AN ANALYSIS OF LOOP CHECKING MECHANISMS FOR LOGIC PROGRAMS

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An Analysis of Loop Checking Mechanisms for Logic Programs

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Abstract

We systematically study loop checking mechanisms for logic programs by considering their soundness, completeness, relative strength and related concepts. We introduce a natural concept of a *simple loop check* and prove that no sound and complete simple loop check exists, even for programs without function symbols. Then we introduce a number of sound simple loop checks and identify natural classes of PROLOG programs without function symbols for which they are complete. In these classes a limited form of recursion is allowed. As a by-product we obtain an implementation of the closed world assumption of Reiter [19] and a query evaluation algorithm for these classes of logic programs .

1. Introduction

1.1. Motivation

PROLOG has been advocated as a programming language which allows us to write executable specifications. Unfortunately, when interpreting correct specifications written in the form of a logic program as a PROLOG program, a divergence usually arises. This is due to the fact that the PROLOG interpreter

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uses a depth-first search and consequently can enter an infinite branch and miss a solution.

The problem of detecting such a possibility of divergence is obviously undecidable as PROLOG has the full power of recursion theory. Consequently this problem has been taken care of by developing a number of useful heuristics on how to avoid a possibility of non-termination. However, the resulting program can be very different from the original specification.

Another possible approach to this problem has been based on modifying the underlying computation mechanism that searches through the corresponding SLD-trees by adding a capability of pruning. Pruning an SLD-tree means that at some point the interpreter is forced to stop its search through a certain part of the tree, typically an infinite branch. Every method of pruning SLD-trees considered so far has been based on excluding some kind of repetition in the SLD-derivations, because such a repetition makes the interpreter enter an infinite loop. That is why pruning SLD-trees has been called *loop checking*. Such modifications of PROLOG interpreters were considered in the literature (see e.g. [3], [4], [8], [18], [20], [21] and [22]), but no results were proved about them, with notable exceptions of [20], [21] and [22].

1.2. Plan of the paper

In this paper we systematically study loop checking mechanisms. To this end, after providing in section 2 a sufficiently general definition of a loop check, we introduce in section 3 the relevant concepts, like soundness (no computed answer substitution to a goal is missed), completeness (all resulting derivations are finite) and relative strength. We also introduce there a natural subclass of loop checks, called *simple* loop checks, obtained when their definition does not depend on the analyzed logic programs. We prove among others that no sound and complete simple loop check exists even in the absence of function symbols.

In the remainder of the paper we study a number of intuitive simple loop checks. We can divide them into three groups, which are studied in the sections 4,5 and 6 respectively. For each group we prove the appropriate soundness results and identify one or more natural classes of programs without function symbols for which the loop checks in the group are complete. The loop checks in all three groups appear to be complete for *restricted* programs without function symbols. Restricted programs allow a restricted form of recursion (hence the name).

The first group consists of loop checks based on the *equality* between goals, respectively resultants, of the derivations and is studied in section 4. We call these loop checks *equality checks*.

The second group loop checks based on the *inclusion* between goals, respectively resultants, of the derivations and is studied in section 5. We call these loop checks *subsumption checks*. Subsumption checks are stronger than the corresponding equality checks and therefore they prune SLD-derivations earlier than their counterparts. This makes it more difficult to establish their soundness but opens a possibility for completeness for a larger class of programs than restricted ones.

We show that subsumption checks are complete for logic programs without function symbols in which no variables are introduced in the clause bodies (so called *nvi programs*). Also, the subsumption checks are complete for logic programs without function symbols in which a variable occurs at most once in every clause body (so called *svo programs*). These completeness theorems make use of a simple version of Kruskal's Tree Theorem, called Higman's Lemma [12]. While the use of this theorem to establish termination of term rewriting systems is well-known (see e.g. [9] or [13]), we have not encountered any applications of this theorem in the area of logic programming.

The third group is based on a simple loop check introduced by Besnard [3] and is studied in section 6. These checks test for equality of atoms in a certain context (a goal or a resultant). Therefore we call them *context checks*. We prove that for certain selection rules, the subsumption checks are stronger than the context checks.

As mentioned above, we prove that context checks are complete for restricted programs without function symbols. Besnard [3] claims without proof that a context check is also complete for nvi programs without function symbols.

1.3. Example

To better understand the relevance of the problems studied here, consider the following example. Let P be the following simple-minded PROLOG program computing in the relation *tc* the transitive closure of the relation *r*:

$P = \{ tc(x,y) \leftarrow r(x,y).$

$\quad tc(x,y) \leftarrow r(x,z),tc(z,y). \}$

Suppose we add to P the following facts about r: $r(a,a)\leftarrow$, $r(a,b)\leftarrow$, $r(b,c)\leftarrow$, $r(d,a)\leftarrow$. Then if we ask:

- tc(a,b) we get the answer 'yes';
- tc(a,c) the program gets into an infinite loop (whereas we should get the answer 'yes');
- tc(a,d) the program gets into an infinite loop (whereas we should get the answer 'no');
- tc(b,d) we get the answer 'no'.

Thus P is not the right program for computing the transitive closure. One solution is to write a different program, which is not straightforward - see for example the program in [7], section 7.2. In fact, Kunen [14] recently proved that any such program must use either function symbols or negated literals.

In our solution, we change the underlying interpreter by adding to it an equality check, and retain the above program, which turns out to be restricted. (In contrast, this solution cannot be applied to an alternative version of P obtained by replacing the second clause by $tc(x,y) \leftarrow tc(x,z),tc(z,y)$, as the resulting program is not any more restricted.)

1.4. Applications

As a by-product of these considerations we obtain an implementation of the *closed world assumption* of Reiter [19] and of a query evaluation mechanism for various classes of definite deductive databases. The closed world assumption (CWA in short) is a way of inferring negative information in deductive databases. Reiter [19] showed that in the case of definite deductive databases (DB in short) it does not introduce inconsistency. However, even though CWA is correctly defined for DB, there is still the problem of how it can be implemented, since it calls for the use of the following rule (or rather metarule):

$$\text{if } DB \nvdash \varphi \text{ then } DB \vdash \neg\varphi,$$

that is: deduce $\neg\varphi$ if φ cannot be proved from DB using first order logic.

The problem is how to determine for a particular ground atom (or *fact* in short) that there is no proof of it. The soundness and completeness results proved in section 4 show that when DB is a restricted program, to infer $\neg A$ for a fact A it suffices to use Clark's [5] *negation as (finite) failure* rule augmented with an appropriate equality check.

A more general problem is that of query processing in DB: given an atom A, compute the set $[A]_{DB}$ of all its ground instances $A\theta$ such that $DB \vdash A\theta$. Indeed, when A is ground and $DB \nvdash A$, the query processing problem reduces to the problem of deducing $\neg A$ by means of CWA. The results proved in section 4 imply that when DB is a restricted program, to compute $[A]_{DB}$ for an atom A, it suffices to collect all computed answer substitutions in the SLD-tree with leftmost selection rule and $\leftarrow A$ as root, pruned by an appropriate equality check.

Similar results concerning CWA and query processing hold for the subsumption and context checks and the corresponding classes of programs for which they are complete.

This paper is an extension of Apt, Bol & Klop [1], where exclusively equality checks were studied.

2. Loop checking

Throughout this paper we assume familiarity with the basic concepts and notations of logic programming as described in [15]. For two substitutions σ and τ, we write $\sigma \leq \tau$ when σ is more general than τ and for two expressions E and F, we write $E \leq F$ when F is an instance of E. We then say that F is *less general* than E.

Throughout this paper, by an SLD-derivation we mean an SLD-derivation in the sense of [15] *or an initial fragment of it*. In SLD-derivations we shall only use idempotent mgu's. It is known that any idempotent mgu is *relevant*, i.e. its domain contains only variables of the atoms that are unified. An SLD-derivation step from a goal G, using a clause C and an idempotent mgu θ, to a goal H is denoted as $G \Rightarrow_{C,\theta} H$.

2.1. Definitions

The purpose of a loop check is to prune every infinite SLD-tree to a finite subtree of it containing the root. One might define a loop check as a function from SLD-trees to SLD-trees, directly giving the pruned tree. However, this would be a very general definition, allowing practically everything. We shall use here a more restricted definition according to which for a program P:

- a node in an SLD-tree of $P \cup \{G\}$ (for some goal G) is *pruned* if all its descendants have been removed. (Note the terminology: the pruned node itself remains in the tree.)
- by pruning some of the nodes we obtain a pruned version of the SLD-tree.
- whether a node is pruned depends only upon its ancestors in the SLD-tree, that is on the SLD-derivation from the root up to this node.

Therefore, we can define a loop check as a function on the SLD-derivations instead of on the SLD-trees. However, for convenience we do not define it as a function from derivations to derivations, but as a set of derivations (depending on the program): the derivations that are pruned exactly at their last node. Such a set of SLD-derivations L(P) can be extended in a canonical way to a function $f_{L(P)}$ from SLD-trees to SLD-trees by pruning in an SLD-tree the nodes in { G | the SLD-derivation from the root to G is in L(P) }. In the remainder of this article, we shall usually make this conversion implicitly.

