x_2) M (x_3 x_4))
(distance A_1 C_1 b)
(distance A_1 B_1 c)
(mid-x M C_1 A_1)
(mid-y M C_1 A_1)
(distance B_1 M m_a))

The result is: $4m_a^2 - 2c^2 + b^2 - 2a^2 = 0$.

Example 41. Determine the radius of the circumcircle of a triangle in terms of its three sides.

((a b c) (e_r x_1 x_2 x_3 x_4)
(B_1 (0 0) C_1 (a 0) A_1 (x_1 x_2) D (x_3 x_4))
(distance A_1 C_1 b)
(distance A_1 B_1 c)
(distance B_1 D e_r)
(distance C_1 D e_r)
(distance A_1 D e_r))

The result is: $(c^4 - (2b^2 + 2a^2)c^2 + b^4 - 2a^2b^2 + a^4)e_r^2 + a^2b^2c^2 = 0$; or

$$e_r^2 = \frac{c^2b^2a^2}{(c-b+a)(c+b+a)(-c+b+a)(c+b-a)}.$$

Example 42. Determine the radius of the inscribed circle of a triangle in terms of its three sides.

((a b c) (i_r x_1 x_2 x_4 x_5 x_6 x_7 x_8 x_9)
(A_1 (0 0) B_1 (a 0) C_1 (x_1 x_2) E (x_4 0) N (x_5 x_6) D (c 0) M (x_7 x_8) I (x_9 i_r))
(distance A_1 C_1 b)

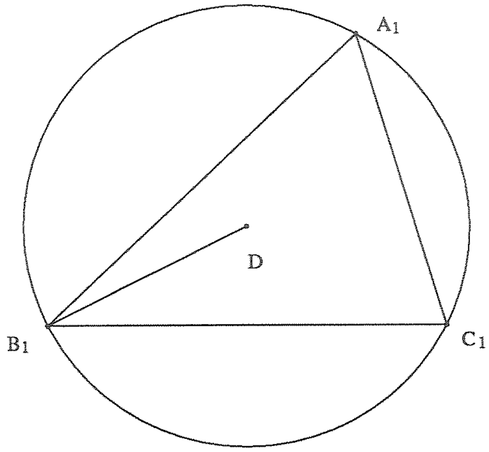

Fig. 41

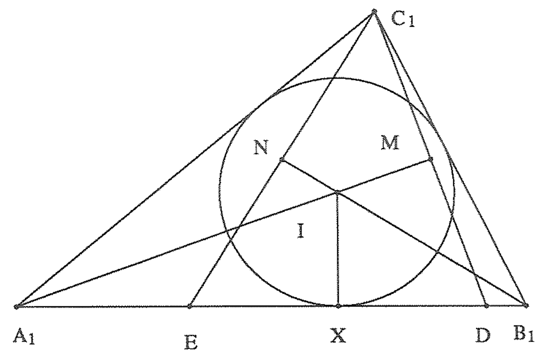

Fig. 42

(distance B_1 C_1 c)
(pp– b (pp– a x_4))
(mid-x N C_1 E)
(mid-y N C_1 E)
(mid-x M C_1 D)
(mid-y M C_1 D)
(collinear A_1 I M)
(collinear B_1 I N))

The result is: $(4c+4b+4a)i_r^2 + c^3 + (-b-a)c^2 + (-b^2+2ab-a^2)c + b^3 - ab^2 - a^2b + a^3 = 0$; or $i_r^2 = \frac{(c+b-a)(-c+b+a)(c-b+a)}{4(c+b+a)}$.

Example 43. Determine the bisectors of a triangle in terms of its three sides.

((a b c) (i_c x_1 x_2 x_3 x_4)
(B_1 (0 0) C_1 (a 0) A_1 (x_1 x_2) D (x_3 x_4))
(distance A_1 C_1 b)
(distance A_1 B_1 c)
(collinear B_1 A_1 D)
(eqtangent B_1 C_1 D D C_1 A_1)
(distance C_1 D i_c))

The result is: $(b^2+2ab+a^2)i_c^2 + abc^2 - ab^3 - 2a^2b^2 - a^3b = 0$ and $(b^2-2ab+a^2)i_c^2 - abc^2 + ab^3 - 2a^2b^2 + a^3b = 0$; or $i_c^2 = \frac{(-c+b+a)(c+b+a)ba}{(b+a)^2}$ and $i_c^2 = \frac{(c-b+a)(c+b-a)ba}{(b-a)^2}$.

Example 44. Determine the radius of the nine point circle of a triangle in terms of its three sides.

((a b c) (n_r x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9)
(B_1 (0 0) C_1 (a 0) A_1 (x_1 x_2) D (x_3 0) E (x_4 x_5) F (x_6 x_7) G (x_8 x_9))
(distance A_1 C_1 b)
(distance A_1 B_1 c)
(mid-x D B_1 C_1)

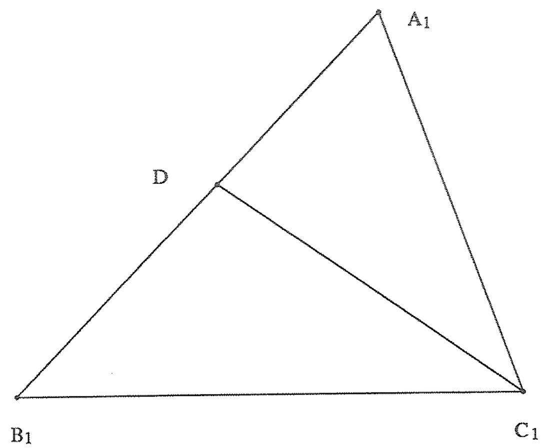

Fig. 43

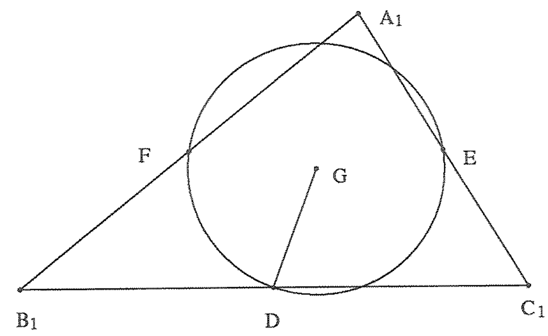

Fig. 44

(mid-x E A_1 C_1)
(mid-y E A_1 C_1)
(mid-x F B_1 A_1)
(mid-y F B_1 A_1)
(distance G D n_r)
(distance G E n_r)
(distance G F n_r))

The result is: $(4c^4 - (8b^2 + 8a^2)c^2 + 4b^4 - 8a^2b^2 + 4a^4)n_r^2 + a^2b^2c^2 = 0$; or

$$n_r^2 = \frac{c^2b^2a^2}{4(-c+b+a)(c-b+a)(c+b-a)(c+b+a)}.$$

Example 45. Determine the length of the Euler segment from the orthocenter to the circumcenter of a triangle in terms of its three sides.

((a b c) (e_u x_1 x_2 x_3 x_4 x_5)
(B_1 (0 0) C_1 (a 0) A_1 (x_1 x_2) D (x_3 x_4) E (x_1 x_5))
(distance A_1 C_1 b)
(distance A_1 B_1 c)
(mid-x D B_1 C_1)
(eqdistant D B_1 D A_1)
(perpendicular E B_1 A_1 C_1)
(distance E D e_u))

The result is: $(c^4 + (-2b^2 - 2a^2)c^2 + b^4 - 2a^2b^2 + a^4)e_u^2 + c^6 + (-b^2 - a^2)c^4 + (-b^4 + 3a^2b^2 - a^4)c^2 + b^6 - a^2b^4 - a^4b^2 + a^6 = 0$; or $e_u^2 = \frac{(c^6+(-b^2-a^2)c^4+(-b^4+3a^2b^2-a^4)c^2+b^6-a^2b^4-a^4b^2+a^6)}{(c-b+a)(-c+b+a)(c+b+a)(c+b-a)}$.

Example 46. Find the relation between the distances from the orthocenter to the centroid and from the circumcenter to the centroid of a triangle.

((x_0 x_1 u_1) (u_2 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9)
(B (0 0) C (x_0 0) A (x_1 x_2) O (x_3 x_4) H (x_1 x_5) M_1 (x_3 0) M_2 (x_6 x_7) G (x_8 x_9))

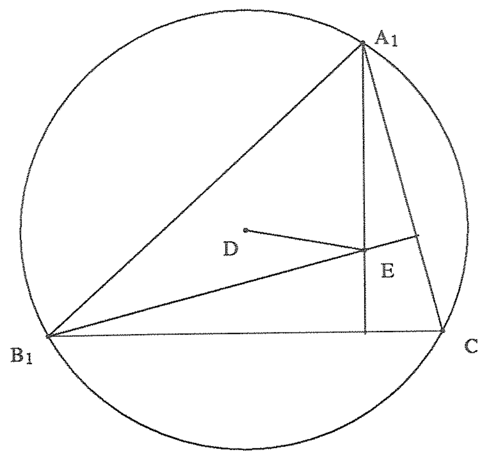

Fig. 45

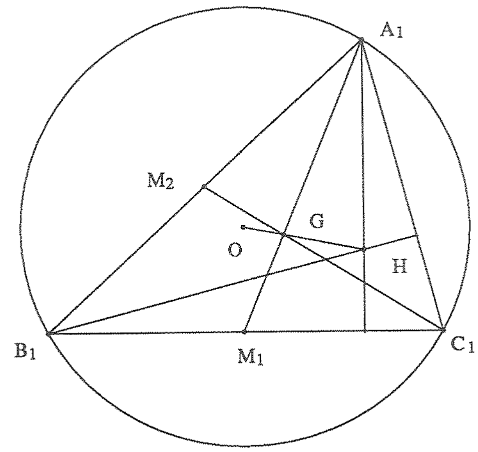

Fig. 46

(mid-x M_1 B C)
(mid-x M_2 B A)
(mid-y M_2 B A)
(eqdistant O B O A)
(perpendicular H B A C)
(collinear G A M_1)
(collinear G C M_2)
(distance H G u_1)
(distance G O u_2))

The result is: $2u_2 + u_1 = 0$ and $2u_2 - u_1 = 0$.

Example 47. Find the relation between the distance from the centroid of a triangle to the orthocenter and the distance from the centroid to the center of the nine-point circle.

((x_0 x_1 u_1) (u_2 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11})
(B (0 0) C (x_0 0) A (x_1 x_2) O (x_3 x_4) H (x_1 x_5) M_1 (x_3 0) M_2 (x_6 x_7) G (x_8 x_9)
F (x_1 0) N (x_{10} x_{11}))
(mid-x M_1 B C)
(mid-x M_2 B A)
(mid-y M_2 B A)
(eqdistant O B O A)
(perpendicular H B A C)
(collinear G A M_1)
(collinear G C M_2)
(eqdistant N F N M_1)
(eqdistant N F N M_2)
(distance H G u_1)
(distance G N u_2))

The result is: $4u_2 - u_1 = 0$ and $4u_2 + u_1 = 0$.

Example 48. The orthocenter H, the centroid G, the circumcenter O, and the center of the

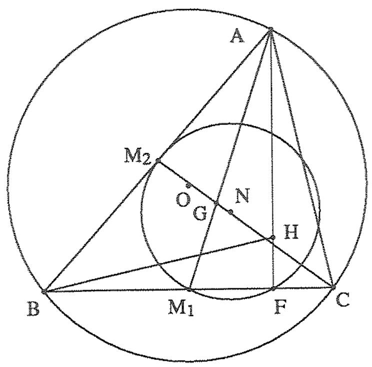

Fig. 47

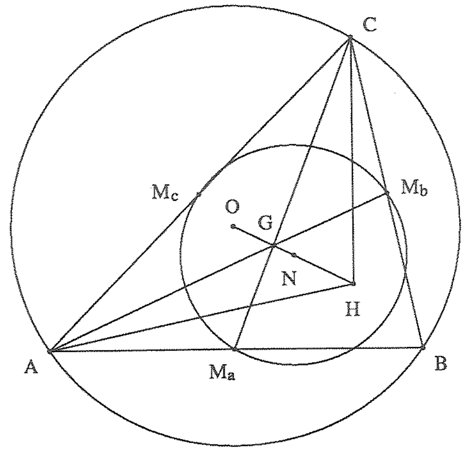

Fig. 48

nine point circle N of a triangle are collinear. Find the cross ratio (HG, NO).

$((x_1\ x_2\ x_3)\ (u_1\ n_r\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}\ x_{11}\ x_{12}\ x_{13})$
$(A\ (0\ 0)\ B\ (x_1\ 0)\ C\ (x_2\ x_3)\ M_a\ (x_4\ 0)\ M_b\ (x_5\ x_6)\ M_c\ (x_7\ x_6)\ N\ (x_8\ x_9)\ O\ (x_4\ x_{10})$
$G\ (x_{11}\ x_{12})\ H\ (x_2\ x_{13}))$
(mid-x M_a A B)
(mid-x M_b B C)
(mid-y M_b B C)
(mid-x M_c A C)
(distance N M_a n_r)
(distance N M_b n_r)
(distance N M_c n_r)
(eqdistant O A O C)
(collinear A G M_b)
(collinear C G M_a)
(perpendicular A H B C)
(c–ratio u_1 H G N O))

The result is: $u_1 + 1 = 0$.

Example 49. Find the distance from the incenter of a triangle to the circumcenter.

$((a\ b\ c)\ (r\ x_1\ x_2\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}\ x_{11}\ x_{12})$
$(A_1\ (0\ 0)\ B_1\ (a\ 0)\ C_1\ (x_1\ x_2)\ E\ (x_4\ 0)\ N\ (x_5\ x_6)\ D\ (c\ 0)\ M\ (x_7\ x_8)\ I\ (x_9\ x_{10})$
$O\ (x_{11}\ x_{12}))$
(distance A_1 C_1 b)
(distance B_1 C_1 c)
(pp– b (pp– a x_4))
(mid-x N C_1 E)
(mid-y N C_1 E)
(mid-x M C_1 D)
(mid-y M C_1 D)
(collinear A_1 I M)

(collinear B_1 I N)
(eqdistant A_1 O B_1 O)
(eqdistant A_1 O C_1 O)
(distance O I r))

The result is: $(c^5+(b+a)c^4-(2b^2+2a^2)c^3-(2b^3+2ab^2+2a^2b+2a^3)c^2+(b^4-2a^2b^2+a^4)c+b^5+ab^4-2a^2b^3-2a^3b^2+a^4b+a^5)r^2+ac^6-(ab+a^2)c^5-(ab^2-3a^2b+a^3)c^4+(2ab^3-a^2b^2-2a^3b+a^4)c^3-(ab^4+a^2b^3-3a^3b^2+a^4b)c^2-(ab^5-3a^2b^4+2a^3b^3+a^4b^2-a^5b)c+ab^6-a^2b^5-a^3b^4+a^4b^3=0$.

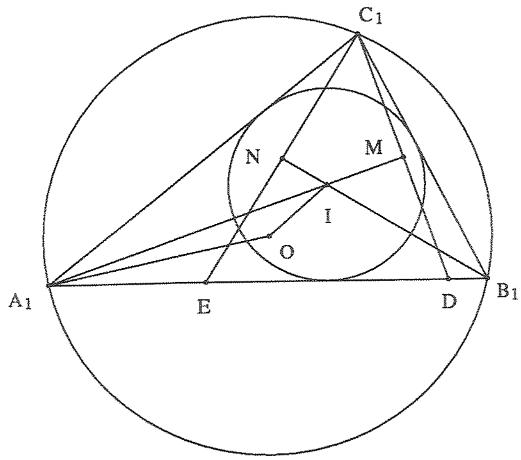

Fig. 49

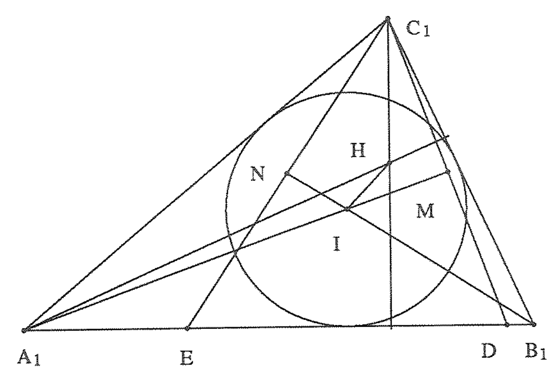

Fig. 50

Example 50. Find the distance from the incenter of a triangle to its orthocenter.

