A COMBINATION OF RITT–WU'S METHOD AND COLLINS' METHOD*

Shang-Ching Chou, Xiao-Shan Gao† and Nicholas McPhee

Department of Computer Sciences
The University of Texas at Austin
Austin, Texas 78712-1188

TR-89-28 October 1989

Abstract

We propose a method for the decision problem in real closed fields. The method is a combination of the Tarski–Seidenberg–Collins (TSC) method and the Ritt–Wu method. To decide whether a system of polynomial equations and inequations has common zeros (solutions) in a real closed field, we first apply Ritt–Wu's decomposition method to delete those components that do not have common zeros in an algebraically closed field. Then we apply the TSC method to the remaining components to complete the solution. Many non-trivial examples have been solved by this method, including the 8_3 configuration problem.

Keywords: Mechanical geometry theorem proving, Tarski–Seidenberg–Collins' method, Ritt–Wu's method, Algebraically closed field, Real closed field, Axioms of geometry, Axioms of order, Unordered metric geometry, Euclidean geometry, the 8_3 configuration problem.

---

* The work reported here was supported in part by the NSF Grant CCR-8702108.

† On leave from Institute of Systems Science, Academia Sinica, Beijing.

1 Introduction

The mechanical method introduced by Wu Wen-Tsün [14], [15] has been successfully used in proving hundreds of non-trivial theorems in Euclidean geometry [1], [2], [5], [10]. The geometry statements that Wu's method can address are those of equality (equation) type, and we shall call the class of such statements the W class. In Euclidean geometry, the W class is a subclass of the class of all elementary sentences (statements), and Wu's method can only be used to confirm (prove) statements in the W class. But to disprove a geometric statement S of the W class in Euclidean geometry, additional conditions are needed, i.e., S should be generic in **R** (for details, see [9]). However, Wu's method is complete for class W in *the theory of metric geometry.* Thus the assertion in [13] that the TSC method is logically more expressive than Wu's method is correct in *Euclidean* geometry, but not in *metric* geometry. This will be discussed in more detail in Section 2.

Inspired by Wu's work, many variants based on the Gröbner basis method, all having roughly the same logical expressiveness, have also been successfully applied to the W class [8], [12], [11].

In Euclidean geometry, the TSC method is much more expressive than Wu's method. However, Wu's method can easily prove many theorems in Euclidean geometry that no implementations of Collins' method could prove within the time and space available. Thus a combination of the two methods could solve problems not in the scope of Wu's method, but which cannot be solved by the TSC method alone within the available time and space. We have been working in this direction since 1986 [4], but solving the 8_3 configuration problem suggests this method has a more promising future than previously believed.

We assume the reader is already familiar with Ritt–Wu's Zero Decomposition Algorithm [15], [17], [6]. At present, we see the TSC method as a black box which can solve polynomial inequality problems in **R**.

2 Metric Geometry and Wu's Method

In this section we will point out the delicate differences between the scopes of Wu's method and the TSC method.

2.1 A Theory of Metric Geometry

In [16], Wu presented a theory of unordered metric geometry. This geometry has four basic predicates: collinear, perpendicular, segment congruent, and angle congruent. These four predicates satisfy five groups of axioms:

Group 1. Axioms of Incidence.
Group 2. Axioms of Parallelism and Pappus' Axiom.
Group 3. Axioms of Perpendicularity.
Group 4. Axioms of Segment Congruence.
Group 5. Axioms of Angle Congruence.

These five groups of Axioms form a theory of metric geometry called Wu's Metric Geometry (*WMG*). Note that there are no axioms of order in *WMG*, so it is not possible in *WMG* to express statements such as "point A is between points B and C on line l".

In [2], all models of the theory *WMG* have been classified. They are Cartesian products H^2 of H, where H is a Hilbert field. A Hilbert field is a field with characteristic zero in which the sum of two squares has a square root. Euclidean geometry, $\mathbf{R}^2$, and complex geometry, $\mathbf{C}^2$, are two typical models of the theory *WMG*.

2.2. A Comparison of the Scopes of Ritt-Wu's Method and the TSC Method

Our major point is that if one wishes to determine whether a geometry statement S of the W class is a theorem in *WMG*, then it is beyond the scope of the TSC method. Wu's method was the first to give a decision procedure for the W class in the theory *WMG*.

Let us look at two examples.

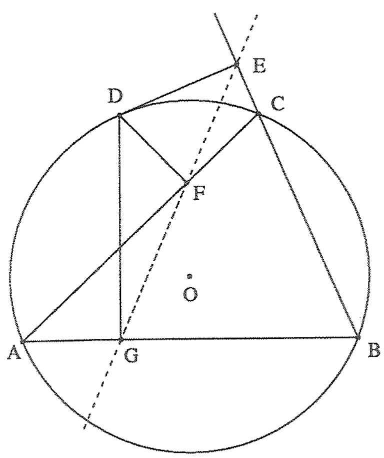

Figure 1

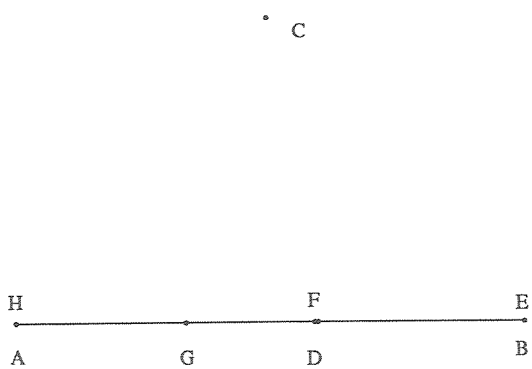

Figure 2

Example (2.1). (Simson's Theorem). Let D be a point on the circumscribed circle (O) of triangle ABC. From D three perpendiculars are drawn to the three sides BC, CA and AB of $\triangle ABC$. Let E, F and G be the three feet respectively. Show that E, F and G are collinear (Figure 1).

Let us specify the exact statement of this theorem in the theory *WMG* as follows

$$\forall A \forall B \forall C \forall O \forall D \forall E \forall F \forall G [H(A,B,C,O,D,E,F,G) \Rightarrow \text{collinear}(E,F,G)]. \tag{2.2}$$

Here $H(A,B,C,O,D,E,F,G)$ is the hypothesis consisting the conjunction of following predicates:

pseudoPerpendicular(AB, DG),
pseudoPerpendicular(AC, DF),
pseudoPerpendicular(BC, DE),
collinear(A, B, G),
collinear(A, C, F),

collinear(B,C,E),
congruent(OA,OB),
congruent(OA,OC),
congruent(OA,OD),
$\neg$collinear(A,B,C),
$\neg$isotropic(AB),
$\neg$isotropic(AC),
$\neg$isotropic(BC).

where the predicate pseudoPerpendicular(AB,DG) is: $A = B \vee D = G \vee (A \neq B \wedge D \neq G \wedge AB \perp DG)$. An isotropic line is a line perpendicular to itself.

The statement of Simson's theorem is in metric geometry, and thus is also a statement in Euclidean geometry. But if we want to ask whether this statement is a theorem in *WMG*, that is, whether Simson's theorem can be inferred from *WMG without using the axioms of order* (betweenness), then it is beyond the scope of the TSC method. This statement has been proved a theorem in *WMG* by Wu's method, and thus a theorem in Euclidean geometry. However, if we delete the conditions that AB, BC, and AC are non–isotropic from the hypotheses, then Wu's method *proves* that the statement is not a theorem in the theory *WMG* (see the Appendix for details). In the following example the opposite is the case.

Example (2.3). (The 8_3 Configuration Problem, see [13]).

(2.4) $\forall A \forall B \forall C \forall D \forall E \forall F \forall G \forall H[\text{MacLane}(A,B,C,D,E,F,G,H) \Rightarrow \text{collinear}(ABC)]$,

where MacLane(A,B,C,D,E,F,G,H) is:

collinear(A,B,D) $\wedge$ collinear(A,C,H) $\wedge$ collinear(A,F,G) $\wedge$ collinear(B,C,E) $\wedge$ collinear(B,G,H) $\wedge$ collinear(C,D,F) $\wedge$ collinear(D,E,G) $\wedge$ collinear(E,F,H).