It is useful to note here that our definition of a loop check excludes more complicated pruning mechanisms for which the decision whether a node in a tree is pruned depends on the so far traversed fragment of the considered tree. Such mechanisms are for example studied in Vieille [22] and Seki & Itoh [21].

We shall also study an even more restricted form of a loop check, called simple loop check, in which the set of pruned derivations is independent of the program P. In other words, a loop check is a function, having a program as input and a simple loop check as output. This leads us to the following definitions.

DEFINITION 2.1.

Let L be a set of SLD-derivations.

RemSub(L) = { D ∈ L | L does not contain a proper subderivation of D }

L is *subderivation free* if L = RemSub(L). □

In order to render the intuitive meaning of a loop check L: ‘every derivation D ∈ L is pruned *exactly* at its last node’, we need that L is subderivation free. Note that RemSub(RemSub(L)) = RemSub(L).

In the following definition, by a *variant* of a derivation D we mean a derivation D' in which in every derivation step, atoms in the same positions are selected and the same programs clause are used. D' may differ from D in the renaming that is applied to these program clauses for reasons of standardizing apart and in the mgu used. It has been shown that in this case every goal in D' is a variant of the corresponding goal in D (see [16]). Thus any variant of an SLD-refutation is also an SLD-refutation and yields the same computed answer substitution up to a renaming.

DEFINITION 2.2.
A *simple loop check* is a computable set L of finite SLD-derivations such that
- for every derivation D: if D ∈ L then for every variant D' of D: D' ∈ L;
- L is subderivation free. □

The first condition here ensures that the choice of variables in the input clauses in an SLD-derivation does not influence its pruning. This is a reasonable demand since we are not interested in the choice of the names of the variables in the derivations.

DEFINITION 2.3.
A *loop check* is a computable function L from programs to sets of SLD-derivations such that for every program P, L(P) is a simple loop check. □

Of course, we can treat a simple loop check L as a loop check, namely as the constant function $\lambda P.L$.

DEFINITION 2.4.
Let L be a loop check. An SLD-derivation D of $P \cup \{G\}$ is *pruned by L* if L(P) contains a subderivation D' of D. □

2.2. Example

EXAMPLE 2.5 (Variant of Atom check).
(This example is based on Example 8 in [3], see also [10]).
A first attempt to formulate the *Variant of Atom* (*VA*) check might be: 'A derivation is pruned at the first goal that contains a variant A of an atom A' that occurred in an earlier goal.' Note that we have to allow here that A and A' are

variants: if we required A = A' then we would violate the first condition in Definition 2.2.

The intuition behind this loop check is the following. We wish to prove A' by resolution. If we find out after some resolution steps that in order to prove A' we need to prove a variant A of A', then there are two possibilities. One is that there is a proof for A. Then this proof could also be used as a proof for A', by applying an appropriate renaming on it. So we do not need the proof of A' that goes via A. The other possibility is that there is no proof for A. In that case, the attempt to prove A' via A cannot be successful. So in both cases there is no reason to continue the attempt to prove A' via A.

The derivation step $\leftarrow B,A \Rightarrow_{B\leftarrow} \leftarrow A$ shows that the first formulation of the VA check is not precise enough: it does not capture the intuition that the proof of A' *goes via* A. The atom A should be the result (after one or more derivation steps) of resolving A', or a further instantiated version of A' (if A' is not immediately selected).

Therefore we define VA =

RemSub({ D | D = ($G_0 \Rightarrow_{C_1,\theta_1} G_1 \Rightarrow \ldots \Rightarrow G_{k-1} \Rightarrow_{C_k,\theta_k} G_k$) such that for some i and j, $0 \le i \le j < k$, G_k contains an atom A that is
- a variant of an atom A' in G_i and
- the result of an attempt to resolve $A'\theta_{i+1}\ldots\theta_j$, the further instantiated version of A', that is selected in G_j }).

We now illustrate the use of this loop check.

Let P = { $A(0) \leftarrow$ (C1),
$B(1) \leftarrow$ (C2),
$A(x) \leftarrow A(y)$ (C3),
$C \leftarrow A(x),B(x)$ (C4) },

let G = $\leftarrow$ C.

That the informal justification of the loop check VA is incorrect, is shown by applying it to two SLD-trees of P∪{G}, via the leftmost and rightmost selection rule respectively, which gives us Figure 2.1. (In this figure and elsewhere a failed node, i.e. a node without a successor in the SLD-tree, is marked by a box around it.)

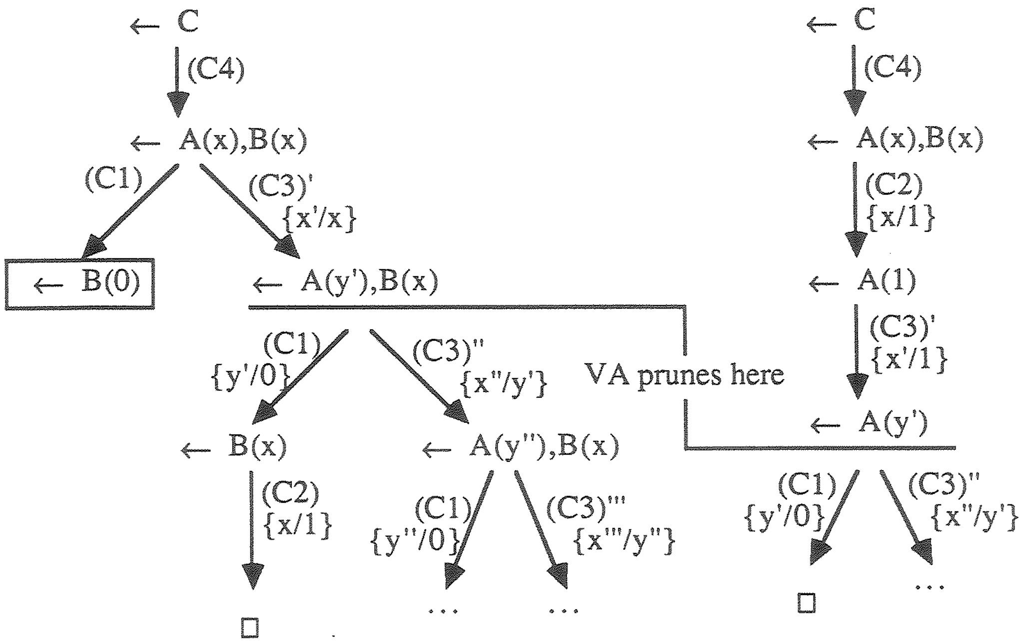


FIGURE 2.1

A detailed analysis shows why the goal $G_3 = \leftarrow A(y')$ in the rightmost tree is pruned by the VA check. Clearly, a variant of A(y') occurs in an earlier goal: A(x) in G_1. So we take i = 1. In G_1, A(x) is not yet selected, so j > i. In fact j = 2, for in G_2 the atom A(1), which is a further instantiated version of A(x), is selected. Indeed, A(y') is the result of resolving A(1). Therefore the derivation is pruned at G_3 by the VA check. (In this case, A(y') is the direct result of resolving A(1), but in general there may be any number of derivation steps between G_j and G_k.) □

Indeed, this loop check has not worked properly here: all successful derivations have been pruned. Clearly, this is an undesirable property for loop checks. On the other hand, all infinite derivations are pruned, as intended. In the next section, we shall give formal definitions of these and related properties of loop checks.

3. Some general considerations

In this section some basic properties of loop checks are introduced and some natural results concerning them are established.

3.1. Soundness and completeness

The most important property is definitely that using a loop check does not result in a loss of success. Since we intend to use pruned trees instead of the original ones, we need at least that pruning a successful tree yields again a successful tree.

Even stronger, because we use here a PROLOG-like interpreter augmented with a loop check as the *only* inference mechanism, we do not want to lose any individual solution. That is, if the original tree contains a successful branch (with some computed answer substitution), then we require that the pruned tree contains a successful branch with a more general answer substitution.

Finally, we would like to retain only shorter derivations and prune the longer ones that give the same result. This leads to the following definitions, where for a derivation D, |D| stands for its length, i.e. the number of goals in it.

DEFINITION 3.1 (Soundness).

i) A loop check L is *weakly sound* if for every program P and goal G, and SLD-tree T of $P \cup \{G\}$: if T contains a successful branch, then $f_{L(P)}(T)$ contains a successful branch.

ii) A loop check L is *sound* if for every program P and goal G, and SLD-tree T of $P \cup \{G\}$: if T contains a successful branch with a computed answer substitution σ, then $f_{L(P)}(T)$ contains a successful branch with a computed answer substitution σ' such that $\sigma' \leq \sigma$.

iii) A loop check L is *shortening* if for every program P and goal G, and SLD-tree T of $P \cup \{G\}$: if T contains a successful branch D with a computed answer substitution σ, then either $f_{L(P)}(T)$ contains D or $f_{L(P)}(T)$ contains a successful branch D' with a computed answer substitution σ' such that $\sigma' \leq \sigma$ and $|D'| < |D|$. □

The following lemma is an immediate consequence of these definitions.