((a b c) (r x_1 x_2 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11})
(A_1 (0 0) B_1 (a 0) C_1 (x_1 x_2) E (x_4 0) N (x_5 x_6) D (c 0) M (x_7 x_8) I (x_9 x_{10})
H (x_1 x_{11}))
(distance A_1 C_1 b)
(distance B_1 C_1 c)
(pp- b (pp- a x_4))
(mid-x N C_1 E)
(mid-y N C_1 E)
(mid-x M C_1 D)
(mid-y M C_1 D)
(collinear A_1 I M)
(collinear B_1 I N)
(perpendicular H A_1 B_1 C_1)
(distance H I r))

The result is: $(c^5+(b+a)c^4-(2b^2+2a^2)c^3-(2b^3+2ab^2+2a^2b+2a^3)c^2+(b^4-2a^2b^2+a^4)c+b^5+ab^4-2a^2b^3-2a^3b^2+a^4b+a^5)r^2+(b+a)c^6+(b^2-ab-a^2)c^5-(2b^3+ab^2-a^2b)c^4-(2b^4-2ab^3+2a^3b-a^4)c^3+(b^5-ab^4+4a^3b^2-2a^5)c^2+(b^6-ab^5+a^2b^4-2a^3b^3+a^5b)c+ab^6-a^2b^5+a^4b^3-2a^5b^2+a^7=0$.

Example 51. Find the distance from the incenter of a triangle to its centroid.

((a b c) (r x_1 x_2 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11} x_{12} x_{13} x_{14} x_{15})

$(A_1\ (0\ 0)\ B_1\ (a\ 0)\ C_1\ (x_1\ x_2)\ E\ (x_4\ 0)\ N\ (x_5\ x_6)\ D\ (c\ 0)\ M\ (x_7\ x_8)\ I\ (x_9\ x_{10})$
$F\ (x_{11}\ 0)\ H\ (x_{12}\ x_{13})\ G\ (x_{14}\ x_{15}))$
(distance A_1 C_1 b)
(distance B_1 C_1 c)
(pp– b (pp– a x_4))
(mid-x N C_1 E)
(mid-y N C_1 E)
(mid-x M C_1 D)
(mid-y M C_1 D)
(collinear A_1 I M)
(collinear B_1 I N)
(mid-x F A_1 B_1)
(mid-x H A_1 C_1)
(mid-y H A_1 C_1)
(collinear G C_1 F)
(collinear G B_1 H)
(distance G I r))

The result is: $(9c^2+(18b+18a)c+9b^2+18ab+9a^2)r^2-2c^4-(b-5a)c^3+(2b^2-ab-4a^2)c^2-(b^3+ab^2-5a^2b+a^3)c-2b^4+5ab^3-4a^2b^2-a^3b+a^4=0$.

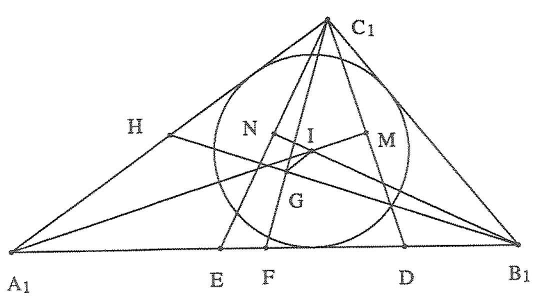

Fig. 51

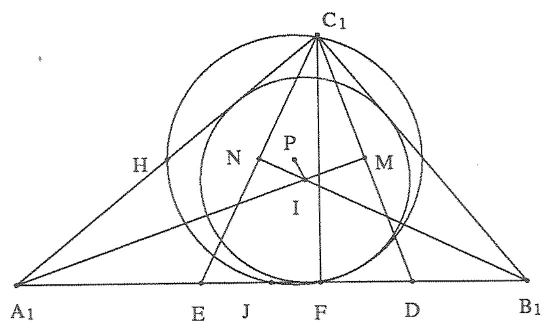

Fig. 52

Example 52. Find the distance from the incenter of a triangle to the center of the nine-point circle.

$((a\ b\ c)\ (r\ x_1\ x_2\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}\ x_{11}\ x_{12}\ x_{13}\ x_{14}\ x_{15})$
$(A_1\ (0\ 0)\ B_1\ (a\ 0)\ C_1\ (x_1\ x_2)\ E\ (x_4\ 0)\ N\ (x_5\ x_6)\ D\ (c\ 0)\ M\ (x_7\ x_8)\ I\ (x_9\ x_{10})$
$F\ (x_1\ 0)\ J\ (x_{11}\ 0)\ H\ (x_{12}\ x_{13})\ P\ (x_{14}\ x_{15}))$
(distance A_1 C_1 b)
(distance B_1 C_1 c)
(pp– b (pp– a x_4))
(mid-x N C_1 E)
(mid-y N C_1 E)

(mid-x M C_1 D)
(mid-y M C_1 D)
(collinear A_1 I M)
(collinear B_1 I N)
(mid-x J A_1 B_1)
(mid-x H A_1 C_1)
(mid-y H A_1 C_1)
(eqdistant P F P J)
(eqdistant P F P H)
(distance I P r))

The result is: $(4c^5+(4b+4a)c^4+(-8b^2-8a^2)c^3+(-8b^3-8ab^2-8a^2b-8a^3)c^2+(4b^4-8a^2b^2+4a^4)c+4b^5+4ab^4-8a^2b^3-8a^3b^2+4a^4b+4a^5)r^2-c^7+(b+3a)c^6+(3b^2-4ab-3a^2)c^5+(-3b^3-3ab^2+9a^2b-a^3)c^4+(-3b^4+8ab^3-5a^2b^2-8a^3b+5a^4)c^3+(3b^5-3ab^4-5a^2b^3+11a^3b^2-a^4b-3a^5)c^2+(b^6-4ab^5+9a^2b^4-8a^3b^3-a^4b^2+4a^5b-a^6)c-b^7+3ab^6-3a^2b^5-a^3b^4+5a^4b^3-3a^5b^2-a^6b+a^7=0$.

Example 53. Find the relation of the distance from the circumcenter O of a triangle ABC to the side AB and the distance from the orthocenter H to the vertex C.

((x_1 x_2 u_1) (u_2 x_3 x_4 x_5)
(A (0 0) B (x_1 0) C (x_2 x_3) O (x_4 u_1) H (x_2 x_5))
(eqdistant O A O B)
(eqdistant O A O C)
(perpendicular A H B C)
(pp– u_2 (pp– x_3 x_5)))

The result is: $u_2-2u_1=0$.

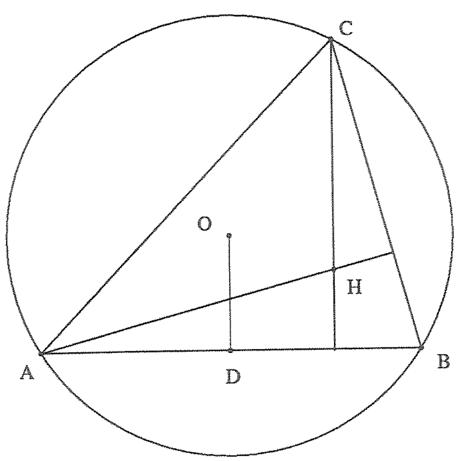

Fig. 53

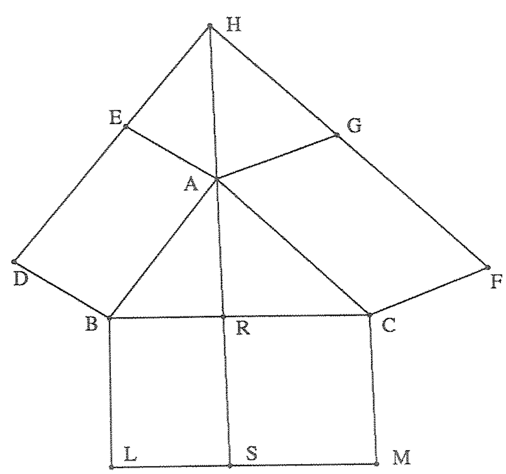

Fig. 54

Example 54. Let ABC be any triangle and $ABDE$, $ACFG$ any parallelograms described on AB and AC. Let DE and FG meet in H and draw BL and CM equal and parallel to HA. Find the relation among $area(BCML)$, $area(ABDE)$, and $area(ACFG)$.

((x_1 x_2 x_3 x_4 x_8 u_1 u_2) (u_3 x_5 x_6 x_7 x_9 x_{10} x_{11} x_{12} x_{13} x_{16} x_{17} x_{18})

(B (0 0) C (x_1 0) A (x_2 x_3) D (x_4 x_5) E (x_6 x_7) F (x_8 x_9) G (x_{10} x_{11}) H (x_{12} x_{13})
L (x_{17} x_{16}) M (x_{18} x_{16}))
(parallel A E B D)
(parallel A B D E)
(parallel A G C F)
(parallel A C G F)
(collinear H G F)
(collinear H E D)
(parallel B L H A)
(parallel B L C M)
(x-ratio 1 H A B L)
(area u_1 B L C)
(area u_2 A C F)
(area u_3 A D B)
non-deg (pp− x_6 x_4) (pp− x_{12} x_2) u_3)

The result is: $u_3 + u_2 - u_1 = 0$.

Example 55. In a triangle ABC, let p and q be the radii of two circles through A, touching side BC at B and C, respectively. Find the relation among p, q, and r.

((x_1 u_1 u_2) (r x_2 x_3 x_4 x_5)
(B (0 0) C (x_1 0) A (x_2 x_3) Q (0 u_1) P (x_1 u_2) O (x_4 x_5))
(eqdistant A Q Q B)
(eqdistant A P P C)
(distance O A r)
(distance O B r)
(distance O C r))

The result is: $r^2 - u_1 u_2 = 0$.

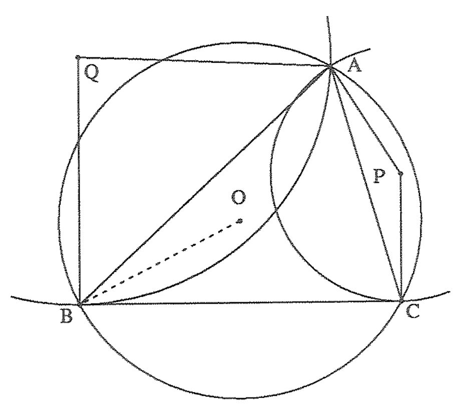

Fig. 55

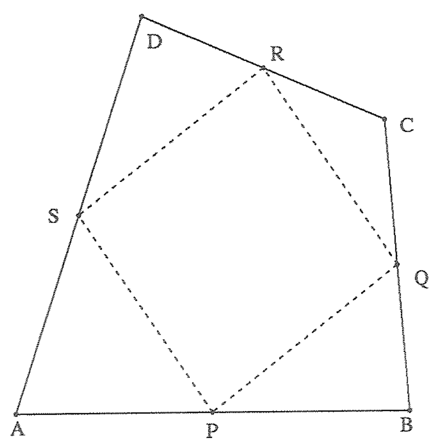

Fig. 56

Example 56. Find the relation between the area of a quadrangle and the area of the figure formed by joining the midpoints of the sides of the quadrangle.

((x1 x2 x3 x4 u1) (u2 x5 x6 x7 x8 x9 x10 x11 x12)
(A (0 0) B (x1 0) C (x2 x3) D (x4 x5) P (x6 0) Q (x7 x8) R (x9 x10) S (x11 x12))
(mid-x P A B)
(mid-x Q B C)
(mid-y Q B C)
(mid-x R C D)
(mid-y R C D)
(mid-x S A D)
(mid-y S A D)
(area u1 A B C D)
(area u2 P Q R S))

The result is: $2u_2 - u_1 = 0$.

Example 57. If a quadrangle $ABCD$ has its opposite sides AD and BC (extended) meeting at W, while X and Y are the midpoints of the diagonals AC and BD, find the relation between $area(WXY)$ and $area(ABCD)$.

((x1 x2 x3 x4 u1) (u2 x5 x6 x7 x8 x9 x10)
(B (0 0) C (x1 0) D (x2 x3) A (x4 x5) W (x6 0) X (x7 x8) Y (x9 x10))
(collinear A D W)
(mid-x X A C)
(mid-y X A C)
(mid-x Y B D)
(mid-y Y B D)
(area u1 A B C D)
(area u2 X Y W))

The result is: $4u_2 - u_1 = 0$.

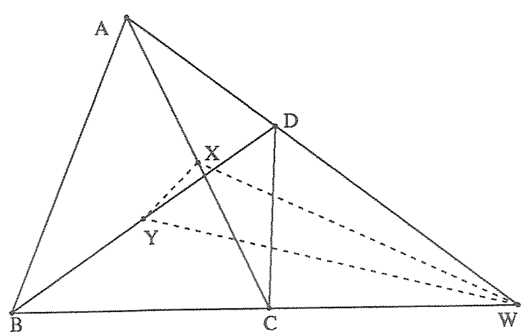

Fig. 57

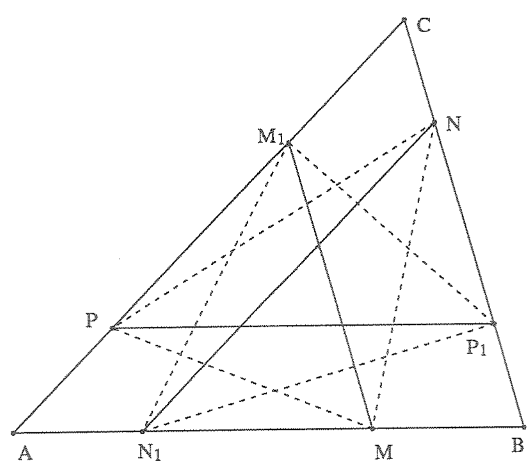

Fig. 58

Example 58. Let M, N, and P be points on the sides AB, BC and AC of a triangle ABC. If

M_1, N_1 and P_1 are points on sides AC, BA, and BC of a triangle ABC such that $MM_1 \parallel BC$, $NN_1 \parallel CA$ and $PP_1 \parallel AB$, find the relation between the $area(MNP)$ and $area(M_1N_1P_1)$.

$((x_1\ x_2\ x_3\ x_4\ x_5\ u_1)\ (u_2\ x_6\ x_7\ x_8\ x_9\ x_{10}\ x_{11}\ x_{12})$
$(A\ (0\ 0)\ B\ (x_1\ 0)\ C\ (x_2\ x_3)\ M\ (x_4\ 0)\ N_1\ (x_5\ 0)\ N\ (x_6\ x_7)\ P_1\ (x_8\ x_9)\ P\ (x_{10}\ x_9)$
$M_1\ (x_{11}\ x_{12}))$
(collinear $A\ C\ P$)
(collinear $A\ C\ M_1$)
(collinear $B\ C\ N$)
(collinear $B\ C\ P_1$)
(parallel $M\ M_1\ B\ C$)
(parallel $N\ N_1\ A\ C$)
(area $u_1\ M\ N\ P$)
(area $u_2\ M_1\ N_1\ P_1$))

The result is: $u_2 - u_1 = 0$.

Example 59. Let l be a line passing through the vertex of M of a parallelogram $MNPQ$ and intersecting the lines NP, PQ, NQ in points R, S, T_1. Find the relation among MR, MS, and MT_1.