This is the exact statement in [13], where it is incorrectly claimed to be a theorem in Euclidean geometry (see Figure 2 for a counterexample). The translation technique in [13] was not careful enough to specify sufficient non-degenerate conditions.[1] On the contrary, what was called Wu's 'careless' technique in [13] did find the following degenerate conditions: e.g., $A = H$, $B = E$, $D = F$ (See Figure 2). *Without excluding those conditions it is not a theorem, even in Euclidean geometry.* Are there other cases in which the formula (2.4) is also not valid? We will use our proposed method to give a complete solution of this problem in Section 4.3. The mistake made in [13] is another example of Wu's assertion that non-degenerate conditions are very hard to find; it is almost impossible for humans to identify sufficient non-degenerate conditions for many problems such as this (for more examples see, e.g., [6].).

Now we return to our main theme. Under the non-degenerate conditions that all 8 points $A,...,H$ are distinct, we have proved that the 8_3 theorem is *not* a theorem in the theory *WMG*. The TSC method is unable to reach this conclusion. On the other hand, to prove it is a theorem in Euclidean geometry is beyond the scopes of Wu's original method or any techniques based on the Gröbner basis method. Our proposed combination of Ritt–Wu's method and the TSC method proves it is a theorem in Euclidean geometry. Like Morley's trisector theorem,

---

[1] In our paper [7] we will explain in detail our method in [2] for generating non-degenerate conditions in geometry forms and prove a theorem stating that for a subclass of the W class these automatically generated non-degenerate conditions are sufficient.

Thèbault's conjecture etc., the 8_3 configuration problem is one of the most challenging problem in mechanical geometry theorem proving.

In the next Section, we propose the new method, and in Section 4 we will present the complete proof of the 8_3 theorem using this method.

3. A Combination of Collins' Method and Ritt–Wu's Method

3.1. The Formulation of the Problem

All polynomials mentioned in this paper are over $\mathbf{Q}$, and it will thus be assumed that all fields contain $\mathbf{Q}$.

The problem we want to solve can be formulated as follows. Let HS be a set consisting of polynomial equations, inequations, and inequalities in the variables $y_1, ..., y_s$. We want to decide whether HS has a solution in $\mathbf{R}$, or in a real closed field, i.e., whether $\exists y_1, ..., y_s \in \mathbf{R}(HS)$ is valid. For a polynomial inequality in HS, say, $g > 0$, we can use Seidenberg's technique to convert it to a polynomial equation by introducing a new variable z: in a real closed field, $g > 0 \iff \exists z(gz^2 - 1 = 0)$. Thus, without loss of generality, we can assume that HS contains equations and inequations only. Let ES be the set of all equation polynomials in HS, and IS be the set of all inequation polynomials. Also without loss of generality, we can assume ES and IS do not contain the zero polynomial.

We denote $F\text{–}Zero(PS)$ as the common zeros of all polynomials in PS in the field F, i.e.,

$$F\text{–}Zero(PS) = \{(a_1, ..., a_s) \in F^s \mid f(a_1, ..., a_s) = 0 \text{ for all } f \in PS\}.$$

If we do not want F to be explicitly mentioned, we just denote $F\text{–}Zero(PS)$ by $Zero(PS)$. Following Wu, for polynomial sets ES and IS we denote $Zero(ES/IS)$ to be the set difference $Zero(ES) - Zero(IS)$. Thus our problem is equivalent to deciding whether $\mathbf{R}\text{–}Zero(ES/IS)$ is empty.

The rationale behind our method is the following observations:

— If $\mathbf{C}\text{–}Zero(ES/IS)$ is empty then $\mathbf{R}\text{–}Zero(ES/IS)$ is empty.

— Using Wu's method or the Gröbner basis method to decide the emptiness of $\mathbf{C}\text{–}Zero(ES/IS)$ is easier than using the TSC method to decide the emptiness of $\mathbf{R}\text{–}Zero(ES/IS)$. Using Ritt–Wu's decomposition, we can break $\mathbf{R}\text{–}Zero(ES/IS)$ into many similar, but simpler problems which hopefully can be solved using the TSC method.

Note that in our previous work [6], we wanted to decide whether

$$Zero(PS/DS) \subset Zero(g)$$

where PS and DS are a sets of polynomials representing the hypothesis equations and the non-degenerate conditions of the geometry problem. This is equivalent in our new formulation to deciding whether $Zero(PS/DS \cup \{g\})$ is empty. The latter is usually superior to the former because it can reduce branching in Ritt–Wu's decomposition. This is the case in the 8_3 configuration problem.

Now we present our method, assuming the reader is already familiar with Ritt–Wu's decomposition, especially with the improved version developed by us in [6].

3.2. Proposed Method

Step 1. Use Ritt–Wu's zero decomposition algorithm to decompose

$$Zero(ES/IS) = \bigcup_{1\le i\le k} Zero(PD(ASC_i)/IS),$$

where the ASC_i are irreducible ascending chains and $PD(ASC)$ (see [6]) is defined by

$$PD(ASC) = \{g \mid prem(g; ASC) = 0\}.$$

Step 2. If $k = 0$, then $\mathbf{C}$–$Zero(ES/IS)$ is empty, hence $\mathbf{R}$–$Zero(ES/IS)$ is empty.

Step 3. Otherwise, use the TSC method either to decide whether each of the $Zero(PD(ASC_i)/IS)$ is empty, or to reduce these problems to smaller subproblems which can then be attacked by returning to Step 1.

In the next section we use the 8_3 theorem to illustrate how the method works. the 8_3 theorem is a very challenging problem and our solution of the problem marks a milestone in our work. In our opinion, a new method is significant if it can solve many problems the previous methods couldn't solve within reasonable time and space limits and solve at least one remarkable problem. About 35 problems in [3] involving equality can also be solved by this new method.

4. A Mechanical Proof of the 8_3 Theorem

4.1. the Exact Statement of the 8_3 Theorem

As was pointed out in Section 2, the 'careful' technique of [13] still missed non-degenerate conditions, as Figure 2 shows. What are the non-degenerate conditions for the 8_3 Theorem? A natural choice is the condition $D(A,B,C,D,E,F,G,H)$ that all 8 points A, B, C, D, E, F, G and H are distinct (in Section 4.3 we show how weaker conditions can be generated mechanically). Thus the new statement of the 8_3 theorem is

$$\forall A \cdots \forall H\ [(\text{MacLane}(A,\cdots,H) \wedge D(A,\cdots,H) \Rightarrow \text{collinear}(A,B,C)]. \tag{4.1}$$

We will prove that formula (4.1) is not a theorem in the theory *WMG*, but is a theorem in Euclidean geometry.