LEMMA 3.2. *Let L be a loop check.*

i) If L is shortening, then L is sound.

ii) If L is sound, then L is weakly sound. □

The purpose of a loop check is to reduce the search space for top-down interpreters. We would like to end up with a finite search space. This is the case when every infinite derivation is pruned.

DEFINITION 3.3 (Completeness).
A loop check L is *complete* if every infinite SLD-derivation is pruned by L. □

We must point out here that in these definitions we have overloaded the terms 'soundness' and 'completeness'. These terms do not refer here only to loop checks, but also to interpreters for logic programs (with or without a loop check). Such an interpreter is sound if any answer it gives is correct w.r.t. the intended model or the intended theory of the program. An interpreter is complete if it finds every correct answer within a finite time.

3.2. Interpreters and loop checks

When a top-down interpreter is augmented with a loop check, we obtain a new interpreter. The soundness and completeness of this new interpreter depends on the soundness and completeness of the old one, as well as on the soundness and completeness of the loop check. However, these relations are not trivial. For example, it is not true that adding a complete loop check to a complete interpreter yields again a complete interpreter.

These relationships are expressed in the following lemma's. We refer here to two interpreters: one searching the SLD-tree depth-first left-to-right (as the PROLOG interpreter does), and one searching breadth-first. Without a loop check, both interpreters are sound w.r.t. CWA. The breadth-first interpreter is also complete (but not complete w.r.t. CWA).

LEMMA 3.4. *Let P be a program, A a ground atom and L a weakly sound loop check. Then for every SLD-tree T of* $P \cup \{\leftarrow A\}$, $P \vdash_{CWA} \neg A$ *iff* $f_{L(P)}(T)$ *contains no successful branches.*

PROOF. We know by the soundness and strong completeness of SLD-resolution (see [2],[15]) that $P \vdash_{CWA} \neg A \Leftrightarrow P \nvdash A \Leftrightarrow$ T contains no successful branch.

$\Rightarrow$ T contains no successful branch and $f_{L(P)}(T)$ is a subtree of T, so $f_{L(P)}(T)$ contains no successful branch either.

$\Leftarrow$ Since L is weakly sound, a successful branch in T would yield a successful branch in $f_{L(P)}(T)$. But $f_{L(P)}(T)$ contains no successful branch, hence T contains no successful branch either. □

Thus an interpreter augmented with a weakly sound loop check remains sound w.r.t. CWA. Since $f_{L(P)}(T)$ may be infinite, nothing can be said about completeness.

LEMMA 3.5. *Let P be a program, A an atom and L a sound loop check. Then for every SLD-tree T of $P \cup \{\leftarrow A\}$ and for every ground substitution θ, $P \vdash A\theta$ iff $f_{L(P)}(T)$ contains a successful branch with a computed answer substitution τ such that $\tau \leq \theta$.*

PROOF. We have by the strong completeness of SLD-resolution $P \vdash A\theta \Leftrightarrow$ T contains a successful branch with a computed answer substitution σ such that $\sigma \leq \theta$.

$\Rightarrow$ T contains this successful branch, and since L is sound, $f_{L(P)}(T)$ contains a successful branch with a computed answer substitution τ such that $\tau \leq \sigma$. Hence $\tau \leq \theta$.

$\Leftarrow$ $f_{L(P)}(T)$ contains a successful branch with a computed answer substitution $\tau \leq \theta$, so T contains this branch as well. □

Thus an interpreter augmented with a sound loop check remains sound. Moreover, a breadth-first interpreter remains complete .

COROLLARY 3.6. *Let P be a program, A a ground atom and L a weakly sound and complete loop check. Then for every SLD-tree T of $P \cup \{\leftarrow A\}$, $P \vdash_{CWA} \neg A$ iff $f_{L(P)}(T)$ is finite and contains no successful branches.*

PROOF. By Lemma 3.4 and the Completeness Definition 3.3. □

Thus an interpreter augmented with a weakly sound and complete loop check becomes complete w.r.t. CWA.

COROLLARY 3.7. *Let P be a program, A an atom and L a sound and complete loop check. Then for every SLD-tree T of $P \cup \{\leftarrow A\}$ and for every ground substitution θ, $P \vdash A\theta$ iff $f_{L(P)}(T)$ is finite and contains a successful branch with a computed answer substitution τ such that $\tau \leq \theta$.*

PROOF. By Lemma 3.5 and the Completeness Definition 3.3. □

Thus a depth-first interpreter augmented with a sound and complete loop check becomes complete . This also means that a sound and complete loop check can be used to implement query processing as defined in the introduction. Indeed, given a program P and an atom A with an SLD-tree T of $P \cup \{\leftarrow A\}$, it suffices to traverse the finite tree $f_{L(P)}(T)$ and collect all computed answer substitutions.

3.3. Comparing loop checks

After studying the relationships between loop checks and interpreters, we shall now analyze a relationship between loop checks themselves. In general, it can be quite difficult to compare loop checks. However, some of them can be compared in a natural way: if every loop that is detected by one loop check, is detected at the same derivation step or earlier by another loop check, then the latter one is *stronger* than the former.

DEFINITION 3.8.

Let L_1 and L_2 be loop checks.

L_1 is *stronger than* L_2 if for every program P and goal G, every SLD-derivation $D_2 \in L_2(P)$ of $P \cup \{G\}$ contains a subderivation D_1 such that $D_1 \in L_1(P)$. □

In other words, L_1 is stronger than L_2 if every SLD-derivation that is pruned by L_2 is also pruned by L_1. Note that the definition implies that every loop check is stronger than itself.

The following theorem will prove to be very useful. It will enable us to obtain soundness and completeness results for loop checks which are related by the 'stronger than' relation, by proving soundness and completeness for only one of them.

THEOREM 3.9 (Relative Strength). *Let L_1 and L_2 be loop checks, and let L_1 be stronger than L_2.*

i) If L_1 is weakly sound, then L_2 is weakly sound.

ii) If L_1 is sound, then L_2 is sound.

iii) If L_1 is shortening, then L_2 is shortening.

iv) If L_2 is complete then L_1 is complete.

PROOF. i)-iii) If an SLD-tree T contains a successful branch, then $f_{L_1(P)}(T)$ contains a successful branch that satisfies the conditions of Definition 3.1. Since L_1 is stronger than L_2, $f_{L_1(P)}(T)$ is a subtree of $f_{L_2(P)}(T)$, so this branch is also contained in $f_{L_2(P)}(T)$.

iv) Every infinite SLD-derivation is pruned by L_2, so it is also pruned by L_1. □

Now we have a clearer view of the situation. Very strong loop checks prune derivations in an 'early stage'. If they prune too early, then they are unsound. Since this is undesirable, we must look for weaker loop checks. But a loop check should preferably be not too weak, for then it might fail to prune some infinite derivations (in other words, it might be incomplete). Of course, the 'stronger than' relation is not linear. Moreover, loop checks exist that are neither sound nor complete.

3.4 Sound and complete loop checks

A question now arises: do there exist sound and complete loop checks? Obviously, there cannot be such a loop check for logic programs in general, as logic programming has the full power of recursion theory. (Remember that according to the definition, a loop check is computable.) So when studying completeness we shall rule out programs that compute over an infinite domain. We shall do so by restricting our attention to programs without function symbols, so called *function-free* programs. This restriction leads to a finite Herbrand Universe, but other solutions (typed functions, bounded term-size property [11]) are also possible here.

Note that our definitions so far referred to arbitrary programs and SLD-derivations. In the sequel, we shall consider certain classes of programs (like function-free programs) and SLD-derivations (like the derivations via the leftmost selection rule). The definitions we introduced can be extended in an obvious way so that we can use terminology like 'complete w.r.t. the leftmost selection rule for function-free restricted programs'.

As stated above, we shall study completeness only for function-free programs. So our question can be reformulated as: is there a sound and complete loop check for function-free programs? Before answering this question for loop checks in general, we shall answer it for simple loop checks.