$((u_1\ u_2\ u_3\ r_1)\ (r_2\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6)$
$(M\ (0\ 0)\ N\ (u_1\ 0)\ P\ (u_2\ u_3)\ Q\ (x_1\ u_3)\ S\ (x_2\ u_3)\ R\ (x_3\ x_4)\ T_1\ (x_5\ x_6))$
(parallel $M\ Q\ N\ P$)
(collinear $T_1\ Q\ N$)
(collinear $R\ N\ P$)
(collinear $M\ T_1\ S$)
(collinear $M\ R\ S$)
(x-ratio $r_1\ M\ T_1\ M\ R$)
(x-ratio $r_2\ M\ T_1\ M\ S$)
non-deg x_2)

The result is: $r_2 + r_1 - 1 = 0$.

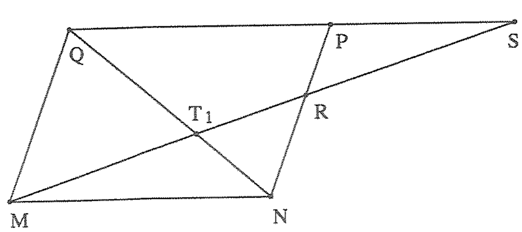

Fig. 59

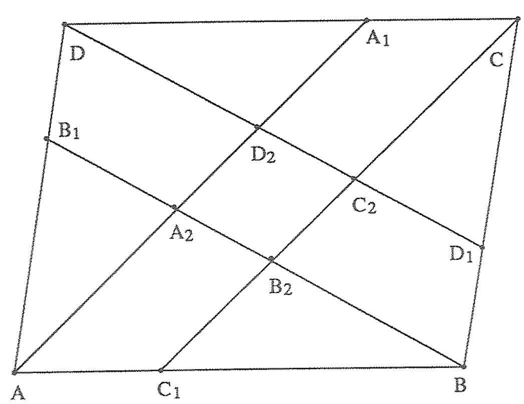

Fig. 60

Example 60. Let A_1, B_1, C_1, and D_1 be points on the sides CD, DA, AB, and BC of a parallelogram $ABCD$ such that $CA_1/CD = DB_1/DA = AC_1/AB = BD_1/BC = 1/3$. Find the relation between the area of the quadrilateral formed by the lines AA_1 BB_1, CC_1, and DD_1 the area of parallelogram $ABCD$.

$((x_1\ x_2\ u_1)\ (u_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}\ x_{11}\ x_{12}\ x_{13}\ x_{14}\ x_{15}\ x_{16}\ x_{17}\ x_{18})$
$(A\ (0\ 0)\ B\ (x_1\ 0)\ C\ (x_2\ x_3)\ D\ (x_4\ x_3)\ C_1\ (x_5\ 0)\ D_1\ (x_6\ x_7)\ A_1\ (x_8\ x_3)\ B_1\ (x_9\ x_{10})$
$A_2\ (x_{11}\ x_{12})\ B_2\ (x_{13}\ x_{14})\ C_2\ (x_{15}\ x_{16})\ D_2\ (x_{17}\ x_{18}))$
(parallel $B\ C\ A\ D$)
(x-ratio 3 $A\ B\ A\ C_1$)
(x-ratio 3 $B\ C\ B\ D_1$)
(y-ratio 3 $B\ C\ B\ D_1$)
(x-ratio 3 $C\ D\ C\ A_1$)
(x-ratio 3 $D\ A\ D\ B_1$)
(y-ratio 3 $D\ A\ D\ B_1$)
(collinear $A_2\ A\ A_1$)
(collinear $A_2\ B\ B_1$)
(collinear $B_2\ B\ B_1$)
(collinear $B_2\ C\ C_1$)
(collinear $C_2\ C\ C_1$)
(collinear $C_2\ D\ D_1$)
(collinear $D_2\ D\ D_1$)
(collinear $D_2\ A\ A_1$)
(area $u_1\ A\ B\ C$)
(area $u_2\ A_2\ B_2\ C_2$))

The result is: $13u_2 - u_1 = 0$.

Example 61. Let A_1, B_1, and C_1 be points on the sides BC, CA, and AB of a triangle ABC such that $BA_1/BC = CB_1/CA = AC_1/AB = r$. Find the relation between the area of the triangle determined by lines AA_1, BB_1 and CC_1 and the area of triangle ABC.

$((x_1\ x_2\ r\ u_1)\ (u_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}\ x_{11}\ x_{12}\ x_{13}\ x_{14})$
$(A\ (0\ 0)\ B\ (x_1\ 0)\ C\ (x_2\ x_3)\ C_1\ (x_4\ 0)\ A_1\ (x_5\ x_6)\ B_1\ (x_7\ x_8)\ C_2\ (x_9\ x_{10})\ A_2\ (x_{11}\ x_{12})$
$B_2\ (x_{13}\ x_{14}))$
(collinear $B\ C\ A_1$)
(collinear $A\ C\ B_1$)
(x-ratio $r\ A\ C_1\ A\ B$)
(x-ratio $r\ B\ A_1\ B\ C$)
(x-ratio $r\ C\ B_1\ C\ A$)
(collinear $A\ A_1\ C_2$)
(collinear $A\ A_1\ A_2$)
(collinear $B\ B_1\ A_2$)
(collinear $B\ B_1\ B_2$)
(collinear $C\ C_1\ B_2$)
(collinear $C\ C_1\ C_2$)
(area $u_1\ A\ B\ C$)
(area $u_2\ A_2\ B_2\ C_2$))

The result is: $(r^2 - r + 1)u_2 + (-4r^2 + 4r - 1)u_1 = 0$.

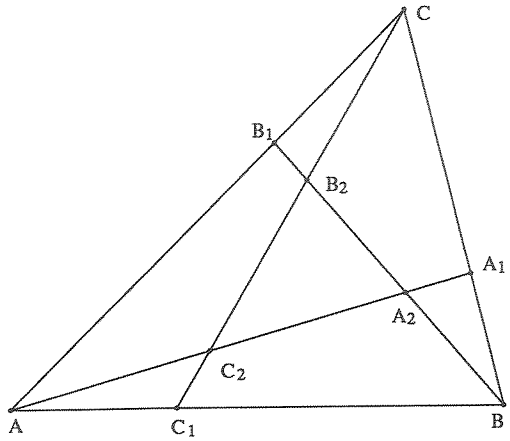

Fig. 61

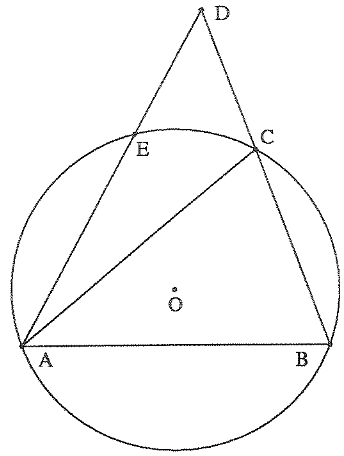

Fig. 62

Example 62. Let ABC be a triangle with $AC \equiv AB$. D is a point on BC. Line AD meets the circumcircle of ABC at E. Find the relation among AB, AD, and AE.

> $((x_1\ u_1\ u_2)\ (u_3\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7)$
> $(B\ (x_1\ 0)\ C\ (x_2\ 0)\ A\ (0\ x_3)\ D\ (x_4\ 0)\ E\ (x_5\ x_6)\ O\ (0\ x_7))$
> (pp+ x_1 x_2)
> (eqdistant $A\ O\ B\ O$)
> (eqdistant $E\ O\ O\ B$)
> (collinear $A\ E\ D$)
> (distance $A\ B\ u_1$)
> (distance $A\ D\ u_2$)
> (distance $A\ E\ u_3$)
> non-deg x_5)

The result is: $u_2u_3 + u_1^2 = 0$ and $u_2u_3 - u_1^2 = 0$.

Example 63. Let R be a point on the circle with diameter AB. At a point P of AB a perpendicular is drawn meeting BR at N, AR at M, and meeting the circle at Q. Find the relation among PQ, PM, and PN.

> $((x_1\ u_1\ u_2)\ (u_3\ x_2\ x_3\ x_4\ x_5)$
> $(O\ (0\ 0)\ A\ (x_1\ 0)\ B\ (x_2\ 0)\ P\ (x_3\ 0)\ R\ (x_4\ x_5)\ M\ (x_3\ u_1)\ Q\ (x_3\ u_2)\ N\ (x_3\ u_3))$
> (pp+ x_2 x_1)
> (eqdistant $R\ O\ A\ O$)
> (collinear $N\ R\ B$)
> (collinear $A\ M\ R$)
> (eqdistant $Q\ O\ A\ O$)
> non-deg u_3)

The result is: $u_1u_3 - u_2^2 = 0$.

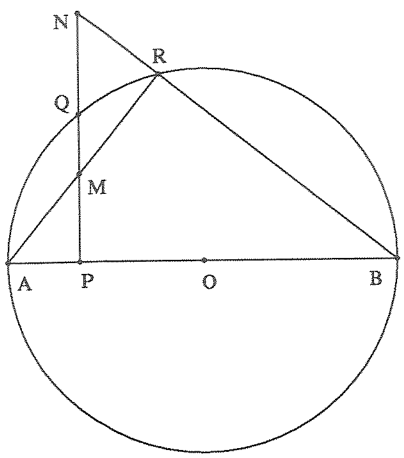

Fig. 63

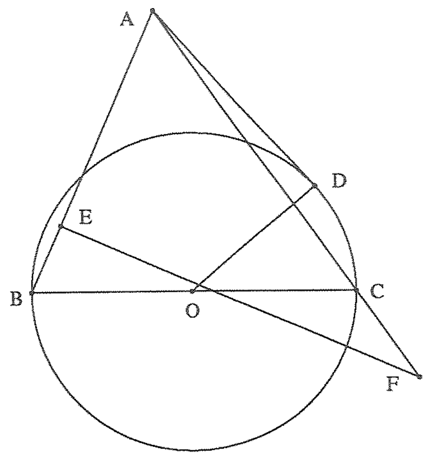

Fig. 64

Example 64. Let ABC be a triangle. Through A a line is drawn tangent to the circle with diameter BC at D. Let E be a point on AB such that $AD \equiv AE$. The perpendicular to B at E meets AC at F. Find the relation among AE, AB, AC, and AF.

((x1 x3 u1) (u2 x2 x4 x5 x6 x7 x8 x9 x10)
(O (0 0) B (x1 0) C (x2 0) A (x3 x4) D (x5 x6) E (x7 x8) F (x9 x10))
(pp+ x1 x2)
(eqdistant D O B O)
(perpendicular O D A D)
(eqdistant A E A D)
(collinear B A E)
(perpendicular F E A B)
(collinear A C F)
(x-ratio u1 A B A E)
(x-ratio u2 A F A C))

The result is: $u_2 - u_1 = 0$.

Example 65. Through point F on the circle with diameter AB a tangent to the circle is drawn meeting the two lines, perpendicular to AB at A and B, at D and E. Find the relation among OA, DF, EF.

((r u1) (u2 x1 x2 x3)
(O (0 0) A (r 0) B (x1 0) F (x2 x3) D (r u1) E (x1 u2))
(pp+ x1 r)
(eqdistant F O A O)
(collinear D F E)
(perpendicular O F D E))

The result is: $u_1 u_2 - r^2 = 0$.

Example 66. Find the relation among the three lines joining the vertices of an equilateral triangle to a point on its circumcircle.

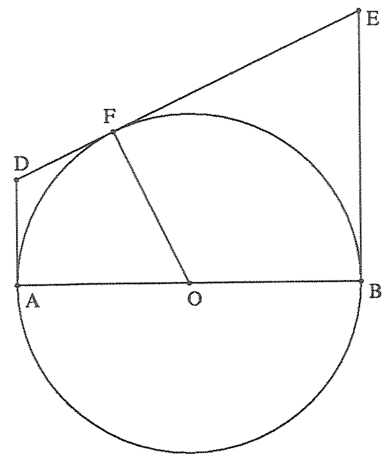

Fig. 65

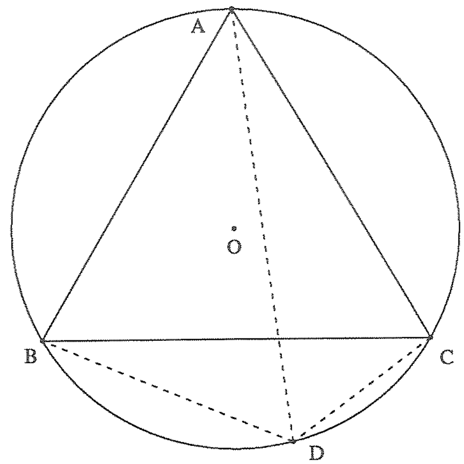

Fig. 66

```
((u1 u2) (u3 x0 x1 x2 x3 x4 x5 x6)
 (O (0 x1) B (x2 0) C (x3 0) A (0 x4) D (x5 x6))
 (pp- (pp∧ x0 2) 3)
 (pp- x2 (pp* x0 x1))
 (pp+ x2 x3)
 (pp- x4 (pp* 3 x1))
 (eqdistant D O B O)
 (distance D A u1)
 (distance D B u2)
 (distance D C u3))
```

The result is: $u_3 + u_2 + u_1 = 0$, $u_3 - u_2 - u_1 = 0$, $u_3 + u_2 - u_1 = 0$, and $u_3 - u_2 + u_1 = 0$.

Example 67. Three parallel lines drawn through the vertices of a triangle ABC meet the respectively opposite sides in the points X, Y, and Z. Find the relation between $area(XYZ)$ and $area(ABC)$.

```
((x1 x2 x4 u1) (u2 x3 x5 x6 x7 x8)
 (B (0 0) C (x1 0) A (x2 x3) X (x4 0) Y (x5 x6) Z (x7 x8))
 (collinear Z A B)
 (collinear Y A C)
 (parallel A X Z C)
 (parallel A X B Y)
 (area u1 A B C)
 (area u2 X Z Y))
```

The result is: $u_2 - 2u_1 = 0$.

Example 68. The parallel to the side AC through the vertex B of the triangle ABC meets the tangent to the circumcircle (O) of ABC at C in B_1, and the parallel through C to AB meets the tangent to (O) at B in C_1. Find the relation among BC, BC_1 , and B_1C.

```
((x1 x2 u1 u2) (u3 x3 x4 x5 x6 x7 x8 x9)
```

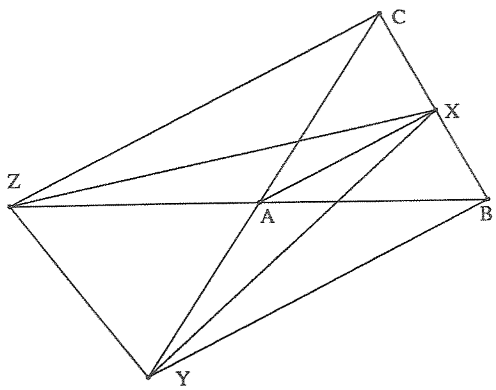

Fig. 67

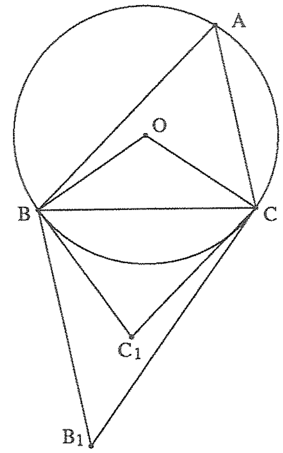

Fig. 68

$(B\ (0\ 0)\ C\ (u_1\ 0)\ A\ (x_2\ x_3)\ O\ (x_4\ x_5)\ B_1\ (x_6\ x_7)\ C_1\ (x_8\ x_9))$
(mid-x O B C)
(eqdistant O B O A)
(parallel B_1 B A C)
(perpendicular B_1 C C O)
(parallel C_1 C A B)
(perpendicular C_1 B B O)
(distance B C_1 u_2)
(distance B_1 C u_3))

The result is: $u_2u_3 + u_1^2 = 0$ and $u_2u_3 - u_1^2 = 0$.