Without loss of generality, we can let $A = (0,0)$, $B = (y_1, 0)$, $D = (y_2, 0)$, $C = (y_3, y_8)$, $E = (y_4, y_9)$, $F = (y_5, y_{10})$, $G = (y_6, y_{11})$, and $H = (y_7, y_{12})$, and use the following variable order $y_1 < \cdots < y_{12}$. Noting that the condition collinear(A,B,D) is trivially satisfied by our choice of coordinates, the set of equation polynomials for the hypothesis of (4.1) is $ES =$

$\{y_3y_{12} - y_7y_8,$ # collinear(A,C,H)
$y_5y_{11} - y_6y_{10},$ # collinear(A,F,G)

$(y_3 - y_1)y_9 + (-y_4 + y_1)y_8,$ # collinear(B, C, E)
$(y_6 - y_1)y_{12} + (-y_7 + y_1)y_{11},$ # collinear(B, G, H)
$(y_3 - y_2)y_{10} + (-y_5 + y_2)y_8,$ # collinear(C, D, F)
$(y_4 - y_2)y_{11} + (-y_6 + y_2)y_9,$ # collinear(D, E, G)
$(y_5 - y_4)y_{12} + (-y_7 + y_4)y_{10} + (y_7 - y_5)y_9\}.$ # collinear(E, F, H)

The set of inequation polynomials for the hypothesis (all 8 points are distinct) of (4.1) is $IS = \{\, y_1,\ y_8z + y_3,\ y_2,\ y_{11}z + y_6,\ y_{12}z + y_7,\ y_9z + y_4,\ y_{10}z + y_5,\ y_8z + y_3 - y_1,\ y_{11}z + y_6 - y_1,\ y_{12}z + y_7 - y_1,\ y_9z + y_4 - y_1,\ y_{10}z + y_5 - y_1,\ y_2 - y_1,\ y_8z + y_3 - y_2,\ y_{11}z + y_6 - y_2,\ y_{12}z + y_7 - y_2,\ y_9z + y_4 - y_2,\ y_{10}z + y_5 - y_2,\ (y_{11} - y_8)z + y_6 - y_3,\ (y_{12} - y_8)z + y_7 - y_3,\ (y_9 - y_8)z + y_4 - y_3,\ (y_{10} - y_8)z + y_5 - y_3,\ (y_{12} - y_{11})z + y_7 - y_6,\ (y_{11} - y_9)z + y_6 - y_4,\ (y_{11} - y_{10})z + y_6 - y_5,\ (y_{12} - y_9)z + y_7 - y_4,\ (y_{12} - y_{10})z + y_7 - y_5,\ (y_{10} - y_9)z + y_5 - y_4\, \}$.

Here z is a new (existentially quantified) variable. Thus, e.g., for the last polynomial, $(y_{10} - y_9)z + y_5 - y_4 \neq 0$ for some new indeterminate z if and only if $y_{10} - y_9 \neq 0$ or $y_5 - y_4 \neq 0$, i.e., $E \neq F$.

The conclusion that A, B, and C are collinear is $g = y_1y_8 = 0$. The formula (4.1) is equivalent to $Zero(ES/IS) \subset Zero(y_1y_8)$, or $Zero(ES/IS \cup \{y_1y_8\})$ is empty. In the next subsection we prove the following claims:

— **C**–$Zero(ES/IS \cup \{y_1y_8\})$ is not empty, thus (4.1) is not a theorem in the theory WMG. [2]

— **R**–$Zero(ES/IS \cup \{y_1y_8\})$ is empty, thus (4.1) is a theorem in Euclidean geometry.

The geometric meaning of these claims is that (4.1) is a theorem in Euclidean geometry, but *it cannot be proved without using axioms of order.*

4.2. The Proof of the 8_3 Theorem

Using Ritt–Wu's zero decomposition algorithm, we have the decomposition

$$Zero(ES/IS \cup \{y_1y_8\}) = Zero(PD(ASC_1)/IS \cup \{y_1y_8\}), \tag{4.2}$$

where the irreducible ascending chain $ASC_1 =$

$f_7 = y_3y_{12} - y_7y_8$
$f_6 = (y_4 - y_2)y_{11} + (-y_6 + y_2)y_9$
$f_5 = (y_3 - y_2)y_{10} + (-y_5 + y_2)y_8$
$f_4 = (y_3 - y_1)y_9 + (-y_4 + y_1)y_8$
$f_3 = ((y_2y_3 - y_1y_2)y_5 + (-y_1y_3 + y_1y_2)y_4 + (-y_2 + y_1)y_3^2)y_7 + (((-y_2 + y_1)y_3)y_4 - y_1y_3^2 + y_1y_2y_3)y_5 + (y_2y_3^2 - y_1y_2y_3)y_4$
$f_2 = (((y_2 - y_1)y_4 + (-y_2 + y_1)y_3)y_5 + (-y_2y_3 + y_1y_2)y_4 + y_2^2y_3 - y_1y_2^2)y_6 + ((y_2y_3 - y_2^2)y_4 - y_1y_2y_3 + y_1y_2^2)y_5$

[2] In [13] it is claimed that Wu's method is to solve the "finding problem", i.e., to find a polynomial d such that $Zero(ES) \subset Zero(d{\cdot}g)$ and $Zero(ES) \not\subset Zero(d)$. This is a misunderstanding. In [13] it is also claimed that such a d polynomial does not exist for the 8_3 configuration problem, but one can easily find such a d polynomial, e.g., $d = (((y_2 - y_1)y_4 + (-y_2 + y_1)y_3)y_5 + (-y_2y_3 + y_1y_2)y_4 + y_2^2y_3 - y_1y_2^2)y_6 + ((y_2y_3 - y_2^2)y_4 - y_1y_2y_3 + y_1y_2^2)y_5$, so that $d \cdot g \in ES$ and $Zero(ES) \not\subset Zero(d)$.

$f_1 = ((y_2^2 - y_1y_2 + y_1^2)y_4^2 + ((y_1y_2 - 2y_1^2)y_3 - 2y_1y_2^2 + y_1^2y_2)y_4 + y_1^2y_3^2 - y_1^2y_2y_3 + y_1^2y_2^2)y_5^2 + (((-2y_2^2 + y_1y_2)y_3 + y_1y_2^2 - 2y_1^2y_2)y_4^2 + (-y_1y_2y_3^2 + (3y_1y_2^2 + 3y_1^2y_2)y_3 - y_1^2y_2^2)y_4 - y_1^2y_2y_3^2 - y_1^2y_2^2y_3)y_5 + (y_2^2y_3^2 - y_1y_2^2y_3 + y_1^2y_2^2)y_4^2 + (-y_1y_2^2y_3^2 - y_1^2y_2^2y_3)y_4 + y_1^2y_2^2y_3^2.$

It took 13020.8 seconds on a Symbolics 3600 to complete the above decomposition. The above simple form (only one component) is due to the dimension theorem (see Theorem (4.4) in [6]). Our program produced 77 components, but all other 76 components are of lower dimensions ($< 12 - 7 = 5$) than $PD(ASC_1)$; thus they are redundant. By Theorem (4.8) in [6], $\mathbf{C}$–$Zero(ES/IS \cup \{y_1y_8\})$ is non-empty, thus we have proved:

Theorem (4.3). Formula (4.1) is not a theorem in the theory of Wu's metric geometry *WMG*.

Notice that the TSC method cannot reach this conclusion because it cannot address problems in the complex plane.

Now to decide whether (4.1) is a theorem in Euclidean geometry, we want to decide whether $\mathbf{R}$–$Zero(ES/IS \cup \{y_1y_8\}) = \mathbf{R}$–$Zero(PD(ASC_1)/IS \cup \{y_1y_8\})$ is empty. Now we resort to the TSC method. Let $lv(f)$ be the highest variable in polynomial f, $lc(f)$ be the leading, or initial, coefficient of $lv(f)$ in f, and $deg(f,x)$ be the degree of x in f. Thus $lv(f_1) = y_5$, $deg(f_1, lv(f_1)) = deg(f_1, y_5) = 2$, and $lc(f_1) = (y_2^2 - y_1y_2 + y_1^2)y_4^2 + ((y_1y_2 - 2y_1^2)y_3 - 2y_1y_2^2 + y_1^2y_2)y_4 + y_1^2y_3^2 - y_1^2y_2y_3 + y_1^2y_2^2$. If $\{f_1, ..., f_r\}$ is an ascending chain of polynomials such that $deg(f_i, lv(f_i)) = 1$ then $\{f_1, ..., f_r\}$ always has a solution in any field. In our example, since $deg(f_i, lv(f_i)) = 1$ for $i > 1$, the only polynomial which can cause the emptiness of $\mathbf{R}$–$Zero(PD(ASC_1)/IS \cup \{y_1y_8\})$ is f_1. The discriminate of f_1 in y_5 is $-3(y_3 - y_2)^2(y_4 - y_3)^2(y_4 - y_1)^2y_2^2y_1^2$. Thus (by the TSC method), $\mathbf{R}$–$Zero(PD(ASC_1)/IS \cup \{y_1y_8\})$ is non-empty only when $y_3 - y_2 = 0$, or $y_4 - y_3 = 0$, or $y_4 - y_1 = 0$, or $y_2 = 0$, or $y_1 = 0$, $I_1 = lc(f_1) = 0$. We can now use Ritt–Wu's zero decomposition on the resulting sets:

$$Zero(\{y_3 - y_2\} \cup ES/IS \cup \{y_1y_8\}) \tag{4.4}$$

$$Zero(\{y_4 - y_3\} \cup ES/IS \cup \{y_1y_8\}) \tag{4.5}$$

$$Zero(\{y_4 - y_1\} \cup ES/IS \cup \{y_1y_8\}) \tag{4.6}$$

$$Zero(\{y_2\} \cup ES/IS \cup \{y_1y_8\}) \tag{4.7}$$

$$Zero(\{y_1\} \cup ES/IS \cup \{y_1y_8\}). \tag{4.8}$$

$$Zero(\{I_1\} \cup ES/IS \cup \{y_1y_8\}) \tag{4.9}$$

(4.7) and (4.8) are immediately confirmed by the program to be empty, since $y_i \in IS$ for $i = 1, 2$, so $y_1 \neq 0$ iff $B \neq A$, and $y_2 \neq 0$ iff $D \neq A$. (4.5) and (4.6) have also been confirmed to be empty in $\mathbf{C}$ using Ritt–Wu's zero decomposition algorithm in 1.5 seconds and 1.7 seconds, respectively.

For (4.4) we have the following decomposition (in 368.4 seconds):

$$(4.10)\qquad Zero(\{y_3-y_2\}\cup ES/IS\cup\{y_1y_8\}) = Zero(\{y_3-y_2\}\cup PD(ASC_{1.1})/IS\cup\{y_1y_8\})$$

where the irreducible ascending chain $ASC_{1.1}$ =

$f_{1.8} = y_2y_{12} - y_7y_8$
$f_{1.7} = (y_4-y_2)y_{11} + (-y_6+y_2)y_9$
$f_{1.6} = ((y_4-y_2)y_6)y_{10} + (-y_2y_6+y_2^2)y_9$
$f_{1.5} = (y_2-y_1)y_9 + (-y_4+y_1)y_8$
$f_{1.4} = ((y_1y_4+y_2^2-2y_1y_2)y_6+(-y_2^2+y_1y_2-y_1^2)y_4+y_1^2y_2)y_7+(-y_1y_2y_4+y_1^2y_2)y_6+y_1y_2^2y_4-y_1^2y_2^2$
$f_{1.3} = (y_1^2y_4^2+(y_1y_2^2-3y_1^2y_2)y_4+y_2^4-3y_1y_2^3+3y_1^2y_2^2)y_6^2+((-y_1y_2^2-y_1^2y_2)y_4^2+(-2y_2^4+4y_1y_2^3+2y_1^2y_2^2)y_4+y_1y_2^4-3y_1^2y_2^3)y_6+(y_2^4-y_1y_2^3+y_1^2y_2^2)y_4^2+(-y_1y_2^4-y_1^2y_2^3)y_4+y_1^2y_2^4$
$f_{1.2} = y_5 - y_2$
$f_{1.1} = y_3 - y_2.$

As in ASC_1, $\mathbf{R}$–$Zero(PD(ASC_{1.1}/IS\cup\{y_1y_8\})$ is non-empty only when $f_{1.3}$ has real solutions. $f_{1.3}$ has discriminate $-3(y_2-y_1)^2(y_4-y_2)^4y_2^2y_1^2$ and initial coefficient $I_{1.3} = y_1^2y_4^2 + (y_1y_2^2-3y_1^2y_2)y_4 + y_2^4 - 3y_1y_2^3 + 3y_1^2y_2^2$. We can again use Ritt–Wu's decomposition on the following sets:

$$(4.11)\qquad Zero(\{y_2-y_1, y_3-y_2\}\cup ES/IS\cup\{y_1y_8\}).$$

$$(4.12)\qquad Zero(\{y_4-y_2, y_3-y_2\}\cup ES/IS\cup\{y_1y_8\}).$$

$$(4.13)\qquad Zero(\{y_2, y_3-y_2\}\cup ES/IS\cup\{y_1y_8\}).$$

$$(4.14)\qquad Zero(\{y_1, y_3-y_2\}\cup ES/IS\cup\{y_1y_8\}).$$

$$(4.15)\qquad Zero(\{I_{1.3}, y_3-y_2\}\cup ES/IS\cup\{y_1y_8\}).$$

(4.11)–(4.14) are all empty in $\mathbf{C}$. The emptiness of (4.12) is non-trivial. It took 1.8 seconds to confirm. For (4.15) we have (in 150.5 seconds):

$$(4.16)\qquad Zero(\{I_{1.3}, y_3-y_2\}\cup ES/IS\cup\{y_1y_8\}) = Zero(PD(ASC_{1.1.1})/IS\cup\{y_1y_8\})$$

where the irreducible ascending chain $ASC_{1.1.1}$ =

$f_{1.1.9} = y_2y_{12} - y_7y_8$
$f_{1.1.8} = y_4 - y_2)y_{11} + (-y_6+y_2)y_9$
$f_{1.1.7} = ((y_4-y_2)y_6)y_{10} + (-y_2y_6+y_2^2)y_9$
$f_{1.1.6} = (y_2-y_1)y_9 + (-y_4+y_1)y_8$
$f_{1.1.5} = ((y_1y_4+y_2^2-2y_1y_2)y_6+(-y_2^2+y_1y_2-y_1^2)y_4+y_1^2y_2)y_7+(-y_1y_2y_4+y_1^2y_2)y_6+y_1y_2^2y_4-y_1^2y_2^2$
$f_{1.1.4} = ((2y_1^2y_2-y_1^3)y_4+y_1y_2^3-2y_1^2y_2^2)y_6+(-3y_1^2y_2^2+2y_1^3y_2)y_4+y_2^5-4y_1y_2^4+6y_1^2y_2^3-2y_1^3y_2^2$
$f_{1.1.3} = y_5 - y_2$
$f_{1.1.2} = y_1^2y_4^2 + (y_1y_2^2-3y_1^2y_2)y_4 + y_2^4 - 3y_1y_2^3 + 3y_1^2y_2^2$
$f_{1.1.1} = y_3 - y_2.$

The only polynomial which can cause the emptiness of

$$Zero(PD(ASC_{1.1.1})/IS \cup \{y_1y_8\})$$

is $f_{1.1.2}$, whose discriminate is $-3(y_2 - y_1)^2y_2^2y_1^2$ and whose initial coefficient is y_1^2. We can again use Ritt–Wu's decomposition on the following sets:

$$Zero(\{y_2 - y_1, I_{1.3}, y_3 - y_2\} \cup ES/IS \cup \{y_1y_8\}$$

$$Zero(\{y_2, I_{1.3}, y_3 - y_2\} \cup ES/IS \cup \{y_1y_8\}$$

$$Zero(\{y_1, I_{1.3}, y_3 - y_2\} \cup ES/IS \cup \{y_1y_8\}.$$

All are trivially empty.