THEOREM 3.10. *There is no weakly sound and complete simple loop check for function-free programs.*

PROOF. Let L be a simple loop check that is complete for function free programs. Consider the following infinite SLD-derivation D, obtained by repeatedly using the clause $A(x) \leftarrow A(y),S(y,x)$ (using the leftmost selection rule).

$$\leftarrow A(x_0),B(x_0)$$
$$\Downarrow$$
$$\leftarrow A(x_1),S(x_1,x_0),B(x_0)$$
$$\Downarrow$$
$$\leftarrow A(x_2),S(x_2,x_1),S(x_1,x_0),B(x_0)$$
$$\Downarrow$$
$$\leftarrow A(x_3),S(x_3,x_2),S(x_2,x_1),S(x_1,x_0),B(x_0)$$
$$\Downarrow$$
$$\ldots$$

FIGURE 3.1

Since L is a complete loop check, this derivation is pruned by L and since L is simple, the goal at which pruning takes place is independent of the program used for this derivation. Suppose this derivation is pruned by L at the goal $\leftarrow A(x_n),S(x_n,x_{n-1}),\ldots,S(x_1,x_0),B(x_0)$.

Now let $P = \{ S(i,i+1) \leftarrow \mid 0 \leq i < n \} \cup \{ A(0) \leftarrow, A(x) \leftarrow A(y),S(y,x), B(n) \leftarrow \}$. Extending the above derivation to an SLD-tree of $P \cup \{G\}$ (still using the leftmost selection rule, see Figure 3.2), we see that every goal of the derivation has two descendents, obtained by applying the clauses $A(x) \leftarrow A(y),S(y,x)$ and $A(0) \leftarrow$ respectively. The derivation of Figure 3.1 shows the effect of repeatedly applying A(x)←A(y),S(y,x). After applying A(0)← at some goal, a derivation becomes deterministic: if there are initially m S-atoms, then these atoms are resolved from left to right by the clauses S(0,1)←,..., S(m-1,m)←.

$$
\begin{array}{lll}
\leftarrow A(x_0),B(x_0) & & \Rightarrow \boxed{\leftarrow B(0)} \\
\Downarrow & & \\
\leftarrow A(x_1),S(x_1,x_0),B(x_0) & \Rightarrow \leftarrow S(0,x_0),B(x_0) & \Rightarrow \boxed{\leftarrow B(1)} \\
\Downarrow & & \\
\leftarrow A(x_2),S(x_2,x_1),S(x_1,x_0),B(x_0) & \Rightarrow \leftarrow S(0,x_1),S(x_1,x_0),B(x_0) & \\
 & \quad \Rightarrow \leftarrow S(1,x_0),B(x_0) & \Rightarrow \boxed{\leftarrow B(2)} \\
\Downarrow & & \\
\ldots & & \\
\Downarrow & & \\
\leftarrow A(x_n),S(x_n,x_{n-1}),\ldots,S(x_1,x_0),B(x_0) & \Rightarrow \ldots \text{ n intermediate goals} \ldots & \Rightarrow \leftarrow B(n) \\
 & & \quad \Rightarrow \square \\
\Downarrow & & \\
\ldots & &
\end{array}
$$

FIGURE 3.2

Finally, the goal ←B(m) is left. Since of all goals of the form ←B(i) ($i \geq 0$) only the goal ←B(n) can be refuted, exactly n S-atoms are needed. Therefore the only successful branch of this SLD-tree of P∪{G} goes via the goal $\leftarrow A(x_n),S(x_n,x_{n-1}),\ldots,S(x_1,x_0),B(x_0)$. As exactly this goal is pruned by L, L has pruned the only successful branch of this SLD-tree. Hence L is not weakly sound. □

However, taking the program into account gives us an opportunity to define for function-free programs a shortening (so a fortiori sound) loop check which is complete. Moreover, this loop check is stronger than *every* other shortening loop check. Strange as it may seem, this one is also impractical. It is like solving a puzzle by trial and error. One can save effort if one can avoid the trials that lead to an error. Assuming that the puzzle is solvable (as our 'puzzle', finding the correct answers to a given goal, is), it is possible to find out *exactly* which trials to avoid. How this can be done is formalized in the proof of Theorem 3.13 (1). However, solving the puzzle is the first step of the method described, so it can only be of theoretical importance.

For convenience, we shall write $S(P,G,\sigma)$ for the set of successful SLD-derivations of $P \cup \{G\}$ with a computed answer substitution τ such that $\tau \leq \sigma$. We say that a derivation D is a *shortest derivation in $S(P,G,\sigma)$* if $D \in S(P,G,\sigma)$ and $|D| = \min \{ |D'| \mid D' \in S(P,G,\sigma) \}$.

DEFINITION 3.11 (STRONG check).
For a function-free program P, $\mathrm{STRONG}(P) = \mathrm{RemSub}(\{ D = G \Rightarrow \ldots \mid$ for no σ, D is an initial fragment of a shortest derivation in $S(P,G,\sigma\})$. □

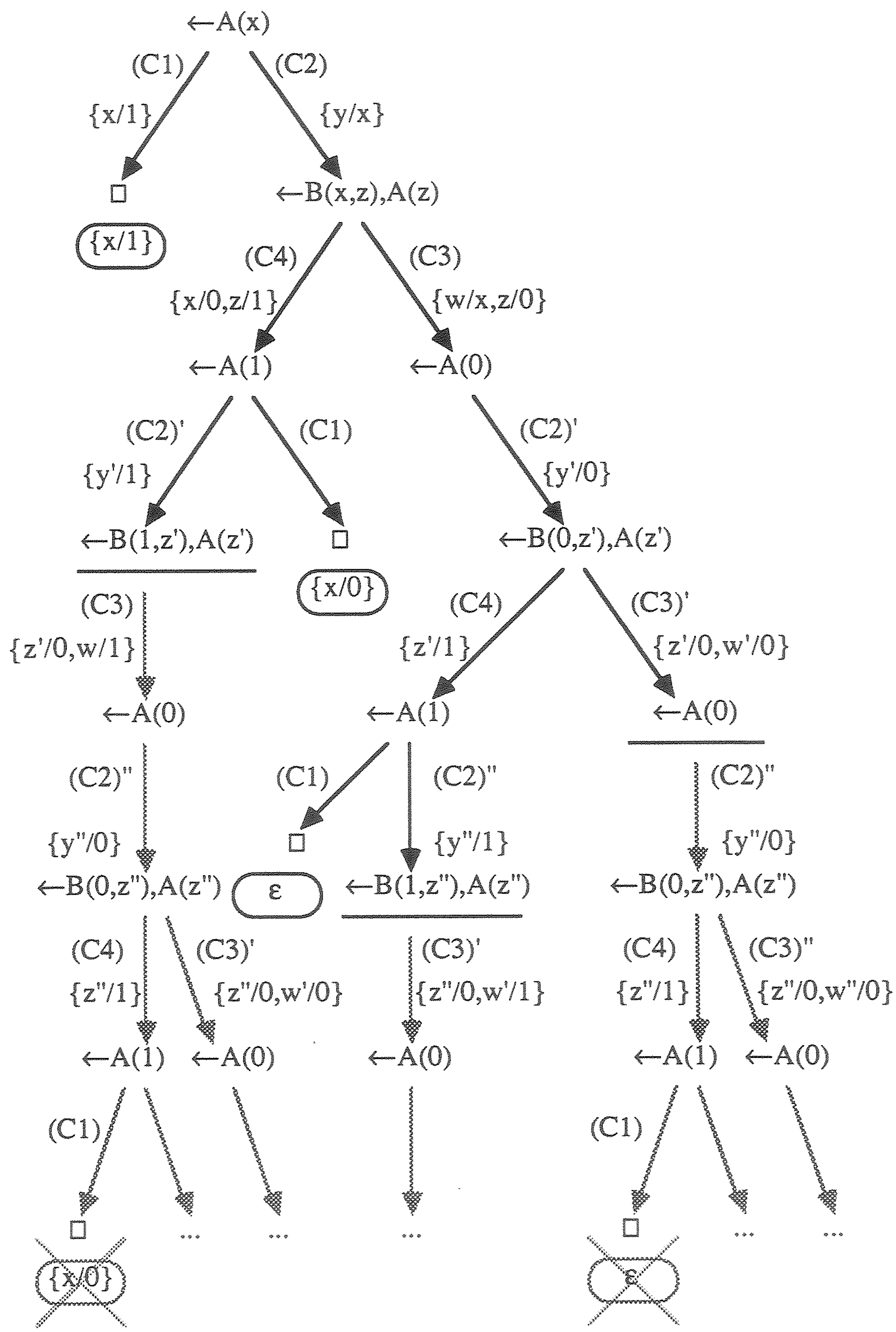
Note that an SLD-tree pruned by STRONG consists not only of the shortest refutation(s) of $P \cup \{G\}$ for any computed answer substitution σ, but also of the derivations that follow the path of such a derivation but 'make a wrong decision', that is a step deviating from such a refutation. After such a step, the derivation is immediately pruned by STRONG. This effect is a consequence of the fact that pruning a node in a tree implies removing *all* descendents, so we cannot remove the descendents caused by a 'wrong step' while retaining the others. The following example shows the effect of pruning an SLD-tree by STRONG.

EXAMPLE 3.12.

Let P = { A(1) ← (C1),
A(y) ← B(y,z),A(z) (C2),
B(w,0) ← (C3),
B(0,1) ← (C4) },

and let G = ←A(x).