Example 69. CF is the internal bisector of $\angle C$; AJ and BK are perpendicular to AF respectively. Find the cross-ratio (CF, KJ).

$((x_1\ x_2\ x_3)\ (u_1\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9)$
$(C\ (0\ 0)\ B\ (x_1\ 0)\ A\ (x_2\ x_3)\ F\ (x_4\ x_5)\ K\ (x_6\ x_7)\ J\ (x_8\ x_9))$
(point-line-dis F A C x_5)
(collinear F A B)
(collinear F C K)
(collinear J F C)
(perpendicular B K F C)
(perpendicular A J F C)
(c–ratio u_1 C F K J))

The result is: $u_1 + 1 = 0$.

Example 70. Find the relation among two sides of a triangle, the altitude to the third and the circumdiameter.

$((r\ u_1\ u_2)\ (u_3\ x_1\ x_2\ x_3\ x_4)$
$(A\ (0\ 0)\ B\ (x_1\ 0)\ C\ (x_2\ u_1)\ O\ (x_3\ x_4))$
(distance A O r)

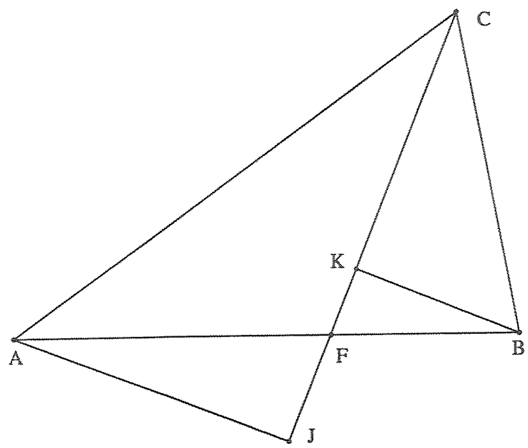

Fig. 69

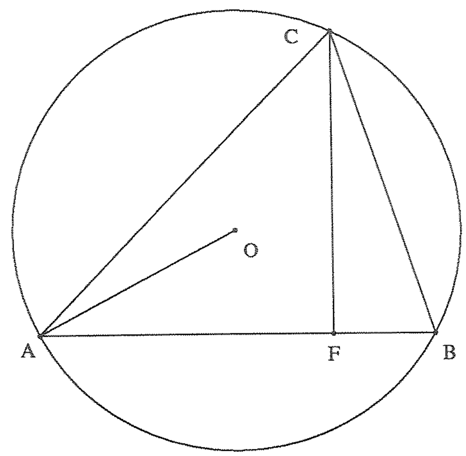

Fig. 70

(distance B O r)
(distance C O r)
(distance A C u_2)
(distance B C u_3)
non-deg x_1)

The result is: $u_2u_3 + 2ru_1 = 0$ and $u_2u_3 - 2ru_1 = 0$.

Example 71. A line is drawn through the centroid of a triangle. Find the relation among the distances of the line from the three midpoints of the triangle.

((x_1 x_2 u_1 u_2) (u_3 x_3 x_4 x_5 x_6 x_7 x_8 x_9)
(G (0 0) D_1 (x_1 0) E_1 (x_2 0) F_1 (x_3 0) D (x_1 u_1) E (x_2 u_2) F (x_3 u_3) A (x_4 x_5)
B (x_6 x_7) C (x_8 x_9))
(mid-x D B C)
(mid-y D B C)
(mid-x F A B)
(mid-y F A B)
(mid-x E A C)
(mid-y E A C)
(collinear A D G)
(collinear B E G))

The result is: $u_3 + u_2 + u_1 = 0$.

Example 72. Y is the tangent point of the incircle of a right triangle. Determine the area of the triangle in terms of AY and CY.

((u_1 u_2) (u_3 x_1 x_2 x_3 x_4)
(B (0 0) C (x_1 0) A (x_2 x_3) I (u_1 x_4))
(point-line-dis I A B x_4)
(point-line-dis I A C x_4)
(perpendicular B A C A)

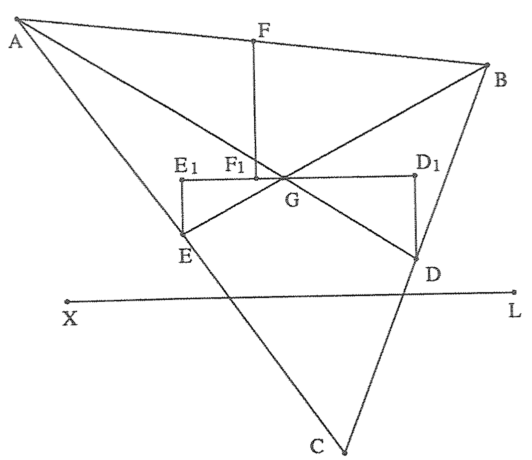

Fig. 71

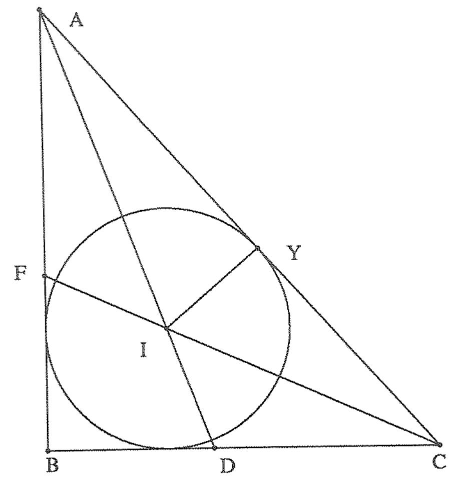

Fig. 72

(pp– u_2 (pp– x_1 u_1))
(area u_3 A B C)
non-deg x_3)

The result is: $u_3 + u_1u_2 = 0$ and $u_3 - u_1u_2 = 0$.

Example 73. If p, q, and r are the distances of a point inside a triangle ABC from the sides of the triangle, find the relation among p/h_a, q/h_b, and r/h_c.

((x_1 x_2 x_3 u_1 u_2) (u_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11} x_{12} x_{13})
(A (0 0) B (x_1 0) C (x_2 x_3) D (x_4 x_5) E (x_6 x_7) O (x_8 x_9) E_1 (x_{10} x_{11}) D_1 (x_{12} x_{13}))
(collinear D B C)
(perpendicular A D B C)
(collinear D_1 B C)
(perpendicular O D_1 B C)
(collinear E A C)
(perpendicular B E A C)
(collinear E_1 A C)
(perpendicular O E_1 A C)
(pp– (pp* u_1 x_3) x_9)
(x-ratio u_2 O D_1 A D)
(x-ratio u_3 O E_1 B E))

The result is: $u_3 + u_2 + u_1 - 1 = 0$.

Example 74. Find the relation among the two segments into which a side of a triangle is divided by The corresponding vertex of the orthic triangle and the sides of the orthic triangle passing through the vertex considered.

((u_1 u_2 u_3) (u_4 x_1 x_2 x_3 x_4 x_5)
(D (0 0) B (u_1 0) C (u_2 0) A (0 x_1) E (x_2 x_3) F (x_4 x_5))
(collinear A B F)
(perpendicular C F A B)

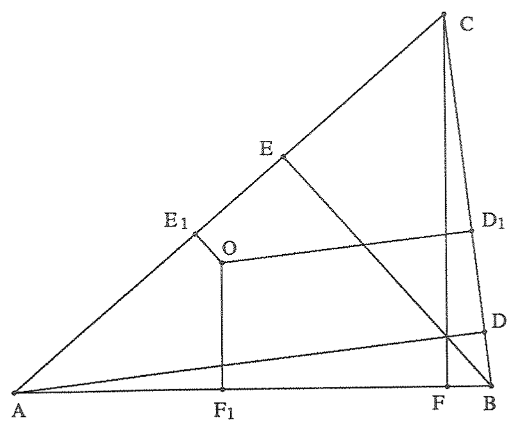

Fig. 73

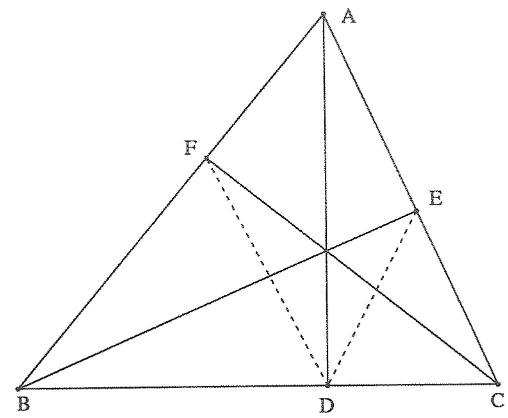

Fig. 74

(collinear A E C)
(perpendicular B E A C)
(distance F D u_3)
(distance E D u_4))

The result is: $u_3u_4 + u_1u_2 = 0$ and $u_3u_4 - u_1u_2 = 0$.

Example 75. Find the relation among the tangent from a vertex of a triangle to the nine-point circle; the altitude issued from that vertex, and the distance of the opposite side from the circumcenter.

((x_1 u_1 u_2) (u_3 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9)
(B (0 0) C (x_1 0) A (x_2 u_1) A_1 (x_3 0) B_1 (x_4 x_5) C_1 (x_6 x_7) N (x_8 x_9) O (x_3 u_2))
(mid-x A_1 B C)
(mid-x B_1 A C)
(mid-y B_1 A C)
(mid-x C_1 A B)
(mid-y C_1 A B)
(eqdistant N A_1 N B_1)
(eqdistant N A_1 N C_1)
(eqdistant O B O A)
(pp– (pp* u_3 u_3) (eqdistant A N N A_1)))

The result is: $u_3^2 - u_1u_2 = 0$.

Example 76. The parallels through the vertices A, B, and C of the triangle ABC to the medians of this triangle issued from the vertices B, C, and A, respectively, form a triangle $A_1B_1C_1$. Find the relation between the areas of the two triangles.

((x_1 x_2 u_1) (u_2 y_2 x_3 x_4 x_5 x_6 x_7 x_{11} x_{12} x_{13} x_{14} x_{15} x_{16})
(B (0 0) C (x_1 0) A (x_2 y_2) A_1 (x_3 0) B_1 (x_4 x_5) C_1 (x_6 x_7) A_2 (x_{11} x_{12}) B_2 (x_{13} x_{14})
C_2 (x_{15} x_{16}))
(mid-x A_1 B C)

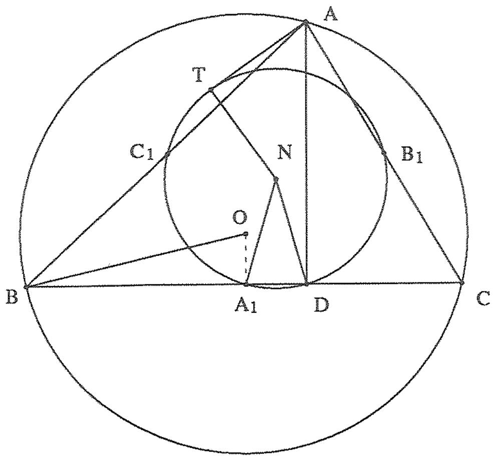

Fig. 75

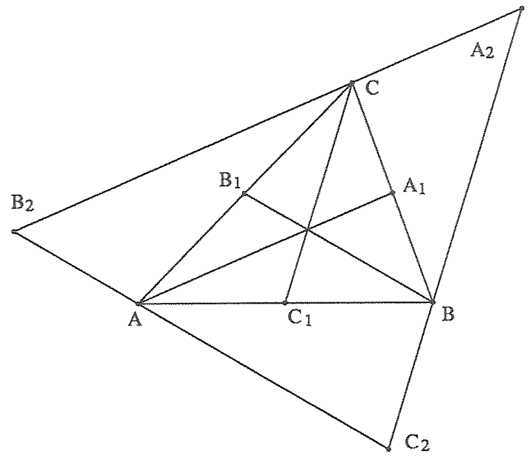

Fig. 76

(mid-x B_1 A C)
(mid-y B_1 A C)
(mid-x C_1 A B)
(mid-y C_1 A B)
(collinear A B_2 C_2)
(parallel A B_2 B B_1)
(collinear B C_2 A_2)
(parallel B A_2 C C_1)
(collinear C A_2 B_2)
(parallel C A_2 A A_1)
(area u_1 A B C)
(area u_2 A_2 B_2 C_2))

The result is: $u_2 - 3u_1 = 0$.

Example 77. E and F are the middle points of the diagonal of a quadrilateral $ABCD$. Find the relation among the six sides of the quadrilateral and EF.

((x_0 x_1 u_1 u_2 u_3) (u_4 x_2 x_3 x_4 x_5 x_6 x_7 x_8)
(A (0 0) B (x_0 0) C (x_1 x_2) D (x_3 x_4) E (x_5 x_6) F (x_7 x_8))
(mid-x E A C)
(mid-y E A C)
(mid-x F D B)
(mid-y F D B)
(pp– u_1 (sq-distance A B) (sq-distance B C) (sq-distance C D) (sq-distance D A))
(distance D B u_2)
(distance C A u_3)
(distance E F u_4))

The result is: $4u_4^2 + u_3^2 + u_2^2 - u_1 = 0$.

Example 78. Find the relation among the diagonals of a quadrilateral and the two lines joining the midpoints of the two pairs of opposite sides of the quadrilateral.

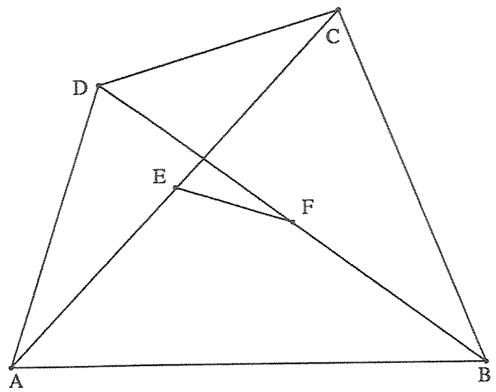

Fig. 77

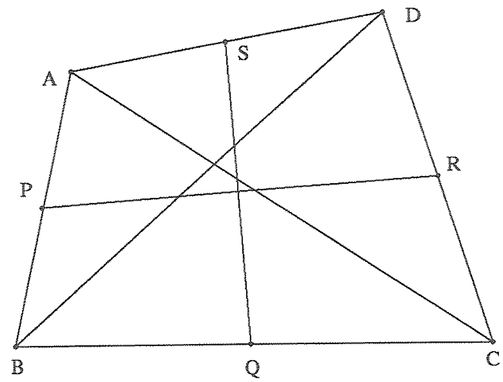

Fig. 78

$((x_0\ x_1\ u_1\ u_2\ u_3)\ (u_4\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}\ x_{11})$
$(B\ (0\ 0)\ C\ (x_0\ 0)\ D\ (x_1\ x_2)\ A\ (x_3\ x_4)\ Q\ (x_5\ 0)\ R\ (x_6\ x_7)\ S\ (x_8\ x_9)\ P\ (x_{10}\ x_{11}))$
(mid-x $Q\ B\ C$)
(mid-x $R\ D\ C$)
(mid-y $R\ D\ C$)
(mid-x $S\ A\ D$)
(mid-y $S\ A\ D$)
(mid-x $P\ A\ B$)
(mid-y $P\ A\ B$)
(distance $B\ D\ u_1$)
(distance $A\ C\ u_2$)
(distance $P\ R\ u_3$)
(distance $S\ Q\ u_4$))

The result is: $2u_4^2 + 2u_3^2 - u_2^2 - u_1^2 = 0$.

Example 79. A line through the centroid G of the triangle ABC meets AB in M and AC in N; find the relation among AN, MB, AM, and NC.