Now for (4.9), using Ritt–Wu's decomposition, we have (in 8443.5 seconds):

$$\begin{aligned} & Zero(\{I_1\} \cup ASC_1 \cup ES/IS \cup \{y_1y_8\}) = \\ Zero(PD(ASC_{1.2})/IS \cup \{y_1y_8\}) & \bigcup Zero(PD(ASC_{1.3})/IS \cup \{y_1y_8\}) \end{aligned} \tag{4.17}$$

where $ASC_{1.2}$ =

$f_{2.8} = y_1y_{12} - y_7y_8$
$f_{2.7} = (y_2 - y_1)y_{11} + (y_6 - y_2)y_9$
$f_{2.6} = (y_2 - y_1)y_{10} + (y_5 - y_2)y_8$
$f_{2.5} = (y_5y_6 - y_2y_5)y_9 + ((-y_5 + y_2)y_6)y_8$
$f_{2.4} = ((y_5^2 - y_1y_5 - y_1y_2 + y_1^2)y_6 - y_2y_5^2 + (2y_1y_2 - y_1^2)y_5)y_7 + (-y_1y_5^2 + y_1y_2y_5)y_6 + y_1^2y_5^2 - y_1^2y_2y_5$
$f_{2.3} = (y_5^2 - y_1y_5 + y_1^2)y_6^2 + (-y_1y_5^2 - y_1^2y_5)y_6 + y_1^2y_5^2$
$f_{2.2} = y_4 - y_1$
$f_{2.1} = y_3 - y_1$,

and $ASC_{1.3}$ =

$f_{3.8} = y_3y_{12} - y_7y_8$
$f_{3.7} = (y_4 - y_2)y_{11} + (-y_6 + y_2)y_9$
$f_{3.6} = (y_3 - y_2)y_{10} + (-y_5 + y_2)y_8$
$f_{3.5} = (y_3 - y_1)y_9 + (-y_4 + y_1)y_8$
$f_{3.4} = ((y_2y_3 - y_1y_2)y_5 + (-y_1y_3 + y_1y_2)y_4 + (-y_2 + y_1)y_3^2)y_7 + (((-y_2 + y_1)y_3)y_4 - y_1y_3^2 + y_1y_2y_3)y_5 + (y_2y_3^2 - y_1y_2y_3)y_4$
$f_{3.3} = (((y_2 - y_1)y_4 + (-y_2 + y_1)y_3)y_5 + (-y_2y_3 + y_1y_2)y_4 + y_2^2y_3 - y_1y_2^2)y_6 + ((y_2y_3 - y_2^2)y_4 - y_1y_2y_3 + y_1y_2^2)y_5$
$f_{3.2} = ((y_2^2 - 4y_1y_2 + y_1^2)y_4 + (2y_1y_2 - y_1^2)y_3 - y_1y_2^2 + 2y_1^2y_2)y_5 + ((-y_2^2 + 2y_1y_2)y_3 + 2y_1y_2^2 - y_1^2y_2)y_4 - y_1y_2y_3^2 - y_1^2y_2^2$
$f_{3.1} = (y_2^2 - y_1y_2 + y_1^2)y_4^2 + ((y_1y_2 - 2y_1^2)y_3 - 2y_1y_2^2 + y_1^2y_2)y_4 + y_1^2y_3^2 - y_1^2y_2y_3 + y_1^2y_2^2$.

Now the only polynomial which can cause the emptiness of $Zero(PD(ASC_{1.2})/IS \cup \{y_1y_8\})$ is $f_{2.3}$ whose initial coefficient is $I_{2.3} = (y_5^2 - y_1y_5 + y_1^2)$ and whose discriminate is $-3(y_5 - y_1)^2y_5^2y_1^2$. Thus we can use Ritt–Wu's decomposition for the following sets:

$$Zero(\{I_1, (y_5 - y_1)\} \cup ASC_{1.2} \cup ES/IS \cup \{y_1y_8\}) \tag{4.18}$$

$$Zero(\{I_1, y_5\} \cup ASC_{1.2} \cup ES/IS \cup \{y_1y_8\}) \tag{4.19}$$

$$Zero(\{I_1, y_1\} \cup ASC_{1.2} \cup ES/IS \cup \{y_1 y_8\}) \tag{4.20}$$

$$Zero(\{I_1, I_{2.3}\} \cup ASC_{1.2} \cup ES/IS \cup \{y_1 y_8\}). \tag{4.21}$$

Now (4.20) is empty, and (4.18) has been confirmed to be empty in 8.2 seconds, but for (4.19) we have:

$$Zero(\{I_1, y_5\} \cup ASC_{1.2} \cup ES/IS \cup \{y_1 y_8\}) = Zero(PD(ASC_{1.2.1})/\{y_1 y_8\})$$

where $ASC_{1.2.1} =$

$f_{2.1.9} = y_1 y_{12} - y_7 y_8$
$f_{2.1.8} = (y_2 - y_1) y_{11} - y_2 y_9$
$f_{2.1.7} = (y_2 - y_1) y_{10} - y_2 y_8$
$f_{2.1.6} = ((y_2 - y_1) y_7) y_9 + ((-2y_2 + y_1) y_7 + y_1 y_2) y_8$
$f_{2.1.5} = (3y_2^2 - 3y_1 y_2 + y_1^2) y_7^2 + (-3y_1 y_2^2 + y_1^2 y_2) y_7 + y_1^2 y_2^2$
$f_{2.1.4} = y_6$
$f_{2.1.3} = y_5$
$f_{2.1.2} = y_4 - y_1$
$f_{2.1.1} = y_3 - y_1.$

Now the only polynomial which can cause the emptiness of $Zero(PD(ASC_{1.2.1})/IS \cup \{y_1 y_8\})$ is $f_{2.1.5}$, whose initial coefficient is $I_{2.1.5} = 3y_2^2 - 3y_1 y_2 + y_1^2$ and whose discriminate is $-3(y_2 - y_1)^2 y_2^2 y_1^2$. We only need to decompose the set (in 16.7s):

$$Zero(\{I_1, I_{2.1.5}, y_5\} \cup ASC_{1.2.1} \cup ES/IS \cup \{y_1 y_8\}) = Zero(PD(ASC_{1.2.1.1})/\{y_1 y_8\})$$

where $ASC_{1.2.1.1} =$

$f_{2.1.1.10} = y_1 y_{12} - y_7 y_8$
$f_{2.1.1.9} = (y_2 - y_1) y_{11} - y_2 y_9$
$f_{2.1.1.8} = (y_2 - y_1) y_{10} - y_2 y_8$
$f_{2.1.1.7} = ((y_2 - y_1) y_7) y_9 + ((-2y_2 + y_1) y_7 + y_1 y_2) y_8$
$f_{2.1.1.6} = (6y_2 - 3y_1) y_7 - 3y_1 y_2 + y_1^2$
$f_{2.1.1.5} = y_6$
$f_{2.1.1.4} = y_5$
$f_{2.1.1.3} = y_4 - y_1$
$f_{2.1.1.2} = y_3 - y_1$
$f_{2.1.1.1} = 3y_2^2 - 3y_1 y_2 + y_1^2.$

Then the emptiness of $Zero(PD(ASC_{1.2.1.1})/\{y_1 y_8\})$ can be easily checked.

Also for (4.21) we have (in 31.8 seconds):

$$Zero(\{I_1, I_{2.3}\} \cup ASC_{1.2} \cup ES/IS \cup \{y_1 y_8\}) = Zero(PD(ASC_{1.2.2})/IS \cup \{y_1 y_8\}) \tag{4.22}$$

where $ASC_{1.2.2} =$

$f_{2.2.9} = y_1 y_{12} - y_7 y_8$
$f_{2.2.8} = (y_2 - y_1) y_{11} + (y_6 - y_2) y_9$
$f_{2.2.7} = (y_2 - y_1) y_{10} + (y_5 - y_2) y_8$
$f_{2.2.6} = (y_5 y_6 - y_2 y_5) y_9 + ((-y_5 + y_2) y_6) y_8$

$f_{2.2.5} = (y_2y_6 + (-y_2 + y_1)y_5 - y_1y_2)y_7 + ((-y_2 + y_1)y_5 - y_1^2)y_6 + (y_1y_2 - y_1^2)y_5 + y_1^3$
$f_{2.2.4} = (2y_5 - y_1)y_6 - y_1y_5 + y_1^2$
$f_{2.2.3} = y_5^2 - y_1y_5 + y_1^2$
$f_{2.2.2} = y_4 - y_1$
$f_{2.2.1} = y_3 - y_1.$

Then the emptiness of $Zero(PD(ASC_{1.2.2})/IS \cup \{y_1y_8\})$ can be easily checked.