Consider an SLD-tree of $P \cup \{G\}$ displayed in Figure 3.3. In $S(P,G,\{x/1\})$ a shortest derivation has 2 goals, in $S(P,G,\{x/0\})$ a shortest derivation has 4 goals and in $S(P,G,\varepsilon)$ a shortest derivation has 6 goals. These derivations are retained by STRONG in the considered SLD-tree, the others are pruned (at the horizontal lines in the figure). Among these are successful ones, but not shortest ones. (The tree in Figure 3.3 is extended beyond the sixth level to show this effect.) □



←A(x)
(C1)
{x/1}
(C2)
{y/x}
□
{x/1}
←B(x,z),A(z)
(C4)
{x/0,z/1}
(C3)
{w/x,z/0}
←A(1)
←A(0)
(C2)'
{y'/1}
(C1)
(C2)'
{y'/0}
←B(1,z'),A(z')
□
{x/0}
←B(0,z'),A(z')
(C3)
{z'/0,w/1}
(C4)
{z'/1}
(C3)'
{z'/0,w'/0}
←A(0)
←A(1)
←A(0)
(C2)"
{y"/0}
(C1)
(C2)"
{y"/1}
(C2)"
{y"/0}
←B(0,z"),A(z")
□
ε
←B(1,z"),A(z")
←B(0,z"),A(z")
(C4)
{z"/1}
(C3)'
{z"/0,w'/0}
(C3)'
{z"/0,w'/1}
(C4)
{z"/1}
(C3)"
{z"/0,w"/0}
←A(1)
←A(0)
←A(0)
←A(1)
←A(0)
(C1)
(C1)
□
...
...
...
□
...
...
{x/0}
ε

FIGURE 3.3

THEOREM 3.13. *For function-free programs:*

i) STRONG is a shortening loop check.

ii) STRONG is stronger than any shortening loop check.

iii) STRONG is complete.

PROOF. i a) STRONG is a loop check.

The non-trivial point here is to prove that for every function-free program P, STRONG(P) is computable. Can we, given a derivation $D = G \Rightarrow \ldots$, decide whether or not D is pruned by STRONG and if so, at which node? Indeed we can, using the following procedure.

1. Compute the set of correct answer substitutions for $P \cup \{G\}$ (e.g. bottom up). Since P has no function symbols, this set is finite. Construct (breadth first) an initial fragment of an SLD-tree of $P \cup \{G\}$ that contains (an initial part of) D and for each correct answer substitution a successful branch with a more general computed answer substitution. Such a fragment exists by the strong completeness of SLD-resolution. It has been shown in [17] that a length preserving bijection exists between the successful branches of two different SLD-trees for $P \cup \{G\}$. Therefore in every SLD-tree of $P \cup \{G\}$, for every correct answer substitution σ there exists a derivation $D' \in S(P,G,\sigma)$ with $|D'| = \min \{ |D''| \mid D'' \in S(P,G,\sigma) \}$.

2. For each computed answer substitution, mark the nodes of the shortest successful branches with this computed answer substitution.

3. Prune D at the first node in the tree that is not marked. If such a node does not exist, then D is a subderivation of a shortest successful branch.

i b) STRONG is shortening.

If a successful derivation D of $P \cup \{G\}$ with computed answer substitution σ is pruned by STRONG, then it is not a shortest derivation in $S(P,G,\sigma)$. By construction, there exists a shortest derivation $D' \in S(P,G,\sigma)$ in the SLD-tree. D' is shorter than D and not pruned by STRONG.

ii) STRONG is stronger than any shortening loop check.

Let L be a loop check and let D be a derivation of $P \cup \{G\}$ that is pruned by L. If D is a subderivation of a shortest successful derivation D', then L is not shortening. Otherwise, D is pruned by STRONG.

iii) STRONG is complete.

If D is an infinite SLD-derivation, then only an initial fragment of D is contained in the constructed (finite) part of the SLD-tree. Since the last goal of D that is in the tree is not successful, D contains a 'wrong step' there or earlier. Hence D is pruned by STRONG. □

So far, we have not been very successful in defining useful sound and complete loop checks. In the next section, we shall restrict our attention to simple loop checks. They will be shortening, but as shown above, they cannot be complete (not even for function-free programs). Nevertheless, for each of these loop checks we shall introduce one or more natural classes of programs for which they are complete.

4. Equality checks

4.1. Overview

In this section, we introduce some simple loop checks. For each of them, there exist two versions: the first one is weakly sound, the second one shortening. The second, shortening version is obtained by adding an additional condition to the first one. By this construction, the first version is always stronger than the corresponding second version.

Starting with the Variant of Atom check, we can make three independent modifications of it.

1. Adding this additional condition, dealing with the computed answer substitution 'generated so far'. A neat formulation of this condition can be obtained by the use of *resultants* instead of goals in SLD-derivations. When considering a derivation $G_0 \Rightarrow_{C1,\theta 1} G_1 \Rightarrow \ldots$, to every goal $G_i = \leftarrow S_i$ there corresponds the resultant $R_i = S_0\theta_1 \ldots \theta_i \leftarrow S_i$. Resultants were introduced in [16].
2. Replace *variant* by *instance*. This yields the *Instance of Atom* (*IA*) check. This check is still unsound: it is even stronger than the VA check. Besnard [3] has introduced a weakly sound version of this loop check. This check and related ones (derived from VA; shortening versions) are discussed in section 6.
3. Replace *atom* by *goal*. This yields the *Equals Variant of Goal* (*EVG*) check. Informally, this loop check prunes a derivation as soon as a *goal* occurs that is a variant of an earlier goal. Replacing 'variant' by 'instance' again yields the *Equals Instance of Goal* (*EIG*) check. The shortening versions are called *Equals Variant of Resultant* (*EVR*) and *Equals Instance of Resultant* (*EIR*).

Taking goals instead of atoms as a basis for a loop check yields two independent choices again.

3a. Whereas equality between atoms is unambiguous, equality between goals is much less clear. In SLD-derivations, we regard goals as lists, so both the number and the order of occurrences of atoms is important. However, we may also regard them as multisets, where the order of the occurrences is unimportant. We might even consider regarding them as sets, but that proves to be impractical: the difference between the derivation steps $\leftarrow A,A \Rightarrow \leftarrow A$ and $\leftarrow A \Rightarrow \leftarrow A$ is then no longer visible. Regarding goals as sets in our loop checks would require regarding goals as sets in SLD-derivations, which would result in too many undesirable effects.

So we shall consider *two* EVG checks: EVG_L (for list) and EVG_M (for multiset). The same holds for EIG, EVR and EIR. We shall refer to these eight loop checks as the *equality* checks. They are discussed in the remainder of this section.

3b. Finally, we may replace 'G_2 *is* a variant/instance of G_1' by 'G_2 *is subsumed by* a variant/instance of G_1'. We define 'G_1 subsumes G_2' as '$G_1 \subseteq G_2$'. Thus we can make a distinction between 'subsumed by a variant' and 'subsumed by an instance'. Usually in literature, 'subsumed by a variant' is not considered, 'subsumed by an instance' is simply called 'subsumed' (see e.g. [6]). Subsumption can also be defined for resolvents.

This yields the *subsumption* check. Since this modification is again independent of the others, there are also eight subsumption checks. These checks are discussed in section 5.

4.2. Formal definitions

We now study the equality checks in more detail. At first we give a formal definition of the weakly sound versions. Then we introduce an additional condition that makes these checks shortening. Finally, we identify a natural class of programs for which the equality checks are complete.

In fact, we should give a definition for each equality check. This would yield eight almost identical definitions. Therefore we compress them into two definitions, trusting that the reader is willing to understand our notation. The equality relation between goals regarded as lists is denoted by $=_L$; similarly $=_M$ for multisets. We begin with the weakly sound versions.

DEFINITION 4.1 (Equality checks for Goals).
For Type $\in$ {L,M}, the *Equals Variant/Instance of Goal$_{Type}$* check is the set of SLD-derivations

EVG/EIG_{Type} = RemSub({ D | D = ($G_0 \Rightarrow_{C_1,\theta_1} G_1 \Rightarrow \ldots \Rightarrow G_{k-1} \Rightarrow_{C_k,\theta_k} G_k$) such that for some i, $0 \leq i < k$, there is a renaming/substitution τ such that $G_k =_{Type} G_i\tau$ }). □

For example,

EIG_M = RemSub({ D | D = ($G_0 \Rightarrow_{C_1,\theta_1} G_1 \Rightarrow \ldots \Rightarrow G_{k-1} \Rightarrow_{C_k,\theta_k} G_k$) such that for some i, $0 \leq i < k$, there is a substitution τ such that $G_k =_M G_i\tau$ }).

The informal justification for these loop checks is similar to the one given for the VA check. Suppose that we want to refute a goal G. If we find that in order to refute G we need to refute a variant or instance of G, say $G\tau$, then two cases arise. If there is no solution for $G\tau$, then pruning $G\tau$ is clearly safe. On the other hand, if there is a solution for $G\tau$, then the derivation giving this solution might be used (possibly in a more general form) directly from G.