$((x_1\ x_2\ x_3\ u_1)\ (u_2\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10}\ x_{11}\ x_{12})$
$(B\ (0\ 0)\ C\ (x_1\ 0)\ A\ (x_2\ x_3)\ D\ (x_4\ 0)\ F\ (x_5\ x_6)\ G\ (x_7\ x_8)\ M\ (x_9\ x_{10})\ N\ (x_{11}\ x_{12}))$
(mid-x $D\ B\ C$)
(mid-x $F\ B\ A$)
(mid-y $F\ B\ A$)
(collinear $A\ G\ D$)
(collinear $C\ G\ F$)
(collinear $A\ B\ M$)
(collinear $A\ C\ N$)
(collinear $M\ N\ G$)
(x-ratio $u_1\ M\ B\ A\ M$)
(x-ratio $u_2\ N\ C\ A\ N$))

The result is: $u_2 + u_1 - 1 = 0$.

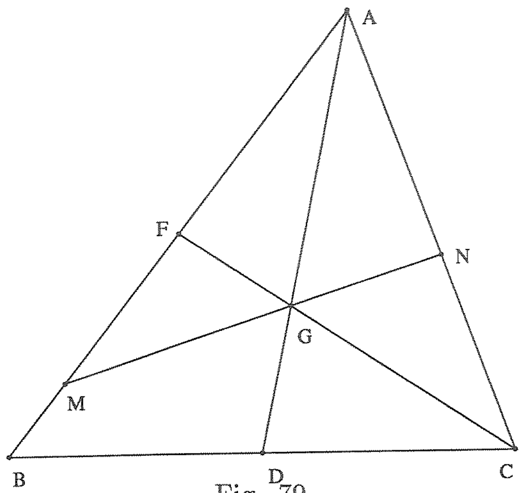

Fig. 79

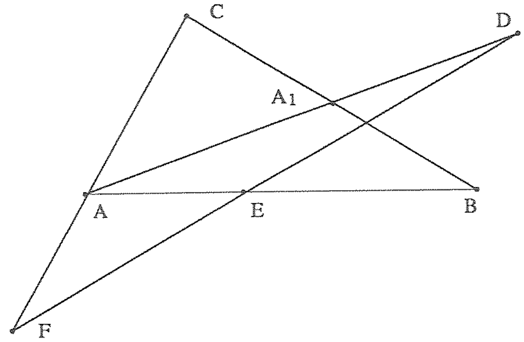

Fig. 80

Example 80. Two equal segments AE and AF are taken on the sides AB and AC of the triangle ABC. The median issued from A intersects EF at D. Find the relation among ED, FD, AB, and AC.

((x3 u1 u2 u3) (u4 x1 x2 x4 x5 x6 x7 x8 x9 x10)
(F (x1 0) E (x2 0) A (0 x3) C (x4 x5) B (x6 x7) A1 (x8 x9) D (x10 0))
(mid-x A F E)
(collinear F A C)
(collinear A E B)
(mid-x A1 C B)
(mid-y A1 C B)
(collinear A A1 D)
(pp– u1 (pp– x10 x2))
(pp– u2 (pp– x10 x1))
(distance A B u3)
(distance A C u4))

The result is: $u_2u_4 + u_1u_3 = 0$ and $u_2u_4 - u_1u_3 = 0$.

Example 81. The parallels to the sides of a triangle ABC through the same point, M, meet the respective medians in the points P, Q, and R. Find the relation among GP/GA, GQ/GB, and GR/GC.

((x1 x2 x3 u1 u2) (u3 x4 x5 x6 x7 x8 x9 x10 x11 x12 x13 x14 x15 x16 x17)
(A (0 0) B (x1 0) C (x2 x3) F (x4 0) D (x5 x6) E (x7 x8) G (x9 x10) M (x11 x12)
R (x13 x12) P (x14 x15) Q (x16 x17))
(mid-x F A B)
(mid-x D B C)
(mid-y D B C)
(mid-x E A C)
(mid-y E A C)
(collinear C G F)
(collinear A G D)

(collinear R C F)
(collinear P A D)
(parallel M P B C)
(collinear Q B E)
(parallel M Q A C)
(x-ratio u_1 G P G A)
(x-ratio u_2 G Q G B)
(x-ratio u_3 G R G C))

The result is: $u_3 + u_2 + u_1 = 0$.

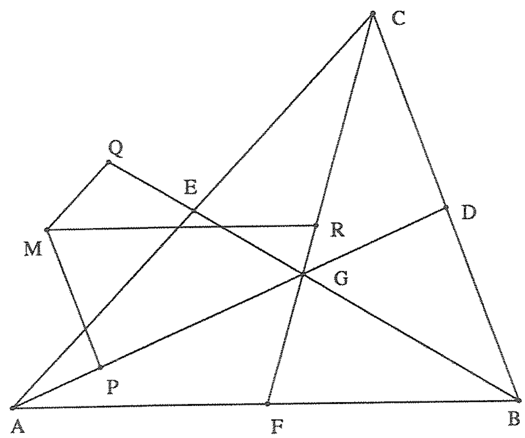

Fig. 81

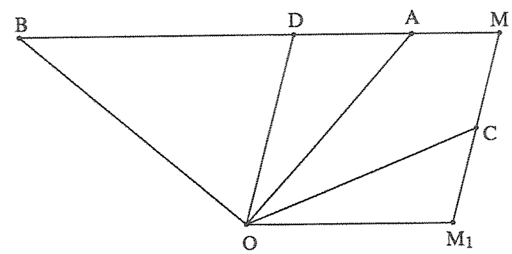

Fig. 82

Example 82. Given the parallelogram $MDOM_1$, the vertex O is joined to the midpoint C of MM_1. If the internal and external bisectors of the angle COD meet MD in A and B, find the relation among MD, MA, and MB.

((x_1 x_2 u_1) (u_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11})
(O (0 0) B (0 x_1) A (x_2 0) D (x_3 x_4) M (x_5 x_6) M_1 (x_7 x_8) D_1 (x_3 x_9) C (x_{10} x_{11}))
(pp+ x_9 x_4)
(collinear B A D)
(collinear B A M)
(parallel M M_1 O D)
(parallel B A O M_1)
(mid-x C M M_1)
(mid-y C M M_1)
(collinear O D_1 C)
(x-ratio u_1 M D M A)
(x-ratio u_2 M D M B)
non-deg (pp− x_5 x_3) (pp− x_5 x_2) x_5)

The result is: $u_1u_2 - 1 = 0$.

Example 83. In the parallelogram $ABCD$, AE is drawn parallel to BD; find the cross ratio

$A(EC, BD)$.

((x1 x2 x3) (u1 x4 x5 x6 x7 x8 x9)
(A (0 0) B (x1 0) C (x2 x3) D (x4 x3) E (x5 x3) F (x6 x7) G (x8 x9))
(parallel A D B C)
(parallel A E D B)
(collinear F A D)
(collinear F E B)
(collinear G A C)
(collinear G E B)
(c–ratio u1 E G F B))

The result is: $u_1 + 1 = 0$.

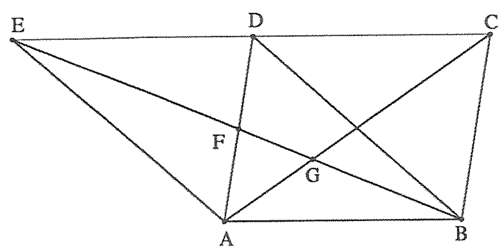

Fig. 83

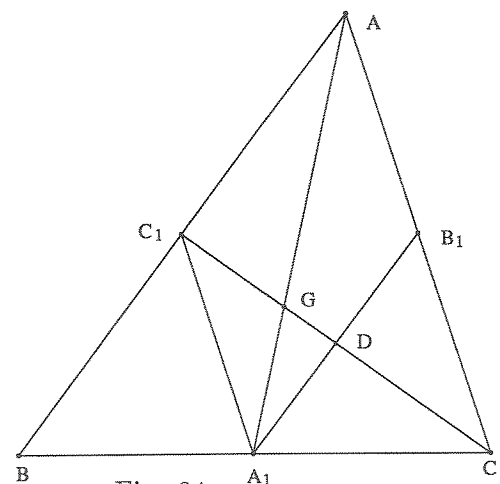

Fig. 84

Example 84. If A_1, B_1, and C_1 are the midpoints of the sides of the triangle ABC, find the cross ratio $A_1(C_1B_1, AC)$.

((x1 x2 x3) (u1 x4 x5 x6 x7 x8 x9 x10 x11 x12)
(B (0 0) C (x1 0) A (x2 x3) A1 (x4 0) B1 (x5 x6) C1 (x7 x8) D (x9 x10) G (x11 x12))
(mid-x A1 B C)
(mid-x B1 A C)
(mid-y B1 A C)
(mid-x C1 A B)
(mid-y C1 A B)
(collinear G A A1)
(collinear G C1 C)
(collinear D A1 B1)
(collinear D C C1)
(c–ratio u1 C1 D G C))

The result is: $u_1 + 1 = 0$.

Example 85. With the usual notations for the triangle ABC, if DF meets BE in K, find the cross ratio (BH, KE).

$((x_1\ x_2\ x_3)\ (u_1\ x_4\ x_5\ x_6\ x_7\ x_8\ x_9\ x_{10})$
$(A\ (0\ 0)\ B\ (x_1\ 0)\ C\ (x_2\ x_3)\ F\ (x_2\ 0)\ D\ (x_4\ x_5)\ H\ (x_2\ x_6)\ E\ (x_7\ x_8)\ K\ (x_9\ x_{10}))$
(collinear $D\ B\ C$)
(perpendicular $A\ D\ B\ C$)
(collinear $H\ A\ D$)
(collinear $K\ D\ F$)
(collinear $K\ H\ B$)
(collinear $E\ A\ C$)
(collinear $E\ B\ H$)
(c–ratio $u_1\ E\ K\ H\ B$))

The result is: $u_1 + 1 = 0$.

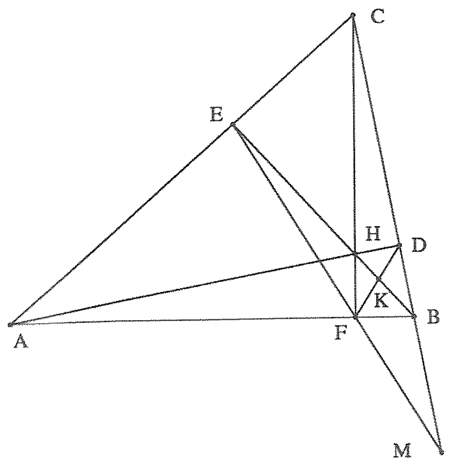

Fig. 85

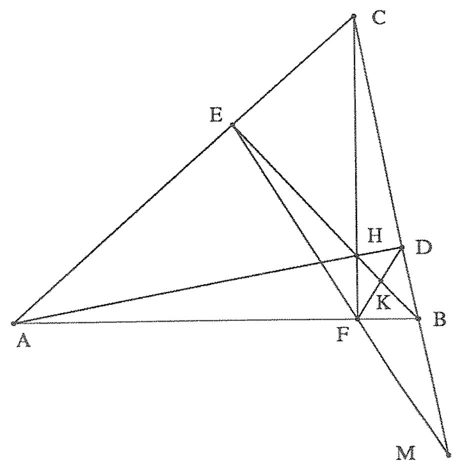

Fig. 86

Example 86. The same as example 85. If EF meets BC in M, find (BC, DM).

$((x_1\ x_2\ x_3\ x_6\ x_9\ x_{10})\ (u_1\ x_4\ x_5\ x_7\ x_8\ x_{11}\ x_{12})$
$(A\ (0\ 0)\ B\ (x_1\ 0)\ C\ (x_2\ x_3)\ F\ (x_2\ 0)\ D\ (x_4\ x_5)\ H\ (x_2\ x_6)\ E\ (x_7\ x_8)\ K\ (x_9\ x_{10})$
$M\ (x_{11}\ x_{12}))$
(collinear $D\ B\ C$)
(perpendicular $A\ D\ B\ C$)
(collinear $E\ A\ C$)
(perpendicular $B\ E\ A\ C$)
(collinear $M\ E\ F$)
(collinear $M\ B\ C$)
(c–ratio $u_1\ C\ B\ D\ M$))

The result is: $u_1 + 1 = 0$.

Example 87. The tangent to a circle at the point C meets the diameter AB, produced, in T_1; find the cross ratio $C(DT_1, AB)$ where T_1D is the other tangent from T_1 to the circle.

((x1 x3) (u1 x2 x4 x5 x6 x7 x8 x9 x10)
(A (x1 0) B (x2 0) O (0 0) C (x3 x4) D (x3 x5) T1 (x6 0) A1 (x7 x8) B1 (x9 x10))
(eqdistant C O A O)
(mid-y O C D)
(mid-x O A B)
(perpendicular T1 C C O)
(collinear A1 T1 D)
(collinear A1 A C)
(collinear B1 T1 D)
(collinear B1 C B)
(c–ratio u1 B1 A1 T1 D))

The result is: $u_1 + 1 = 0$.

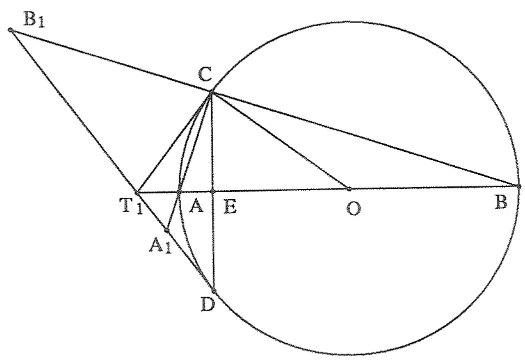

Fig. 87

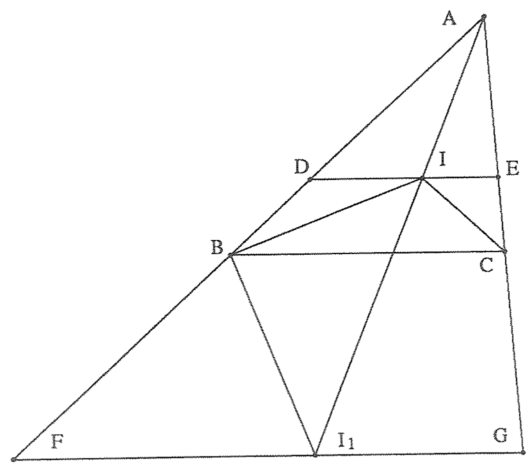

Fig. 88

Example 88. The sides AB, AC intercept the segments DE, FG on the parallels to the side BC through the tritangent centers I and I_1. Find the relation among BC, DE, and FG.

((x1 x2 u1) (u2 x3 x4 x5 x6 x7 x8 x10 x11 x12)
(B (0 0) C (x1 0) A (x2 x3) I (x4 x5) I1 (x6 x7) D (x8 x5) E (x10 x5) F (x11 x7)
G (x12 x7))
(point-line-dis I A B x5)
(point-line-dis I A C x5)
(perpendicular I B B I1)
(collinear A I I1)
(collinear A B D)
(collinear A B F)
(collinear A C E)
(collinear A C G)
(x-ratio u1 B C D E)
(x-ratio u2 B C F G)
non-deg x3)

The result is: $u_2 + u_1 - 2 = 0$.

Example 89. A secant through the vertex A of the parallelogram $ABCD$ meets the diagonal BD and the sides BC, CD in the points E, F and G. Find the relation among AE, AF, and AG.

((x₁ x₂ x₃ u₁) (u₂ x₄ x₅ x₆ x₇ x₉ x₁₀)

$((x_1\ x_2\ x_3\ u_1)\ (u_2\ x_4\ x_5\ x_6\ x_7\ x_9\ x_{10})$
$(A\ (0\ 0)\ B\ (x_1\ 0)\ C\ (x_2\ x_3)\ D\ (x_4\ x_3)\ E\ (x_5\ x_6)\ F\ (x_7\ x_3)\ G\ (x_9\ x_{10}))$
(parallel $A\ D\ B\ C$)
(collinear $E\ B\ D$)
(collinear $G\ C\ B$)
(collinear $A\ E\ F$)
(collinear $A\ F\ G$)
(x-ratio $u_1\ A\ E\ A\ F$)
(x-ratio $u_2\ A\ E\ A\ G$)
non-deg $x_5\ x_7\ x_9$)

The result is: $u_2 + u_1 - 1 = 0$.