Now we come to $ASC_{1.3}$. The only polynomial that can cause the emptiness of $Zero(PD(ASC_{1.3})/IS \cup \{y_1y_8\})$ is $f_{3.1}$, whose discriminate is $-3(y_3 - y_1)^2y_2^2y_1^2$ and whose initial coefficient is $I_{3.1} = y_2^2 - y_1y_2 + y_1^2$. Thus we only need to decompose the follow sets:

$$Zero(\{I_1, y_3 - y_1\} \cup ASC_{1.3} \cup ES/IS \cup \{y_1y_8\}) \tag{4.23}$$

$$Zero(\{I_1, I_{3.1}\} \cup ASC_{1.3} \cup ES/IS \cup \{y_1y_8\}). \tag{4.24}$$

For (4.23) we have (in 271.4 seconds):

$$Zero(\{I_1, y_3 - y_1\} \cup ASC_{1.3} \cup ES/IS \cup \{y_1y_8\}) =$$
$$Zero(PD(ASC_{1.3.1})/IS \cup \{y_1y_8\}) \bigcup Zero(PD(ASC_{1.3.2})/IS \cup \{y_1y_8\}).$$

where $ASC_{1.3.1} =$
$f_{3.1.9} = y_1y_{12} - y_7y_8$
$f_{3.1.8} = (y_2 - y_1)y_{11} - y_2y_9$
$f_{3.1.7} = (y_2 - y_1)y_{10} - y_2y_8$
$f_{3.1.6} = ((y_2 - y_1)y_7)y_9 + ((-2y_2 + y_1)y_7 + y_1y_2)y_8$
$f_{3.1.5} = (3y_2^2 - 3y_1y_2 + y_1^2)y_7^2 + (-3y_1y_2^2 + y_1^2y_2)y_7 + y_1^2y_2^2$
$f_{3.1.4} = y_6$
$f_{3.1.3} = y_5$
$f_{3.1.2} = y_4 - y_1$
$f_{3.1.1} = y_3 - y_1,$

and $ASC_{1.3.2} =$
$f_{3.2.9} = y_1y_{12} - y_7y_8$
$f_{3.2.8} = (y_2 - y_1)y_{11} + (y_6 - y_2)y_9$
$f_{3.2.7} = (y_2 - y_1)y_{10} + (y_5 - y_2)y_8$
$f_{3.2.6} = (y_5y_6 - y_2y_5)y_9 + ((-y_5 + y_2)y_6)y_8$
$f_{3.2.5} = (y_2y_6 + (-y_2 + y_1)y_5 - y_1y_2)y_7 + ((-y_2 + y_1)y_5 - y_1^2)y_6 + (y_1y_2 - y_1^2)y_5 + y_1^3$
$f_{3.2.4} = (2y_5 - y_1)y_6 - y_1y_5 + y_1^2$
$f_{3.2.3} = y_5^2 - y_1y_5 + y_1^2$
$f_{3.2.2} = y_4 - y_1$
$f_{3.2.1} = y_3 - y_1.$

$ASC_{1.3.1} = ASC_{1.2.1}$, so it has already been shown to be empty. For $ASC_{1.3.2}$, since the only non-linear polynomial, $f_{3.2.3}$, has discriminate $-y_1^2$, it is trivial that $Zero(PD(ASC_{1.3.2})/IS \cup \{y_1y_8\})$ is empty.

For (4.24) we have (in 3839.2 seconds):

$$Zero(\{I_1, I_{3.1}\} \cup ASC_{1.3} \cup ES/IS \cup \{y_1 y_8\}) =$$
$$Zero(PD(ASC_{1.3.3})/IS \cup \{y_1 y_8\}) \bigcup Zero(PD(ASC_{1.3.4})/IS \cup \{y_1 y_8\}),$$

where $ASC_{1.3.3} =$

$f_{3.3.9} = y_3 y_{12} - y_7 y_8$
$f_{3.3.8} = (y_4 - y_2) y_{11} + (-y_6 + y_2) y_9$
$f_{3.3.7} = (y_3 - y_2) y_{10} + (-y_5 + y_2) y_8$
$f_{3.3.6} = (y_3 - y_1) y_9 + (-y_4 + y_1) y_8$
$f_{3.3.5} = ((y_2 y_3 - y_1 y_2) y_5 + (-y_1 y_3 + y_1 y_2) y_4 + (-y_2 + y_1) y_3^2) y_7 + (((-y_2 + y_1) y_3) y_4 - y_1 y_3^2 + y_1 y_2 y_3) y_5 + (y_2 y_3^2 - y_1 y_2 y_3) y_4$
$f_{3.3.4} = (((y_2 - y_1) y_4 + (-y_2 + y_1) y_3) y_5 + (-y_2 y_3 + y_1 y_2) y_4 + (y_1 y_2 - y_1^2) y_3 - y_1^2 y_2 + y_1^3) y_6 + ((y_2 y_3 - y_1 y_2 + y_1^2) y_4 - y_1 y_2 y_3 + y_1^2 y_2 - y_1^3) y_5$
$f_{3.3.3} = (3y_3 - 3y_1) y_4 + (-4y_2 + 2y_1) y_3 + y_1 y_2 - 2y_1^2$
$f_{3.3.2} = (y_2 + y_1) y_3^2 + (-6y_1 y_2 + 3y_1^2) y_3 + 2y_1^2 y_2 - 4y_1^3$
$f_{3.3.1} = y_2^2 - y_1 y_2 + y_1^2$

and $ASC_{1.3.4} =$

$f_{3.4.9} = y_3 y_{12} - y_7 y_8$
$f_{3.4.8} = (y_4 - y_2) y_{11} + (-y_6 + y_2) y_9$
$f_{3.4.7} = (y_3 - y_2) y_{10} + (-y_5 + y_2) y_8$
$f_{3.4.6} = (y_3 - y_1) y_9 + (-y_4 + y_1) y_8$
$f_{3.4.5} = ((y_2 y_3 - y_1 y_2) y_5 + (-y_1 y_3 + y_1 y_2) y_4 + (-y_2 + y_1) y_3^2) y_7 + (((-y_2 + y_1) y_3) y_4 - y_1 y_3^2 + y_1 y_2 y_3) y_5 + (y_2 y_3^2 - y_1 y_2 y_3) y_4$
$f_{3.4.4} = (((y_2 - y_1) y_4 + (-y_2 + y_1) y_3) y_5 + (-y_2 y_3 + y_1 y_2) y_4 + (y_1 y_2 - y_1^2) y_3 - y_1^2 y_2 + y_1^3) y_6 + ((y_2 y_3 - y_1 y_2 + y_1^2) y_4 - y_1 y_2 y_3 + y_1^2 y_2 - y_1^3) y_5$
$f_{3.4.3} = ((y_2 + y_1) y_3^2 + (-6y_1 y_2 + 3y_1^2) y_3 + 2y_1^2 y_2 - 4y_1^3) y_5 - y_2 y_3^3 + (2y_1 y_2 - 3y_1^2) y_3^2 + (3y_1^2 y_2 - y_1^3) y_3 + 2y_1^4$
$f_{3.4.2} = (y_2 - 2y_1) y_4 + y_1 y_3 - y_1 y_2 + y_1^2$
$f_{3.4.1} = y_2^2 - y_1 y_2 + y_1^2.$

The last polynomial in both $ASC_{1.3.3}$ and $ASC_{1.3.4}$ is $y_2^2 - y_1 y_2 + y_1^2$ which does not have a real solution when $y_1 \neq 0$. Hence (4.24) is also empty.

Thus $Zero(ES/IS \cup \{y_1 y_8\}) = Zero(PD(ASC_1)/IS \cup \{y_1 y_8\})$ is empty, and we have finally proved:

Theorem (4.25). Formula (4.1) is a theorem in Euclidean geometry. Since it is not a theorem in *WMG*, it cannot be proved *without* using the axioms of order.