We shall prove later in this section that these loop checks are weakly sound. However, they are not sound. To see this, suppose that we find for $G\tau$ a successful derivation D with a computed answer substitution σ. Then using D directly from G gives a computed answer substitution $\tau\sigma$ (maybe a more general substitution, but not necessarily). Therefore success is not lost.

However, the derivation $G = G_i \Rightarrow_{C_{i+1},\theta_{i+1}} \cdots \Rightarrow_{C_k,\theta_k} G_k = G\tau$, followed by D, yields a possibly different computed answer substitution: $\theta_{i+1}\ldots\theta_k\sigma$, thus possibly affecting soundness. (In Example 4.3, we show a specific program and goal for which this difference arises.) Of course, we are only interested in the effect of this difference on the variables of the initial goal G_0. In G_i these variables are renamed to $G_0\theta_1\ldots\theta_i$. So τ and $\theta_{i+1}\ldots\theta_k$ should coincide on the variables of $G_0\theta_1\ldots\theta_i$.

Hence we can make these loop checks sound, and even shortening, by adding the condition $G_0\theta_1\ldots\theta_k = G_0\theta_1\ldots\theta_i\tau$. (Note that in this equality it is irrelevant whether goals are lists or multisets.) It will appear that this condition works not only for EVG and EIG, but for all other loop checks studied in this section, as well.

Finally, note that adding this condition is equivalent to the replacement of the condition $G_k =_{Type} G_i\tau$ by the condition $R_k =_{Type} R_i\tau$, where R_k and R_i are the resultants corresponding to the goals G_k and G_i.

DEFINITION 4.2 (Equality checks for Resultants).
For Type $\in$ {L,M}, the *Equals Variant/Instance of Resultant$_{Type}$* check is the set of SLD-derivations

EVR/EIR_{Type} = RemSub({ D | D = ($G_0 \Rightarrow_{C_1,\theta_1} G_1 \Rightarrow \ldots \Rightarrow G_{k-1} \Rightarrow_{C_k,\theta_k} G_k$) such that for some i, $0 \le i < k$, there is a renaming/substitution τ such that $G_k =_{Type} G_i\tau$ and $G_0\theta_1\ldots\theta_k = G_0\theta_1\ldots\theta_i\tau$ }). □

For example,

EVR_L = RemSub({ D | D = ($G_0 \Rightarrow_{C_1,\theta_1} G_1 \Rightarrow \ldots \Rightarrow G_{k-1} \Rightarrow_{C_k,\theta_k} G_k$) such that for some i, $0 \le i < k$, there is a renaming τ such that $G_k =_L G_i\tau$ and $G_0\theta_1\ldots\theta_k = G_0\theta_1\ldots\theta_i\tau$ }).

The following example shows the difference between the goal-based and resultant-based equality checks. It is so chosen that the other variations (variants or instances, goals regarded as lists or as multisets) do not play a role.

EXAMPLE 4.3.

Let P = { $p(a) \leftarrow$, (C1)
$p(y) \leftarrow p(z)$ (C2) },
let G = $\leftarrow p(x)$.

Without the condition $G_0\theta_1\ldots\theta_k = G_0\theta_1\ldots\theta_i\tau$ we would only obtain the computed answer substitution {x/a}, whereas we should also obtain the empty substitution. This shows that the EVG and EIG loop checks are not sound.

In the leftmost tree $\leftarrow p(z)$ is a variant of $\leftarrow p(x)$, so the derivation is pruned by EVG at that goal. However, the corresponding resultant $p(x) \leftarrow p(z)$ is clearly not a variant of $p(x) \leftarrow p(x)$, therefore the derivation is not yet pruned by EVR. After another application of (C2), the resultant $p(x) \leftarrow p(z')$ occurs, which is a variant of $p(x) \leftarrow p(z)$. At that point the derivation is pruned by EVR.

The rightmost tree in Figure 4.1 shows an 'SLD-tree' in which the goals are replaced by the corresponding resultants. Note that a successful branch in a resultant-based SLD-tree does not end by the empty goal □, but by the instance of the initial goal that was 'proved' by this branch. □

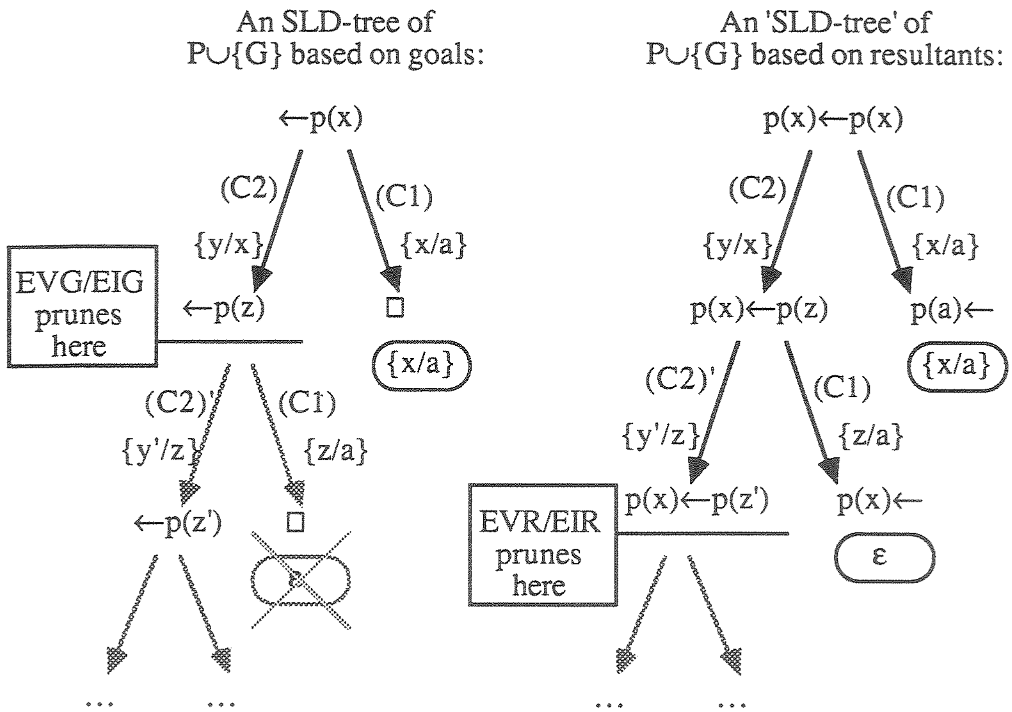


FIGURE 4.1

LEMMA 4.4. *All equality checks are simple loop checks.*
PROOF. Straightforward. □

Figure 4.2. shows the 'stronger than' relationships between the equality checks (and the VA and IA checks) and summarizes their properties. In this figure, an arrow $L_1 \rightarrow L_2$ means that L_2 is stronger than L_1. Proving these 'stronger than' relations is straightforward.

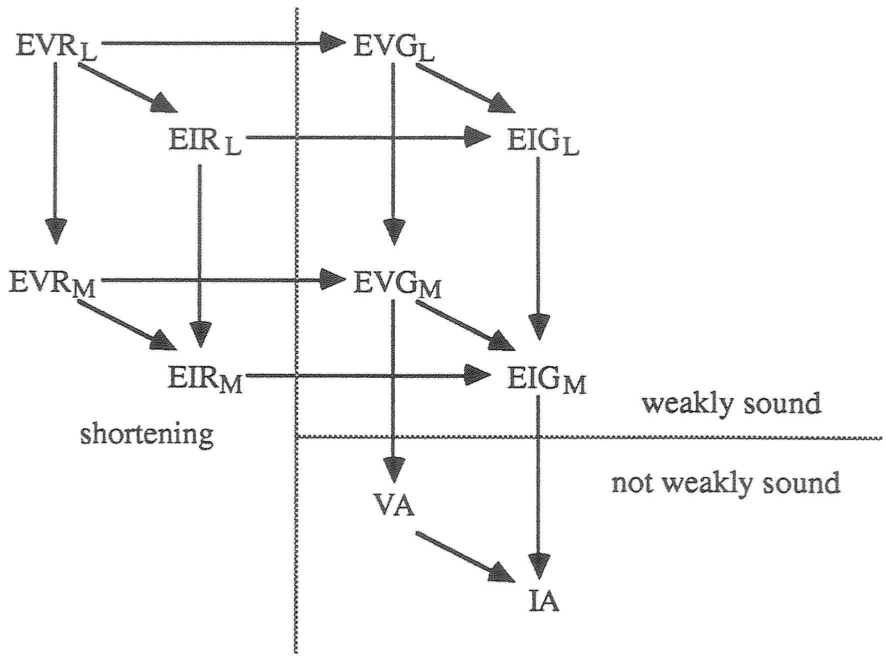


FIGURE 4.2

4.3. Soundness

We now prove that the equality checks based on resultants are shortening and that the equality checks based on goals are weakly sound. According to the Relative Strength Theorem 3.9 it is sufficient to focus on the strongest checks in both classes: the EIR_M and the EIG_M checks. We need the following lemma.