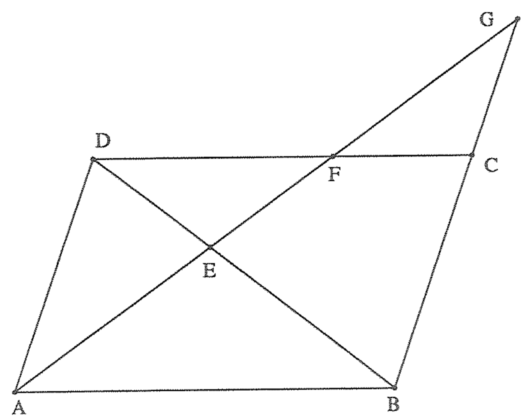

Fig. 89

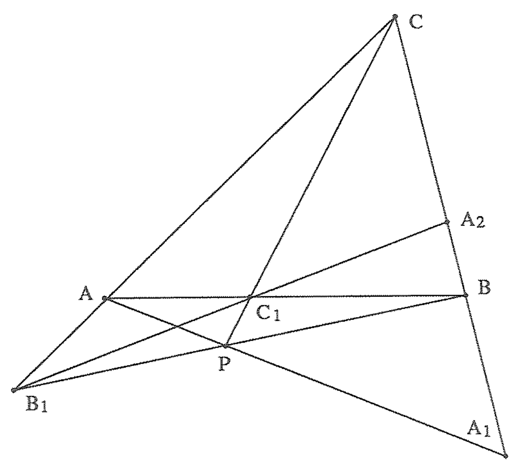

Fig. 90

Example 90. In figure 90, find (BC, A_1A_2).

$((x_1\ x_2\ x_3\ x_4\ x_5)\ (u_1\ x_6\ x_7\ x_8\ x_9\ x_{10}\ x_{11})$
$(B\ (0\ 0)\ C\ (x_1\ 0)\ A\ (x_2\ x_3)\ P\ (x_4\ x_5)\ A_1\ (x_6\ 0)\ B_1\ (x_7\ x_8)\ C_1\ (x_9\ x_{10})\ A_2\ (x_{11}\ 0))$
(collinear $A\ P\ A_1$)
(collinear $B_1\ A\ C$)
(collinear $B_1\ P\ B$)
(collinear $C_1\ A\ B$)
(collinear $C_1\ P\ C$)
(collinear $B_1\ C_1\ A_2$)
(c–ratio $u_1\ C\ B\ A_2\ A_1$))

The result is: $u_1 + 1 = 0$.

Example 91. From a point P on the line joining the two common points A and B of two circles (O) and (O_1) two secants PCE and PFD are drawn to the circles respectively. Find the relation among PC, PE, PF, and PD.

$((x_1\ x_3\ x_7\ u_1\ u_2\ u_3)\ (u_4\ x_2\ x_4\ x_5\ x_6\ x_8\ x_9\ x_{10})$
$(A\ (0\ 0)\ B\ (x_1\ 0)\ P\ (x_2\ 0)\ E\ (x_3\ x_4)\ C\ (x_5\ x_6)\ F\ (x_7\ x_8)\ D\ (x_9\ x_{10}))$
(cocyclic $A\ B\ C\ E$)
(cocyclic $A\ B\ F\ D$)
(collinear $P\ C\ E$)
(collinear $P\ F\ D$)
(distance $P\ C\ u_1$)
(distance $P\ E\ u_2$)
(distance $P\ F\ u_3$)
(distance $P\ D\ u_4$)
non-deg $u_4\ x_1\ x_2\ x_4\ x_6\ x_8\ x_{10}$ (pp− $x_5\ x_3$) (pp− $x_9\ x_7$) (pp− $u_4\ u_3$) (pp+ $u_3\ u_4$)
(pp− $x_3\ x_2$) (pp− $x_5\ x_2$) (pp− $x_7\ x_2$) (pp− $x_9\ x_2$))

The result is: $u_3u_4 + u_1u_2 = 0$ and $u_3u_4 - u_1u_2 = 0$.

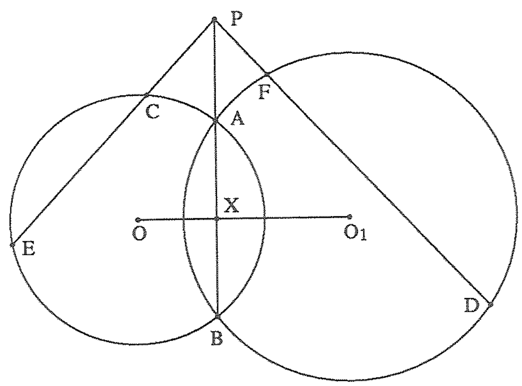

Fig. 91

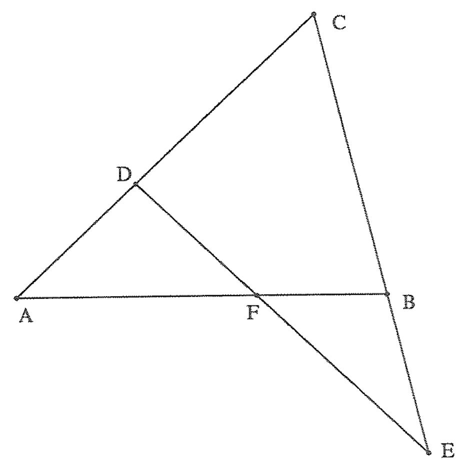

Fig. 92

Example 92. Let D and E be two points on two sides AC and BC of triangle ABC such that $AD = BE$. $F = DE \cap AB$. Find the relation among FD, AC, EF, and BC.

$((x_2\ x_4\ u_1\ u_2)\ (u_4\ x_3\ x_5\ x_6\ x_7\ x_8)$
$(A\ (0\ 0)\ C\ (u_4\ 0)\ B\ (x_2\ x_3)\ D\ (x_4\ 0)\ F\ (x_5\ x_6)\ E\ (x_7\ x_8))$
(collinear $A\ F\ B$)
(collinear $C\ B\ E$)
(collinear $D\ F\ E$)
(eqdistant $A\ D\ B\ E$)
(x-ratio $u_1\ F\ D\ E\ F$)
(distance $B\ C\ u_2$)
non-deg (pp− $x_5\ x_4$) (pp− $x_7\ x_5$) x_3)

The result is: $u_1u_4 - u_2 = 0$ and $u_1u_4 + u_2 = 0$.

Example 93. I and I_1 are the two tritangent centers. Find the cross ratio (AD, II_1).

$((x_1\ x_2\ x_3)\ (r\ x_4\ x_5\ x_6\ x_7\ x_8)$
$(B\ (0\ 0)\ C\ (x_1\ 0)\ A\ (x_2\ x_3)\ I\ (x_4\ x_5)\ D\ (x_6\ 0)\ I_1\ (x_7\ x_8))$
(point-line-dis $I\ A\ B\ x_5$)
(point-line-dis $I\ A\ C\ x_5$)
(collinear $A\ I\ D$)
(collinear $A\ I\ I_1$)
(perpendicular $B\ I\ B\ I_1$)
(c–ratio $r\ A\ D\ I\ I_1$))

The result is: $r + 1 = 0$.

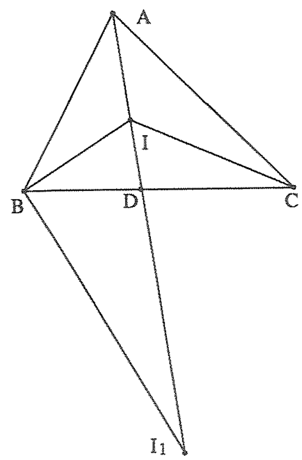

Fig. 93

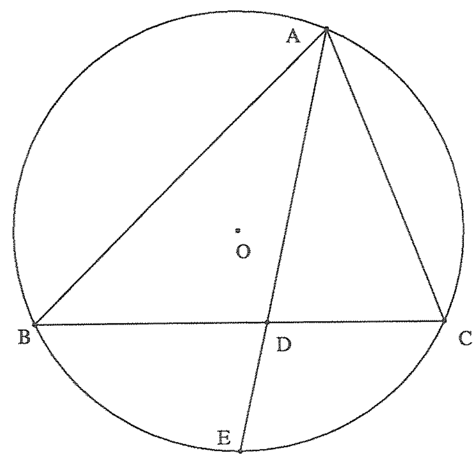

Fig. 94

Example 94. Let D be the intersection of one of the bisectors of $\angle A$ of triangle ABC with side BC, E be the intersection of AD with the circumcircle of ABC. Find the relation among AB, AC, AD, and AE.

$((u_1\ u_2\ u_3)\ (u_4\ x_1\ x_2\ x_3\ x_4\ x_5)$
$(A\ (0\ 0)\ C\ (x_1\ x_2)\ C_1\ (x_1\ x_3)\ B\ (x_4\ x_5)\ D\ (u_1\ 0)\ E\ (u_2\ 0))$
(pp+ $x_3\ x_2$)
(collinear $A\ C_1\ B$)
(collinear $B\ C\ D$)
(cocyclic $A\ B\ E\ C$)
(distance $A\ B\ u_3$)
(distance $A\ C\ u_4$)
non-deg $x_1\ x_2$)

The result is: $u_3u_4 + u_1u_2 = 0$ and $u_3u_4 - u_1u_2 = 0$.

Example 95. In a cyclic orthodiagonal quadrilateral, find the relation between the distance of a side from the circumcenter of the quadrilateral and the opposite side.

$((x_1\ x_2\ u_1)\ (u_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8)$
$(A\ (0\ 0)\ C\ (x_1\ 0)\ B\ (x_2\ x_3)\ D\ (x_2\ x_4)\ O\ (x_5\ x_6)\ P\ (x_7\ x_8))$
(eqdistant $O\ A\ O\ C$)

(eqdistant O A O B)
(eqdistant O A O D)
(mid-x P A B)
(mid-y P A B)
(distance P O u_1)
(distance C D u_2)
non-deg (pp– x_4 x_3))

The result is: $u_2 - 2u_1 = 0$ and $u_2 + 2u_1 = 0$.

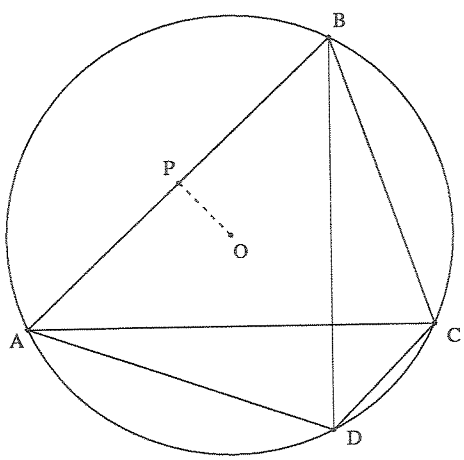

Fig. 95

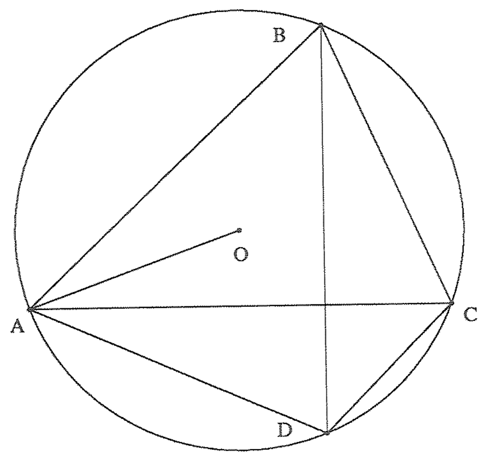

Fig. 96

Example 96. If a quadrilateral is both cyclic and orthodiagonal, find the relation among the two opposite sides and the circumdiameter of the quadrilateral.

((x_1 u_1 u_2) (u_3 x_2 x_3 x_4 x_5 x_6)
(A (0 0) C (x_1 0) B (x_2 x_3) D (x_2 x_4) O (x_5 x_6))
(eqdistant O A O C)
(eqdistant O A O B)
(eqdistant O A O D)
(distance A B u_2)
(distance C D u_3)
(distance O A u_1)
non-deg (pp– x_4 x_3))

The result is: $u_3^2 + u_2^2 - 4u_1^2 = 0$.

Example 97. The lines AP, BQ, CR through the vertices of a triangle ABC parallel, respectively, to the lines OA_1, OB_1 and OC_1 joining any point O to the points A_1, B_1 and C_1 marked in any manner whatever, on the sides of BC, CA and AB meet these sides in the points P, Q, R. Find the relation among OA_1/AP, OB_1/BQ and OC_1/CR.

((x_1 x_2 x_3 x_6 x_8 x_{10} u_1 u_2) (u_3 x_4 x_5 x_9 x_{11} x_{12} x_{13} x_{14} x_{15} x_{16})
(A (0 0) B (x_1 0) C (x_2 x_3) O (x_4 x_5) C_1 (x_6 0) A_1 (x_8 x_9) B_1 (x_{10} x_{11}) R (x_{12} 0)
P (x_{13} x_{14}) Q (x_{15} x_{16}))

(collinear P B C)
(collinear A_1 B C)
(collinear Q A C)
(collinear B_1 A C)
(parallel A P O A_1)
(parallel B Q O B_1)
(parallel C R O C_1)
(x-ratio u_1 O A_1 A P)
(x-ratio u_2 O B_1 B Q)
(x-ratio u_3 O C_1 C R)
non-deg x_{13} (pp– x_6 x_4) (pp– x_8 x_4) (pp– x_{10} x_4))

The result is: $u_3 + u_2 + u_1 - 1 = 0$.

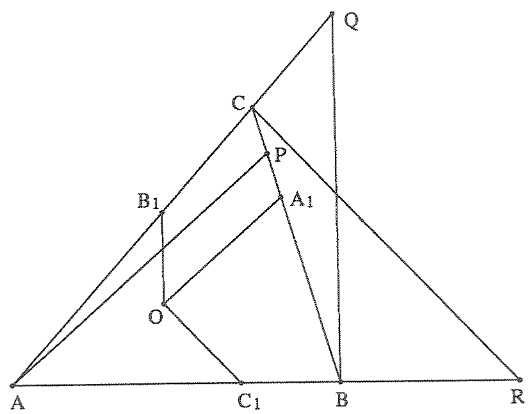

Fig. 97

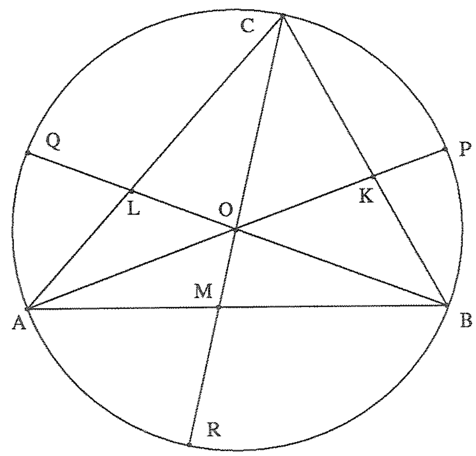

Fig. 98

Example 98. The circumdiameters AP, BQ and CR of a triangle ABC meet the sides BC, CA and AB in the points K, L amd M. Find the relation among KP/AK, LQ/BL, and MR/AM.

((x_1 u_1 u_2) (u_3 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11} x_{12} x_{13} x_{14} x_{15})
(O (0 0) A (x_1 0) P (x_2 0) B (x_3 x_4) Q (x_5 x_6) C (x_7 x_8) R (x_9 x_{10}) K (x_{11} 0)
M (x_{12} x_{13}) L (x_{14} x_{15}))
(mid-x O A P)
(eqdistant B O A O)
(mid-x O B Q)
(mid-y O B Q)
(eqdistant C O A O)
(mid-x O C R)
(mid-y O C R)
(collinear K B C)
(collinear M A B)
(collinear M O C)
(collinear L A C)

(collinear L O B)
(x-ratio u_1 K P A K)
(x-ratio u_2 L Q B L)
(x-ratio u_3 M R C M)
non-deg (pp– x_{11} x_2) (pp– x_{11} x_1) (pp– x_{14} x_5) (pp– x_{14} x_3) (pp– x_{12} x_9)
(pp– x_{12} x_7) (pp– x_7 x_1) (pp– x_3 x_1) x_4 x_8 (pp– x_7 x_3))

The result is: $u_3 + u_2 + u_1 - 1 = 0$.