(4.26). *Remark about the proof.* The first decomposition (4.2) took 13,020.8 seconds. The next two most time-consuming are (4.9) (in 8443.5 seconds) and (4.24) (in 3839.2 seconds), but these two computations were already done in (4.2). Besides, if one first recognizes that the discriminate of $I_1 = (y_2^2 - y_1 y_2 + y_1^2) y_4^2 + ((y_1 y_2 - 2y_1^2) y_3 - 2y_1 y_2^2 + y_1^2 y_2) y_4 + y_1^2 y_3^2 - y_1^2 y_2 y_3 + y_1^2 y_2^2$ is $-3(y_3 - y_1)^2 y_2^2 y_1^2$, then (4.9) would be much easier. The same holds for (4.24).

Alternatively, we can use the decomposition forms (in 15,439.3 seconds):

$$(4.27) \qquad Zero(ES/IS \cup \{y_1 y_8\}) = \bigcup_{1 \le k \le 77} Zero(ASC_k/IS \cup J_k \cup \{y_1 y_8\}),$$

where the J_k are the initial sets of the ascending chains ASC_k, ASC_1 is the same as in (4.2), the rest 76 ascending chains have lengths > 7, and each of them has at least one quadratic polynomial (in the leading variable). For each quadratic polynomial $ax^2 + bx + c$, we only need to check whether $b^2 - 4ac \leq 0$, and don't have to take care of the case when $a = 0$.

4.3. Finding the Weakest Non-Degenerate Conditions

One might think that condition that all eight $A, ..., H$ points are distinct are too strong and try to find some weaker conditions. We can use our proposed method to solve this problem. Our goal now is to find the "weakest" conditions to make Formula (2.4) a theorem in Euclidean geometry. Thus, now we try to decide whether $\mathbf{R}\text{–}Zero(ES/\{y_1 y_8\})$ is empty. Using our program for Ritt–Wu's zero decomposition again (in 15911.1 seconds) we have:

$$Zero(ES/\{y_1 y_8\}) = \bigcup_{1 \leq k \leq 4} Zero(PD(ASC_k)/\{y_1 y_8\}), \tag{4.28}$$

where the ascending chain ASC_1 is the same as the previous subsection, and

$ASC_2 =$

$$\begin{aligned} & y_{12} \\ & y_{11} \\ & y_{10} \\ & y_9 \\ & y_7 \\ & y_5 - y_2 \\ & y_4 - y_1, \end{aligned}$$

$ASC_3 =$

$$\begin{aligned} & y_3 y_{12} - y_6 y_8 \\ & y_3 y_{11} - y_6 y_8 \\ & y_3 y_{10} - y_5 y_8 \\ & y_9 - y_8 \\ & y_7 - y_6 \\ & y_4 - y_3 \\ & y_2, \end{aligned}$$

$ASC_4 =$

$$\begin{aligned} & y_{12} - y_8 \\ & (y_4 - y_1) y_{11} + (-y_5 + y_1) y_9 \\ & (y_3 - y_1) y_{10} + (-y_5 + y_1) y_8 \\ & (y_3 - y_1) y_9 + (-y_4 + y_1) y_8 \\ & y_7 - y_3 \\ & y_6 - y_5 \\ & y_2 - y_1. \end{aligned}$$

Note that our program produced 185 irreducible ascending chains. The dimension theorem [6] cut 185 to 4. By human inspection, it is easy to see that the ASC_2, ASC_3, and ASC_4 correspond to the cases:

Case 1. $D_1(A, \cdots, H)$: $A = H$, $B = E$, $D = F$; A, B, D, and G are collinear (Figure 3).

Case 2. $D_2(A, \cdots, H)$: $A = D$, $C = E$, $H = G$; A, C, G, and F are collinear (Figure 4).

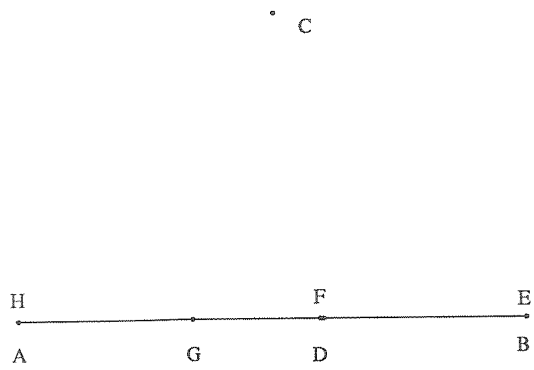

Figure 3

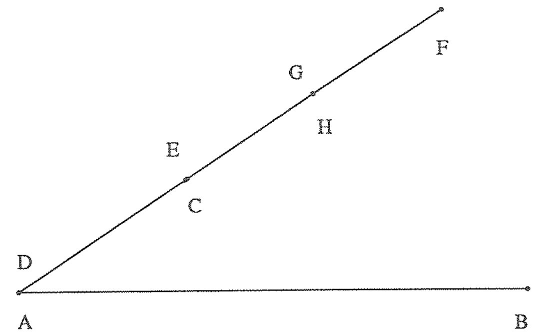

Figure 4

Case 3. $D_3(A, \cdots, H)$: $B = D$, $C = H$, $F = G$; B, C, E, and F are collinear (Figure 5).

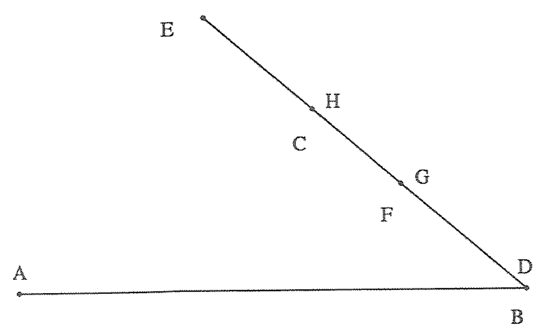

Figure 5

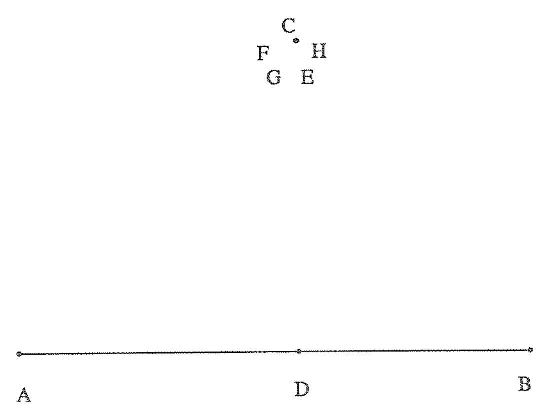

Figure 6

As in Section 4.2, we can continue to decompose

$$Zero(\{y_3 - y_2\} \cup ES/\{y_1 y_8\}) \tag{4.29}$$

$$Zero(\{y_4 - y_3\} \cup ES/\{y_1 y_8\}) \tag{4.30}$$

$$Zero(\{y_4 - y_1\} \cup ES/\{y_1 y_8\}) \tag{4.31}$$

$$Zero(\{y_2\} \cup ES/\{y_1 y_8\}) \tag{4.32}$$

(4.33) $$Zero(\{y_1\} \cup ES/\{y_1 y_8\}).$$

(4.34) $$Zero(\{I_1\} \cup ES/\{y_1 y_8\}).$$

We can repeat the Ritt–Wu's decomposition and the TSC method recursively. Since the above polynomial sets have basic sets of lower ranks than that of ES (see [6]), the process will terminate. The final result is:

(4.35) $$\mathbf{R}\!-\!Zero(ES/\{y_1 y_8\}) = \bigcup_{2 \le k \le 7} \mathbf{R}\!-\!Zero(PD(ASC_k)/\{y_1 y_8\}),$$

where

$ASC_5 =$

$y_{12} - y_8$
$y_{11} - y_8$
$y_{10} - y_8$
$y_9 - y_8$
$y_7 - y_3$
$y_6 - y_3$
$y_5 - y_3$
$y_4 - y_3,$