LEMMA 4.5. *Let P be a program. Let G_1 and G_2 be goals such that $G_1 =_M G_2$. Suppose D is an SLD-derivation of $P \cup \{G_1\}$ with computed answer substitution σ. Then every SLD-tree of $P \cup \{G_2\}$ contains a successful branch of length $|D|$ with a computed answer substitution σ.*

PROOF. By the soundness and strong completeness of SLD-resolution, see [15].

□

THEOREM 4.6. *i) The loop check EIR_M is shortening.*

ii) The loop check EIG_M is weakly sound.

PROOF. Let P be a program and G_0 a goal.

i) Let D be an SLD-refutation of $P \cup \{G_0\}$ with computed answer substitution σ. If D is pruned by EIR_M then we must find in every SLD-tree containing D an

SLD-refutation D' of $P \cup \{G_0\}$ with computed answer substitution σ' such that $\sigma' \leq \sigma$, $|D'| < |D|$ and D' is not pruned by EIR_M. We prove that D' exists by induction on the length l of the refutation D. We have $l \geq 1$. For $l = 1$, D cannot be pruned. Now suppose the theorem is true for every refutation of $P \cup \{G_0\}$ of length $\leq l$. Let D be a refutation of length $l+1$. Suppose that D is pruned by EIR_M. Then we have $D = (G_0 \Rightarrow_{C_1,\theta_1} G_1 \Rightarrow \ldots \Rightarrow G_{i-1} \Rightarrow_{C_i,\theta_i} G_i \Rightarrow_{C_{i+1},\theta_{i+1}} G_{i+1} \Rightarrow \ldots \Rightarrow G_{k-1} \Rightarrow_{C_k,\theta_k} G_k \Rightarrow_{C_{k+1},\theta_{k+1}} G_{k+1} \Rightarrow \ldots \Rightarrow \Box)$, and for some substitution τ: $G_k =_M G_i\tau$ and $G_0\theta_1 \ldots \theta_k = G_0\theta_1 \ldots \theta_i\tau$.

By Lemma 4.5 we have a refutation of $G_i\tau$ with computed answer substitution $\theta_{k+1} \ldots \theta_l$. Now we can obtain an unrestricted (in the sense of [15]) SLD-refutation $D_1 = (G_0 \Rightarrow_{C_1,\theta_1} G_1 \Rightarrow \ldots \Rightarrow G_{i-1} \Rightarrow_{C_i,\theta_i\tau} G_i\tau \Rightarrow \ldots \Rightarrow \Box)$ of $P \cup \{G_0\}$ (that is in the step $G_{i-1} \Rightarrow_{C_i,\theta_i\tau} G_i\tau$ we do not use an mgu), which is shorter than D. Using the Mgu Lemma of [15], we obtain an SLD-refutation D_2 of $P \cup \{G_0\}$ with the same length as D_1 and a computed answer substitution $\sigma_2 \leq \theta_1 \ldots \theta_i\tau\theta_{k+1} \ldots \theta_l$. (Lemma 4.5 and the proof of the Mgu Lemma show that for every SLD-tree containing D, such a derivation D_2 can be constructed.) D_2 is an SLD-refutation of $P \cup \{G_0\}$ which is shorter than D, so by the induction hypothesis there exists an SLD-refutation D_3 of $P \cup \{G_0\}$ with computed answer substitution σ_3 such that $\sigma_3 \leq \sigma_2$ and D_3 is not pruned by EIR_M. Now we can take $D' = D_3$ and we have $G_0\sigma' = G_0\sigma_3 \leq G_0\sigma_2 \leq G_0\theta_1 \ldots \theta_i\tau\theta_{k+1} \ldots \theta_l = G_0\theta_1 \ldots \theta_i\theta_{i+1} \ldots \theta_k\theta_{k+1} \ldots \theta_l = G_0\sigma$, so $\sigma' \leq \sigma$.

ii) Note that the additional condition $G_0\theta_1 \ldots \theta_k = G_0\theta_1 \ldots \theta_i\tau$ was only used to prove that $\sigma' \leq \sigma$. □

COROLLARY 4.7 (Equality Soundness).

i) All equality checks based on resultants are shortening. A fortiori they are sound

ii) All equality checks based on goals are weakly sound.

PROOF. By Theorem 4.6 and the Relative Strength Theorem 3.9. □

4.4. Completeness

For completeness issues, it is sufficient to consider the weakest of the equality checks: the EVR_L check. We know that EVR_L is not complete - Theorem 3.10 presents a counterexample that holds for every simple loop check. However, for the EVR_L check this counterexample can be simplified. The program in Theorem

3.10 consists of a collection of ground facts and one recursive clause. Clearly, this clause is the 'core' of the counterexample. It appears that for EVR_L, we need only this clause for a demonstration of its incompleteness. Moreover, we need only the propositional structure of the clause, i.e. we may remove the arguments.

EXAMPLE 4.8.
Let $P = \{ A \leftarrow A,S \}$.
Then for 'the' SLD-tree T of $P \cup \{\leftarrow A\}$ via the leftmost selection rule, $f_{EVR_L}(T)$ is infinite. Indeed, every descendant of the initial goal has one occurrence of S more than its parent goal, so it cannot be a variant of any of its ancestors. □

Obviously, the problem is that the atom A in the goal is allowed to generate infinitely many S-atoms, which are never selected, thereby making the goal wider and wider. We now introduce a class of programs for which this phenomenon cannot occur and we prove that EVR_L is complete for these programs. The necessary restriction is obtained by allowing at most one recursive call per clause and allowing such a call only after all other atoms in the body of the clause have been completely resolved. In order to avoid unnecessary complications, we shall place the atom that causes the recursive call (if present) at the right end of the body of the clause, and consider only derivations via the leftmost selection rule. For a formal definition, we use the notion of the *dependency graph* D_P of a program P.

DEFINITION 4.9.
The *dependency graph* D_P of a program P is a directed graph whose nodes are the predicate symbols appearing in P and

$(p,q) \in D_P$ iff there is a clause in P using p in its head and q in its body.

D_P^* is the reflexive, transitive closure of D_P. When $(p,q) \in D_P^*$, we say that p *depends on* q. For a predicate symbol p, the *class of p* is the set of predicate symbols p 'mutually depends' on: $cl_P(p) = \{ q \mid (p,q) \in D_P^* \text{ and } (q,p) \in D_P^* \}$.□

DEFINITION 4.10 (Restricted Program).
Given an atom A, let *rel(A)* denote its predicate symbol. Let P be a program.
A clause $A_0 \leftarrow A_1,\ldots,A_n$ ($n \geq 0$) is called *restricted w.r.t. P* if for $i = 1,\ldots,n-1$, $rel(A_i)$ does not depend on $rel(A_0)$ in P. The atoms $A_1,\ldots,A_{n-1}$ are called the *non-recursive* atoms of the clause $A_0 \leftarrow A_1,\ldots,A_n$.
A program P is called *restricted* if every clause in P is restricted w.r.t. P. □

Note that this definition allows at most one recursive call per clause. Thus (disregarding the order of atoms in the bodies) restricted programs include so called linear programs, which contain only one recursive clause and in this clause only a single recursive call occurs. The 'transitive closure' program given in the introduction is restricted. Note also that programs of which all clauses have a body with at most one atom are restricted.

We now prove that EVR_L is complete w.r.t. the leftmost selection rule for restricted programs. An interesting feature of restricted programs is that in each SLD-derivation via the leftmost selection rule, goals have a number of atoms which is bounded by a value depending only on the program and the initial goal. We shall show that this implies that modulo the 'being a variant of' relation, the number of possible goals in such an SLD-derivation is finite.

In the rest of this section, P is a function-free restricted program and G is a goal in L_P. With the *length* of G, $|G|$, we mean the number of atoms in G. The maximum length of the goals in a derivation of $P \cup \{G\}$ can be computed by means of the following *weight*-function, which is defined on goals and predicate symbols (by mutual induction).

DEFINITION 4.11.
Let P be a restricted program. Then the function *weight* is defined as:
i) for a goal $G = \leftarrow A_1,\ldots,A_n$ ($n \geq 1$) in L_P,

$$weight(G) = \max\{weight(rel(A_i))+n-i \mid i=1,\ldots,n\};$$

ii) for a predicate symbol p of P, weight(p) =

$$\begin{aligned}\max(\ &\{\ weight(\leftarrow A_1,\ldots,A_n) \mid \\ &\qquad A\leftarrow A_1,\ldots,A_n \in P,\ n>0,\ rel(A) \in clp(p),\ rel(A_n) \notin clp(p)\ \} \cup \\ &\{\ 1+weight(\leftarrow A_1,\ldots,A_{n-1}) \mid \\ &\qquad A\leftarrow A_1,\ldots,A_n \in P,\ n>1,\ rel(A) \in clp(p),\ rel(A_n) \in clp(p)\ \} \cup \\ &\{\ 1\ \}\ ).\end{aligned}$$

□

Note that in the definition of weight(p), clauses of the form $A \leftarrow B$, with $cl(rel(A)) = cl(rel(B))$ are not considered - they do not affect the length of goals appearing in a derivation. Moreover, if the predicate symbols p and q are mutually dependent, then weight(p) = weight(q).