Example 99. Find the relation between the cross ratios (AB, CD) and (AB, DC).

((x_1 x_2 x_3 u_1) (u_2 x_4)
(A (x_1 0) B (x_2 0) C (x_3 0) D (x_4 0))
(c–ratio u_1 A B C D)
(c–ratio u_2 A B D C))

The result is: $u_1 u_2 - 1 = 0$.

Fig. 99

Fig. 100

Example 100. Find the relation between the cross ratios (AB, CD) and (AC, BD).

((x_1 x_2 x_3 u_1) (u_2 x_4)
(A (x_1 0) B (x_2 0) C (x_3 0) D (x_4 0))
(c–ratio u_1 A B C D)
(c–ratio u_2 A C B D))

The result is: $u_2 + u_1 - 1 = 0$.

Example 101. Find the relation between the cross ratios (AB, CD) and (AC, DB).

((x_1 x_2 x_3 u_1) (u_2 x_4)
(A (x_1 0) B (x_2 0) C (x_3 0) D (x_4 0))
(c–ratio u_1 A B C D)
(c–ratio u_2 A C D B))

The result is: $(u_1 - 1)u_2 + 1 = 0$.

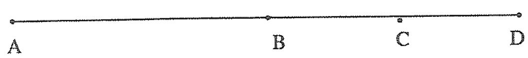

Fig. 101

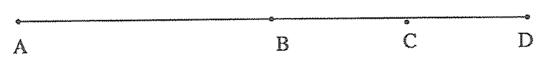

Fig. 102

Example 102. Find the relation between the cross ratios (AB, CD) and (AD, BC).

$((x_1\ x_2\ x_3\ u_1)\ (u_2\ x_4)$
$(A\ (x_1\ 0)\ B\ (x_2\ 0)\ C\ (x_3\ 0)\ D\ (x_4\ 0))$
(c–ratio u_1 $A\ B\ C\ D$)
(c–ratio u_2 $A\ D\ B\ C$))

The result is: $u_1 u_2 - u_1 + 1 = 0$.

Example 103. Find the relation between the cross ratios (AB, CD) and (AD, CB).

$((x_1\ x_2\ x_3\ u_1)\ (u_2\ x_4)$
$(A\ (x_1\ 0)\ B\ (x_2\ 0)\ C\ (x_3\ 0)\ D\ (x_4\ 0))$
(c–ratio u_1 $A\ B\ C\ D$)
(c–ratio u_2 $A\ D\ C\ B$))

The result is: $(u_1 - 1)u_2 - u_1 = 0$.

Example 104. Find the relation between the cross ratios (AB, CD) and (BA, DC).

$((x_1\ x_2\ x_3\ u_1)\ (u_2\ x_4)$
$(A\ (x_1\ 0)\ B\ (x_2\ 0)\ C\ (x_3\ 0)\ D\ (x_4\ 0))$
(c–ratio u_1 $A\ B\ C\ D$)
(c–ratio u_2 $B\ A\ D\ C$))

The result is: $u_2 - u_1 = 0$.

Example 105. Determine the radius of a circle by the four sides of an inscribed quadrilateral.

$((u_1\ u_2\ u_3\ u_4)\ (r\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6)$
$(A\ (0\ 0)\ B\ (u_1\ 0)\ C\ (x_1\ x_2)\ D\ (x_3\ x_4)\ O\ (x_5\ x_6))$

Fig. 103

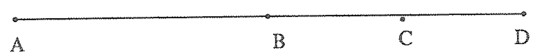

Fig. 104

(distance B C u_2)
(distance C D u_3)
(distance D A u_4)
(mid-x O A B)
(distance A O r)
(distance C O r)
(cocyclic A B C D)
non-deg x_3 x_5)

The result is: $(u_4^4 - (2u_3^2 + 2u_2^2 + 2u_1^2)u_4^2 - 8u_1u_2u_3u_4 + u_3^4 - (2u_2^2 + 2u_1^2)u_3^2 + u_2^4 - 2u_1^2u_2^2 + u_1^4)r^2 + u_1u_2u_3u_4^3 + ((u_2^2 + u_1^2)u_3^2 + u_1^2u_2^2)u_4^2 + (u_1u_2u_3^3 + (u_1u_2^3 + u_1^3u_2)u_3)u_4 + u_1^2u_2^2u_3^2 = 0$ and $(u_4^4 - (2u_3^2 + 2u_2^2 + 2u_1^2)u_4^2 + 8u_1u_2u_3u_4 + u_3^4 - (2u_2^2 + 2u_1^2)u_3^2 + u_2^4 - 2u_1^2u_2^2 + u_1^4)r^2 - u_1u_2u_3u_4^3 + ((u_2^2 + u_1^2)u_3^2 + u_1^2u_2^2)u_4^2 - (u_1u_2u_3^3 + (u_1u_2^3 + u_1^3u_2)u_3)u_4 + u_1^2u_2^2u_3^2 = 0$; or

$$r^2 = \frac{(u_3u_4 + u_1u_2)(u_2u_4 + u_1u_3)(u_1u_4 + u_2u_3)}{(u_4 + u_3 - u_2 + u_1)(-u_4 + u_3 + u_2 + u_1)(u_4 - u_3 + u_2 + u_1)(u_4 + u_3 + u_2 - u_1)}$$

and

$$r^2 = \frac{(u_3u_4 - u_1u_2)(u_2u_4 - u_1u_3)(u_1u_4 - u_2u_3)}{(u_4 + u_3 - u_2 - u_1)(u_4 - u_3 - u_2 + u_1)(u_4 - u_3 + u_2 - u_1)(u_4 + u_3 + u_2 + u_1)}.$$

Example 106. Determine the diagonal of a quadrilateral inscribed in a circle by its four sides.

((u_1 u_2 u_3 u_4) (u_5 x_1 x_2 x_3 x_4)
(A (0 0) B (u_1 0) C (x_1 x_2) D (x_3 x_4))
(distance B C u_2)
(distance C D u_3)
(distance D A u_4)
(distance A C u_5)
(cocyclic A B C D)
non-deg u_5 x_2 x_4)

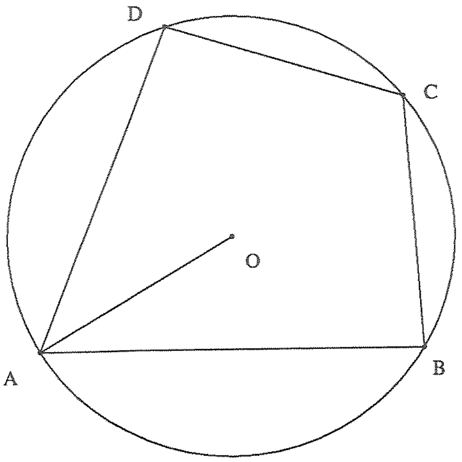

Fig. 105

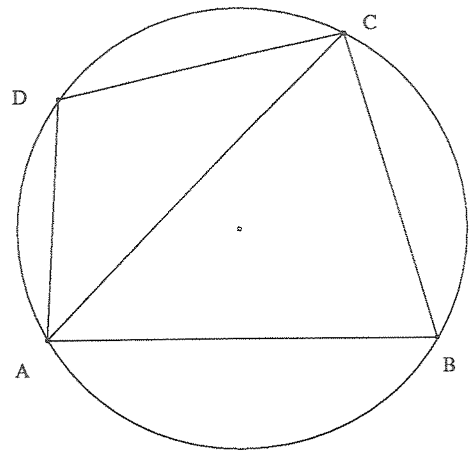

Fig. 106

The result is: $(u_3u_4 + u_1u_2)u_5^2 - u_1u_2u_4^2 + ((-u_2^2 - u_1^2)u_3)u_4 - u_1u_2u_3^2 = 0$ and $(u_3u_4 - u_1u_2)u_5^2 + u_1u_2u_4^2 + ((-u_2^2 - u_1^2)u_3)u_4 + u_1u_2u_3^2 = 0$; or $u_5^2 = \frac{(u_2u_4+u_1u_3)(u_1u_4+u_2u_3)}{(u_3u_4+u_1u_2)}$ and $u_5^2 = \frac{(u_1u_4-u_2u_3)(u_2u_4-u_1u_3)}{(u_1u_2-u_3u_4)}$.

Example 107. Determine the area of a quadrilateral inscribed in a circle by its four sides.

$((u_1\ u_2\ u_3\ u_4)\ (k\ x_1\ x_2\ x_3\ x_4)$
$(A\ (0\ 0)\ B\ (u_1\ 0)\ C\ (x_1\ x_2)\ D\ (x_3\ x_4))$
(distance $B\ C\ u_2$)
(distance $C\ D\ u_3$)
(distance $D\ A\ u_4$)
(cocyclic $A\ B\ C\ D$)
(area $u_5\ A\ B\ C\ D$)
non-deg $u_5\ x_3\ x_4$)

The result is: $16k^2+u_4^4+(-2u_3^2-2u_2^2-2u_1^2)u_4^2-8u_1u_2u_3u_4+u_3^4+(-2u_2^2-2u_1^2)u_3^2+u_2^4-2u_1^2u_2^2+u_1^4 = 0$ and $16k^2+u_4^4+(-2u_3^2-2u_2^2-2u_1^2)u_4^2+8u_1u_2u_3u_4+u_3^4+(-2u_2^2-2u_1^2)u_3^2+u_2^4-2u_1^2u_2^2+u_1^4 = 0$; or $16k^2 = (u_4 + u_3 - u_2 + u_1)(-u_4 + u_3 + u_2 + u_1)(u_4 - u_3 + u_2 + u_1)(u_4 + u_3 + u_2 - u_1)$ and $16k^2 = (u_4 - u_3 + u_2 - u_1)(-u_4 + u_3 + u_2 - u_1)(u_4 + u_3 + u_2 + u_1)(u_4 + u_3 - u_2 - u_1)$.

Locus Problems

Example 108. Two vertices of a triangle ABC move on two fixed lines LM and LN and the three sides pass through three fixed points O, P, Q which lie on a right line. Find the locus of the third vertex.

$((u_1\ u_2\ u_3\ u_4\ u_5\ u_6\ u_7\ x)\ (y\ x_1\ y_1\ x_2\ y_2)$
$(O\ (0\ 0)\ P\ (u_1\ 0)\ Q\ (u_2\ 0)\ M\ (u_3\ 0)\ N\ (u_4\ 0)\ L\ (u_5\ u_6)\ A\ (x\ y)\ B\ (x_1\ y_1)$
$C\ (x_2\ y_2))$
(collinear $O\ B\ C$)
(collinear $L\ B\ M$)
(collinear $L\ C\ N$)

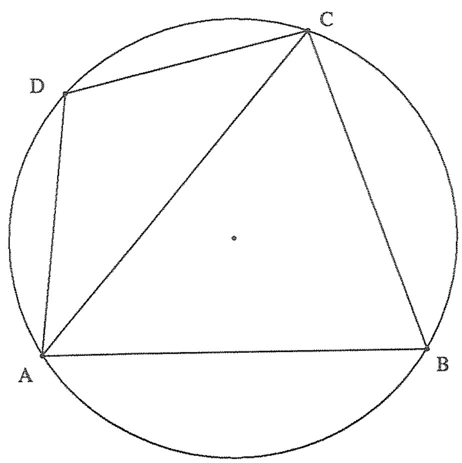

Fig. 107

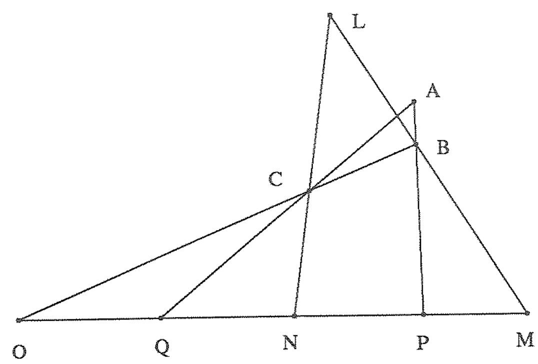

Fig. 108

(collinear A P B)
(collinear A Q C)
non-deg (pp− x_1 u_3))

The result is: $((u_1u_4 - u_2u_3)u_5 + ((u_2 - u_1)u_3 - u_1u_2)u_4 + u_1u_2u_3)y + ((-u_1u_4 + u_2u_3)u_6)x + (((-u_2 + u_1)u_3 + u_1u_2)u_4 - u_1u_2u_3)u_6 = 0$.

Example 109. The locus of a point P from which two orthogonal tangent lines to an ellipse can be drawn is a circle.

((a b x) (y x_1 y_1 x_2 y_2)
(P (x y) T_1 (x_1 y_1) T_2 (x_2 y_2))
(pp− (pp+ (pp∗ a a x_1 x_1) (pp∗ b b y_1 y_1)) 1)
(pp− (pp+ (pp∗ a a x x_1) (pp∗ b b y y_1)) 1)
(pp− (pp+ (pp∗ a a x_2 x_2) (pp∗ b b y_2 y_2)) 1)
(pp− (pp+ (pp∗ a a x x_2) (pp∗ b b y y_2)) 1)
(perpendicular P T_1 P T_2)
non-deg (pp− x_2 x_1) (pp− y_2 y_1))

The result is: $a^2b^2y^2 + a^2b^2x^2 - b^2 - a^2 = 0$.

Example 110. The locus of the middle points of chords, parallel to a given line of a conic is a straight line.

((k a b x) (y z x_1 y_1 x_2 y_2)
(I_1 (x_1 y_1) I_2 (x_2 y_2) M (x y))
(pp− (pp+ (pp∗ a a x_1 x_1) (pp∗ b b y_1 y_1)) 1)
(pp− (pp+ (pp∗ a a x_2 x_2) (pp∗ b b y_2 y_2)) 1)
(pp+ y_1 (pp∗ k x_1) z)
(pp+ y_2 (pp∗ k x_2) z)
(mid-x M I_1 I_2)
(mid-y M I_1 I_2)
non-deg (pp− x_2 x_1) (pp− y_2 y_1))

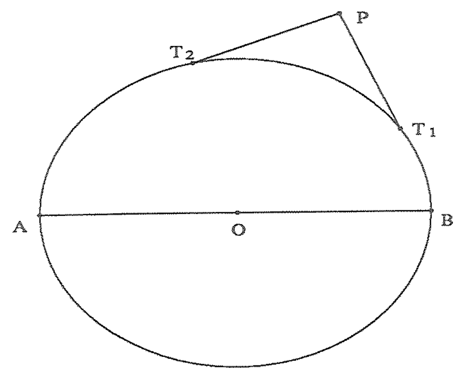

Fig. 109

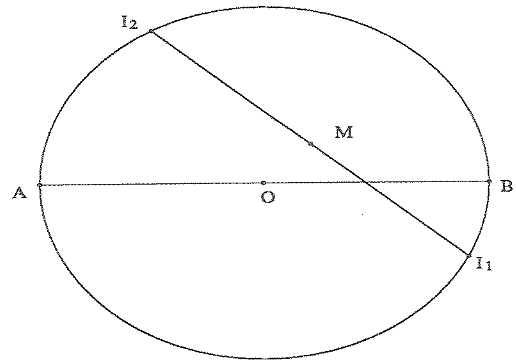

Fig. 110

The result is: $kb^2y - a^2x = 0$.