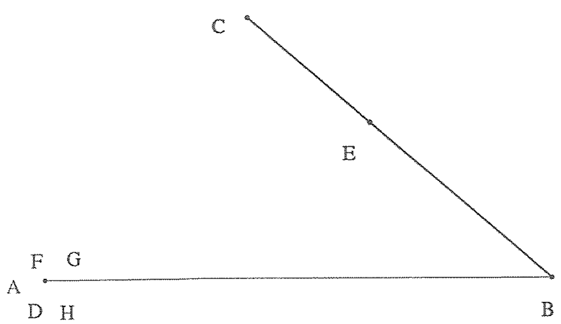

Figure 7

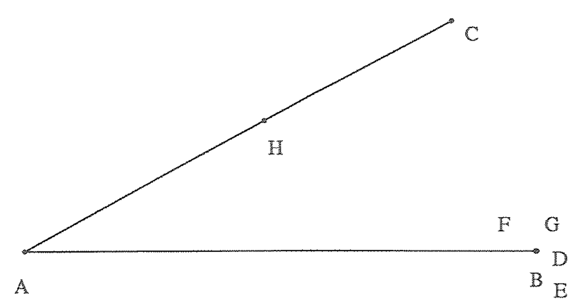

Figure 8

$ASC_6 =$

y_{12}
y_{11}
y_{10}
$(y_3 - y_1)y_9 + (-y_4 + y_1)y_8$
y_7
y_6
y_5
$y_2,$

and $ASC_7 =$

$$\begin{array}{l} y_3 y_{12} - y_7 y_8 \\ y_{11} \\ y_{10} \\ y_9 \\ y_6 - y_1 \\ y_5 - y_1 \\ y_4 - y_1 \\ y_2 - y_1. \end{array}$$

By human inspection, it is easy to find that ASC_i $(i = 5, 6, 7)$ correspond to the following geometric cases.

Case 4. $D_4(A, \cdots, H)$: $C = E = F = G = H$; A, B and D are collinear. (Figure 6).

Case 5. $D_5(A, \cdots, H)$: $A = D = F = G = H$; B, E and C are collinear. (Figure 7).

Case 6. $D_6(A, \cdots, H)$: $B = D = E = F = G$; A, C and H are collinear. (Figure 8).

Thus we have found the "weakest" non-degenerate conditions for the 8_3 theorem:

$$D'(A, \cdots, H) = \neg[D_1(A, \cdots, H) \vee \cdots \vee D_6(A, \cdots, H)],$$

and proved the following theorem.

Theorem (4.36). The formula

$$\forall A \cdots \forall H \ [(\text{MacLane}(A, \cdots, H) \wedge D'(A, \cdots, H) \Rightarrow \text{collinear}(A, B, C)],$$

is a theorem in Euclidean geometry.

Since $(C \neq E \wedge D \neq F \wedge B \neq D) \Rightarrow D'(A, ..., H)$, we have the following 'simpler' form:

Theorem (4.37). The formula

$$\forall A \cdots \forall H \ [(\text{MacLane}(A, \cdots, H) \wedge C \neq E \wedge D \neq F \wedge B \neq D) \Rightarrow \text{collinear}(A, B, C)],$$

is a theorem in Euclidean geometry.

We have verified Theorem (4.37) using the same method as described in Section 4.2. The proof led to the same decomposition (4.2) with a different $IS = \{(y_9 - y_8)z + y_4 - y_3, y_{10}z + y_5 - y_2, y_2 - y_1\}$ (in 8804.1 seconds). This was more complicated than the proof of Theorem (4.1).

Remark (4.38). We have been discussing the 8_3 configuration problem in the form given by [13]. Another way to see this 8_3 configuration is [MacLane(A, B, C, D, E, F, G, H) $\Rightarrow$ (at least one of the other 3-tuples of points is collinear)]. There are 48 other 3-tuples of points, and in this formulation of the problem IS consists of 48 line equations. It took 12,709.0 seconds to complete the decomposition of $Zero(ES/IS)$:

$$Zero(ES/IS) = Zero(PD(ASC_1)/IS),$$

where the irreducible ascending ASC_1 is the same as in (4.2).

5. Conclusions

We propose a new method to combine Ritt–Wu's method with the Tarski–Seidenberg–Collins method, and use it to solve the 8_3 configuration problem. Our solution shows:

(1) Formula (4.1) (the exact statement of MacLane theorem) is not a theorem in the theory of metric geometry *WMG*. We use Ritt–Wu's method to reach this conclusion. The TSC method cannot reach this conclusion.

(2) Formula (4.1) is a theorem in Euclidean geometry. In geometry, this means that it cannot be proved without using the axioms of order. We use a combination of Ritt–Wu's method with the TSC method to solve it.

There is no inclusion relationship between the scopes of Ritt–Wu's method and the TSC method. However, in Euclidean geometry the W class is a subset of all statements that the TSC method can address.

In [13], a set of 'careful' translations (of geometry statements into algebraic expressions) was described, and it was claimed that "provers based on (Ritt–Wu's) characteristic set method cannot take advantage of the careful translations." This claim was incorrect. As shown by this paper and [6], Ritt–Wu's method can deal with inequations easily and efficiently. Especially in Section 4.2 and Remark (4.38), the *IS* consist of 28 and 48 inequations, respectively. It is far beyond the time and space limits available for any known techniques based on the Gröbner basis method. Furthermore, many theorems, such as Thèbault's theorem, Morley's trisector theorem specified in Chapter 4 of [5], cannot be solved by the present techniques based on the Gröbner basis method.

In this paper we only outlined the proposed method. Being not experts in Collins' method, we could not go into the details of Collins' method. Many things need to be done to make the method complete and feasible in implementation.

References

[1] S.C. Chou, "Proving Elementary Geometry Theorems Using Wu's Algorithm", in *Automated Theorem Proving: After 25 years*, Ed. By W.W. Bledsoe and D. Loveland, AMS Contemporary Mathematics Series **29** (1984), 243-286.

[2] S.C. Chou, "Proving and Discovering Theorems in Elementary Geometries Using Wu's Method", PhD Thesis, Department of Mathematics, University of Texas, Austin (1985).

[3] S.C. Chou, "A Step toward Mechanically Proving Geometry Theorems Involving Inequality – Experimental Results", Preprint, March 1986, Institute for Computing Science, University of Texas at Austin.

[4] S.C. Chou, X.S. Gao and D. Arnon, "A Step Toward Mechanical Proving of Geometry Theorems Involving Inequality", Preprint, Department for Computer Sciences, University of Texas at Austin, 1988, to appear in the "*Issues in Robotics and Nonlinear Geometry*" Volume, of *Advances in Computing Research.*

$$
\begin{array}{l}
y_3 y_7 - y_2 y_4 \\
2y_6 - y_1 \\
2y_3 y_5 + y_1 y_2 \\
y_3^2 + y_2^2,
\end{array}
$$

and $ASC_3 =$

$$
\begin{array}{l}
y_3 y_{11} - y_2 y_{10} \\
(2y_1 y_2 - y_1^2) y_{10} - y_2 y_3 y_7 + (y_2^2 - 2y_1 y_2 + y_1^2) y_4 \\
y_3 y_9 + (-y_2 + y_1) y_8 - y_1 y_3 \\
y_3 y_7 + (-y_2 + y_1) y_4 - y_1 y_3 \\
2y_6 - y_1 \\
2y_3 y_5 - y_1 y_2 + y_1^2 \\
y_3^2 + y_2^2 - 2y_1 y_2 + y_1^2.
\end{array}
$$

Thus $\mathbf{C}$–$Zero(ES/\{y_1 y_3, g\})$ is not empty. Therefore Simson's theorem without the assumption that AB, AC, and BC are non-isotropic *is not* a theorem in the theory *WMG*. However, since isotropic lines do not exist in Euclidean geometry, it is still a theorem in Euclidean geometry.