Example 111. The locus of the center of rectangular hyperbolas passing through points A_1, B_1, C_1 is the nine point circle of triangle $A_1B_1C_1$.

```
((x0 x1 y1 x) (y a b c)
 (A1 (0 0) B1 (x0 0) C1 (x1 y1) P (x y))
 (pp+ (pp* y1 y1) (pp* -1 x1 x1) (pp* a x1 y1) (pp* b x1) (pp* c y1))
 (pp+ (pp* -1 x0 x0) (pp* b x0))
 (pp+ (pp* -2 x) (pp* a y) (pp* b))
 (pp+ (pp* 2 y) (pp* a x) (pp* c)))
```

The result is: $2y_1y^2 + (-y_1^2 + x_1^2 - x_0x_1)y + 2y_1x^2 + ((-2x_1 - x_0)y_1)x + x_0x_1y_1 = 0$.

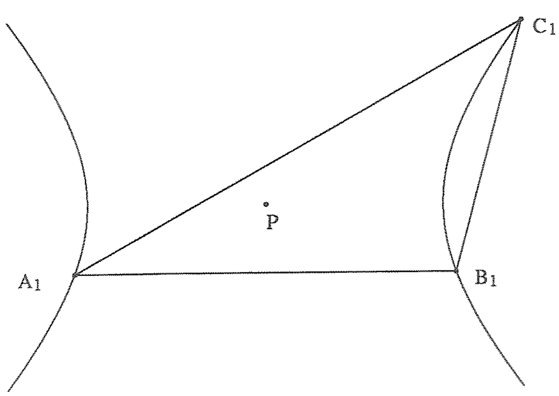

Fig. 111

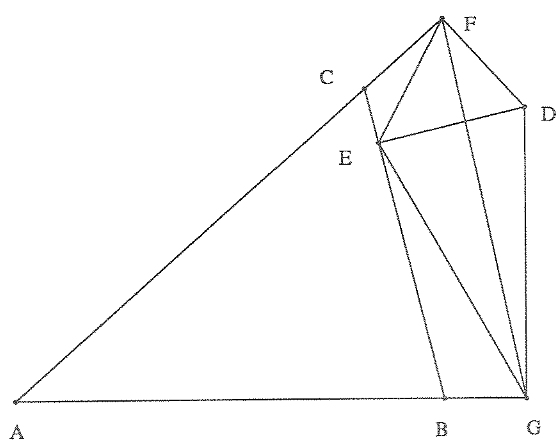

Fig. 112

Example 112. Let ABC be a triangle and D be a moving point. From D three perpendicular lines are drawn to the three sides of the triangle ABC. Let the feet be E, F, and G. If the area of the directed triangle EFG keeps constant, what is the locus of point D?

$((u_0\ u_1\ u_2\ u_3\ x)\ (y\ x_1\ x_2\ x_3\ x_4)$
$(A\ (0\ 0)\ B\ (u_1\ 0)\ C\ (u_2\ u_3)\ D\ (x\ y)\ G\ (x\ 0)\ F\ (x_1\ x_2)\ E\ (x_3\ x_4))$
(collinear $F\ A\ C$)
(perpendicular $D\ F\ A\ C$)
(collinear $E\ B\ C$)
(perpendicular $D\ E\ B\ C$)
(area $u_0\ E\ F\ G$))

The result is: $u_1u_3^3y^2 + (-u_1u_3^4 + (-u_1u_2^2 + u_1^2u_2)u_3^2)y + u_1u_3^3x^2 - u_1^2u_3^3x + 2u_0u_3^4 + (4u_0u_2^2 - 4u_0u_1u_2 + 2u_0u_1^2)u_3^2 + 2u_0u_2^4 - 4u_0u_1u_2^3 + 2u_0u_1^2u_2^2 = 0$.

Example 113. Three similar isoceles triangles, A_1BC, AB_1C, and ABC_1 are erected on the three respective sides, BC, CA, AB, of a triangle ABC, then AA_1, BB_1, and CC_1 are concurrent. Find the locus of the points of concurrency as the areas of the three similar triangles are varied between 0 and infinity.

$((u_1\ u_2\ u_3\ x)\ (y\ x_2\ x_1\ x_4\ x_3)$
$(A\ (0\ 0)\ B\ (u_1\ 0)\ C\ (u_2\ u_3)\ O\ (x\ y)\ C_1\ (x_2\ x_1)\ B_1\ (x_4\ x_3))$
(collinear $C_1\ O\ C$)
(eqdistant $C_1\ A\ C_1\ B$)
(collinear $B_1\ O\ B$)
(eqdistant $B_1\ A\ B_1\ C$)
(eqtangent $B\ A\ C_1\ A\ C\ B_1$))

The result is: $((2u_2 - u_1)u_3)y^2 + ((-2u_3^2 + 2u_2^2 - 2u_1u_2 + 2u_1^2)x + u_1u_3^2 - u_1u_2^2 - u_1^2u_2)y + ((-2u_2 + u_1)u_3)x^2 + ((2u_1u_2 - u_1^2)u_3)x = 0$.

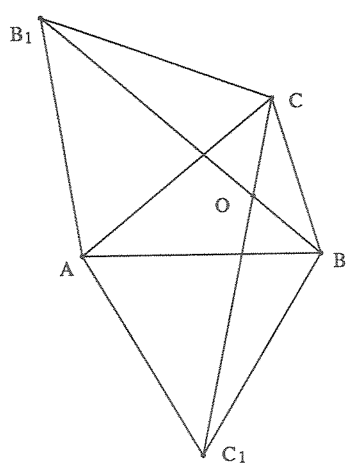

Fig. 113

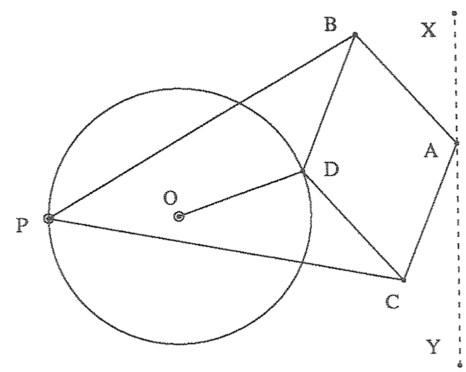

Fig. 114

Example 114. Links CD, AC, AB and BD have equal length, as do links PB and PC. The length of OD equals the distance from P to O. The locations of joints P and O are fixed points on the plane, but the linkage is allowed to rotate about these points. As it does, what is the traces of the joint A?

$((r\ m\ n\ y)\ (x\ x_1\ y_1\ x_2\ y_2)$

$(O\ (0\ 0)\ P\ (r\ 0)\ C\ (x_2\ y_2)\ D\ (x_1\ y_1)\ A\ (x\ y))$
(distance $O\ D\ r$)
(pp– (sq-distance $C\ D$) (pp∗ $n\ n$) (pp∗ $m\ m$))
(pp– (sq-distance $C\ A$) (pp∗ $n\ n$) (pp∗ $m\ m$))
(pp– (sq-distance $P\ C$) (pp∧ (pp+ $n\ r\ r$) 2) (pp∗ $m\ m$))
(collinear $P\ D\ A$)
non-deg (pp– $x_1\ x$))

The result is: $x + 2n + r = 0$.

Example 115. A bar AB slips on a wall OB. Find the locus of a point C on AB.

$((u_1\ u_2\ x)\ (y\ x_1\ y_1)$
$(A\ (x_1\ 0)\ B\ (x\ y)\ C\ (0\ y_1))$
(distance $A\ B\ u_2$)
(distance $C\ B\ u_1$)
(collinear $A\ C\ B$))

The result is: $u_1^2 y^2 + u_2^2 x^2 - u_1^2 u_2^2 = 0$.

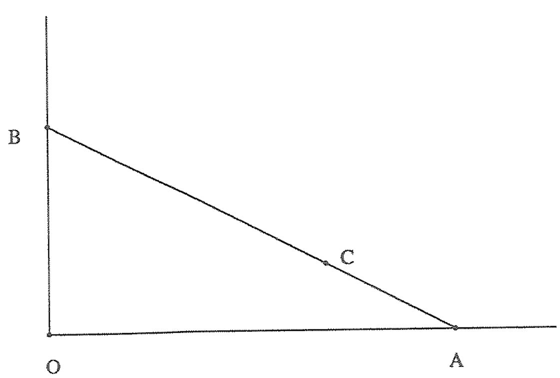

Fig. 115

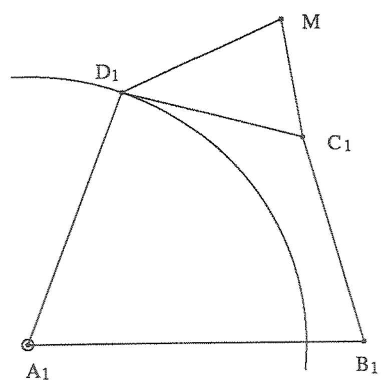

Fig. 116

Example 116. (The Four Bar Linkage) As in Figure 116. The link A_1B_1 is fixed. Link A_1D_1 can rotate around the joint A_1. As it does, what is the locus of joint M?

$((p\ r\ s\ a\ b\ c\ d\ x)\ (y\ c_{osa}\ s_{ina}\ c_{osb}\ s_{inb}\ x_1\ y_1\ x_2\ y_2)$
$(A_1\ (0\ 0)\ B_1\ (p\ 0)\ C_1\ (x_1\ y_1)\ D_1\ (x_2\ y_2)\ M\ (x\ y))$
(distance $C_1\ A_1\ r$)
(distance $D_1\ B_1\ s$)
(distance $C_1\ D_1\ c$)
(distance $C_1\ M\ b$)
(distance $D_1\ M\ a$)
(pp– x_1 (pp– x (pp∗ $b\ c_{osa}$)))
(pp– y_1 (pp– y (pp∗ $b\ s_{ina}$)))
(pp– x_2 (pp– x (pp∗ a (pp– (pp∗ $c_{osa}\ c_{osb}$) (pp∗ $s_{ina}\ s_{inb}$)))))

(pp− y_2 (pp− y (pp∗ a (pp+ (pp∗ c_{osa} s_{inb}) (pp∗ s_{ina} c_{osb})))))
(pp− (pp+ (pp∧ s_{ina} 2) (pp∧ c_{osa} 2)) 1)
(pp− (pp+ (pp∧ s_{inb} 2) (pp∧ c_{osb} 2)) 1))

The result is: the locus is described by a polynomial equation of degree 6.

Example 117. (Cayley) In the four bar linkage, if two bars BD and CD have an equal length. Find the locus of the middle point of the coupler.

((u_1 u_2 x) (y x_1 x_2 x_3 x_4)
(A (0 0) B (u_1 0) C (x_1 x_2) D (x_3 x_4) P (x y))
(distance A C u_1)
(distance C D u_2)
(distance D B u_2)
(mid-x P C D)
(mid-y P C D)
non-deg (pp− x_1 u_1))

The result is: $16y^4 + (32x^2 - 16u_1x - 4u_2^2 - 12u_1^2)y^2 + 16x^4 - 16u_1x^3 + (-4u_2^2 - 12u_1^2)x^2 + (-4u_1u_2^2 + 8u_1^3)x - u_1^2u_2^2 + 4u_1^4 = 0$.

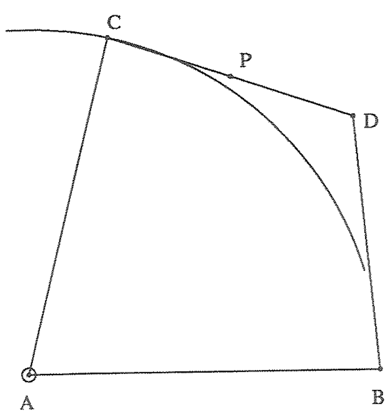

Fig. 117

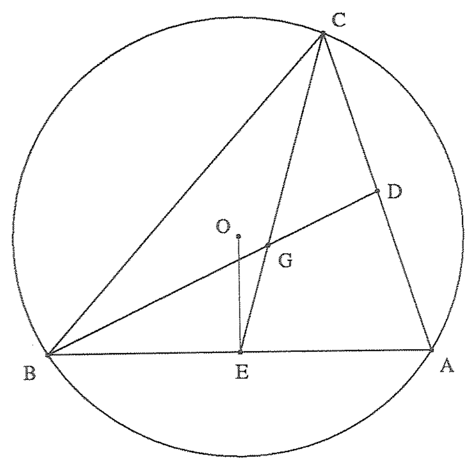

Fig. 118

Example 118. Find the locus equation of the centroid G of a triangle ABC when the side AB is fixed and the point C is moving on a circle passing through A and B.

((u_1 u_2 x) (y x_1 x_2 x_3 x_4 x_5)
(E (0 0) O (0 u_1) A (u_2 0) B (x_1 0) C (x_2 x_3) D (x_4 x_5) G (x y))
(pp+ x_1 u_2)
(eqdistant C O A O)
(mid-x D A C)
(mid-y D A C)
(collinear G C E)
(collinear G D B))

The result is: $y = 0$ and $9y^2 - 6u_1y + 9x^2 - u_2^2 = 0$.

Example 119. Find the locus equation of the orthocenter H of a triangle ABC when the side AB is fixed and the point C is moving on a circle passing through A and B.

> $((u_1\ u_2\ x)\ (y\ x_1\ x_2)$
> $(E\ (0\ 0)\ O\ (0\ u_1)\ A\ (u_2\ 0)\ B\ (x_1\ 0)\ C\ (x\ x_2)\ H\ (x\ y))$
> (pp+ x_1 u_2)
> (eqdistant $C\ O\ A\ O$)
> (perpendicular $A\ H\ B\ C$))

The result is: $y^2 + 2u_1y + x^2 - u_2^2 = 0$.

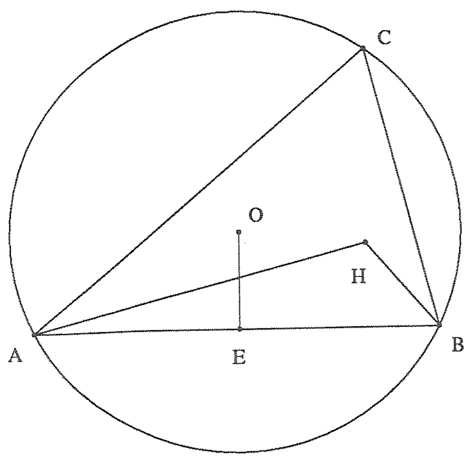

Fig. 119

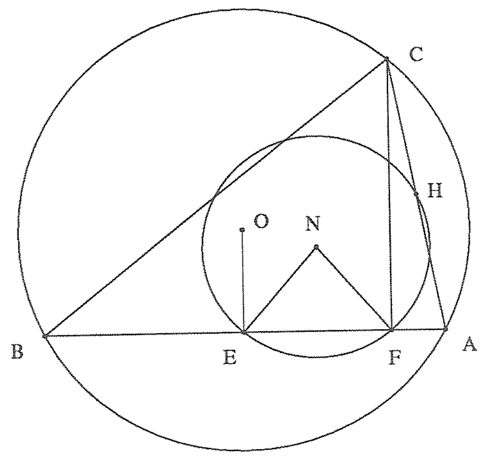

Fig. 120

Example 120. Find the locus equation of the center of the nine point circle N of a triangle ABC when the side AB is fixed and the point C is moving on a circle passing through A and B.

> $((u_1\ u_2\ x)\ (y\ x_1\ x_2\ x_3\ x_4\ x_5)$
> $(E\ (0\ 0)\ O\ (0\ u_1)\ A\ (u_2\ 0)\ B\ (x_1\ 0)\ C\ (x_2\ x_3)\ F\ (x_2\ 0)\ H\ (x_4\ x_5)\ N\ (x\ y))$
> (pp+ x_1 u_2)
> (eqdistant $C\ O\ A\ O$)
> (mid-x $H\ A\ C$)
> (mid-y $H\ A\ C$)
> (eqdistant $N\ E\ N\ F$)
> (eqdistant $N\ E\ N\ H$)
> non-deg x_2)

The result is: $4y^2 + 4x^2 - u_2^2 - u_1^2 = 0$.