The fact that P is restricted ensures that the weight-function is well-defined: if weight(p) is defined in terms of weight(q), then $(q,p) \notin D_P^*$, hence weight(q) is not defined in terms of weight(p). Intuitively, the weight of a goal G majorizes the length of all goals which appear in an SLD-derivation of $P \cup \{G\}$ using leftmost selection rule. More precisely, we have the following lemma's.

LEMMA 4.12. *$|G| \leq weight(G)$.*

PROOF. Let $G = \leftarrow A_1,\ldots,A_n$ $(n \geq 1)$. Then $weight(G) \geq weight(rel(A_1))+n-1 \geq n = |G|$. □

LEMMA 4.13. *Let $G \Rightarrow_C H$ be a derivation step w.r.t. P. Then $weight(G) \geq weight(H)$.*

PROOF. Since the weight of a goal depends only on the predicates appearing in it, and not on the arguments of these predicates, we prove this fact for the case of programs written in the propositional logic. Let $G = \leftarrow A_1,\ldots,A_n$; then $weight(G) = \max\{weight(A_i)+n-i \mid i=1,\ldots,n\}$, and let $C = A_1 \leftarrow B_1,\ldots,B_m$. Then the goal $H = \leftarrow B_1,\ldots,B_m,A_2,\ldots,A_n$ and therefore $weight(H) = \max(\{weight(B_i)+m+n-1-i \mid i=1,\ldots,m\} \cup \{weight(A_{i-m+1})+m+n-1-i \mid i=m+1, \ldots,m+n-1\}) = \max(\{weight(B_i)+m+n-1-i \mid i=1,\ldots,m\} \cup \{weight(A_i)+n-i \mid i=2,\ldots,n\})$. Two cases arise.

i) $weight(H) = \max\{weight(A_i)+n-i \mid i=2,\ldots,n\}$.
Then clearly $weight(H) \leq weight(G)$.

ii) $weight(H) = \max\{weight(B_i)+m+n-1-i \mid i=1,\ldots,m\}$ (hence $m>0$). We show that in this case $weight(H) \leq weight(A_1)+n-1$ (which is $\leq weight(G)$). Subtracting n-1, it suffices to show that $\max\{weight(B_i)+m-i \mid i=1,\ldots,m\} \leq weight(A_1)$. Again two cases arise.

iia) $(rel(B_m),rel(A_1)) \notin D_P^*$. Then because of the existence of C, $weight(A_1) \geq weight(\leftarrow B_1,\ldots,B_m) = \max\{weight(B_i)+m-i \mid i=1,\ldots,m\}$.

iib) $(\mathrm{rel}(B_m),\mathrm{rel}(A_1)) \in D_P{}^*$. Then $\mathrm{weight}(A_1) \geq 1+\mathrm{weight}(\leftarrow B_1,\ldots,B_{m-1}) = 1+\max\{\mathrm{weight}(B_i)+m-1-i \mid i=1,\ldots,m-1\} = \max\{\mathrm{weight}(B_i)+m-i \mid i=1,\ldots,m-1\}$. Also $\mathrm{weight}(B_m)+m-m = \mathrm{weight}(A_1)$, since $\mathrm{rel}(B_m) \in cl_P(\mathrm{rel}(A_1))$. This proves the claim that $\max\{\mathrm{weight}(B_i)+m-i \mid i=1,\ldots,m\} \leq \mathrm{weight}(A_1)$. □

COROLLARY 4.14. *Let $D = G_0 \Rightarrow G_1 \Rightarrow G_2 \Rightarrow \ldots \Rightarrow G_i \Rightarrow \ldots$ be an SLD-derivation. Then for every goal G_i in D: $|G_i| \leq \mathrm{weight}(G_0)$.*

PROOF. By induction on i. The induction basis is provided by Lemma 4.12, the induction step by Lemma 4.13. □

So $\mathrm{weight}(G_0)$ is indeed the desired maximum length of goals occurring in any SLD-derivation of $P\cup\{G_0\}$.

We now present a formalization of the 'being a variant of' relation on resultants. Our presentation here is more general than needed for the completeness proof for the equality checks. However, we need these results in full generality to prove the completeness of the subsumption checks and the context checks.

DEFINITION 4.15.

Let X be a set of variables. We define the relation $\sim_X$ on resultants as $R_1 \sim_X R_2$ if for some renaming ρ, $R_1\rho = R_2$ and for every $x \in X$, $x\rho = x$. Now let G be a goal and let $k\geq 1$. Then the relation $\sim_{X,G,k}$ stands for the restriction of the relation $\sim_X$ to resultants $G_1 \leftarrow G_2$ such that G_1 is an instance of G and $|G_2| \leq k$. □

LEMMA 4.16. *For every set of variables X, goal G and $k\geq 1$, $\sim_{X,G,k}$ is an equivalence relation.*

PROOF. Straightforward. □

For a resultant R, the equivalence class of R w.r.t. the relation $\sim_{X,G,k}$ will be denoted as $[R]_{X,G,k}$, or just [R] whenever X, G and k are clear from the context. The following lemma is crucial for our considerations.

LEMMA 4.17. *Suppose that the language L has no function symbols and finitely many predicate symbols and constants. Then for every finite set of variables X, goal G and $k \geq 1$, the relation $\sim_{X,G,k}$ has only finitely many equivalence classes.*

PROOF. Let C be the number of constants in L, R the number of predicate symbols and let a be the maximum arity of the predicate symbols in L. Let G be a goal of the form $\leftarrow p_1(\ldots),p_2(\ldots),\ldots,p_m(\ldots)$ with $m \geq 1$ and let V be the number of distinct variables in G.

A resultant in an equivalence class of $\sim_{X,G,k}$ is then of the form $p_1(\ldots),p_2(\ldots),\ldots,p_m(\ldots) \leftarrow q_1(\ldots),q_2(\ldots),\ldots,q_n(\ldots)$ with $0 \leq n \leq k$. An equivalence class of $\sim_{X,G,k}$ is completely described by the predicate symbols $q_1,\ldots,q_n$, the arguments of $p_1,\ldots,p_m$ (in accordance with G) and the arguments of $q_1,\ldots,q_n$.

The number of arguments that must be specified in this resultant is V for $p_1,\ldots,p_m$, plus at most $n \cdot a$ for $q_1,\ldots,q_n$. For every argument we may choose either a constant, a variable from X or another (fresh) variable. However, we need at most $V+n \cdot a$ different fresh variables. Therefore the choice of the arguments is limited to $(C+\#X+V+n \cdot a)^{V+n \cdot a}$ possibilities.

Since for a fixed n, the choice of the predicate symbols $q_1,\ldots,q_n$ is limited to R^n possibilities, we have at most $\sum_{n=0}^{k} R^n \cdot (C+\#X+V+n \cdot a)^{V+n \cdot a}$ equivalence classes of $\sim_{X,G,k}$. □

We can now prove the desired theorem.

THEOREM 4.18. *The loop check EVR_L is complete w.r.t. the leftmost selection rule for function-free restricted programs.*

PROOF. Let P be a function-free restricted program and let G_0 be a goal in L_P. Let k = weight(G_0). Consider an infinite SLD-derivation $D = G_0 \Rightarrow_{C_1,\theta_1} G_1 \Rightarrow \ldots \Rightarrow G_{i-1} \Rightarrow_{C_i,\theta_i} G_i \Rightarrow \ldots$ of $P \cup \{G_0\}$. By Corollary 4.14 for every $i \geq 0$: $|G_i| \leq k$. Every goal G_i is a goal in L_P and hence every resultant $G_0\theta_1\ldots\theta_i \leftarrow G_i$ belongs to an equivalence class of $\sim_{\emptyset,G_0,k}$. Since L_P satisfies the conditions of Lemma 4.17, $\sim_{\emptyset,G_0,k}$ has only finitely many equivalence classes, so for some $i \geq 0$ and $j > i$, $(G_0\theta_1\ldots\theta_i \leftarrow G_i)$ and $G_0\theta_1\ldots\theta_j \leftarrow G_j$ are variants. This implies that D is pruned by EVR_L. □

COROLLARY 4.19 (Equality Completeness). *All equality checks are complete w.r.t. the leftmost selection rule for function-free restricted programs.*

PROOF. By Theorem 4.18 and the Relative Strength Theorem 3.9. □

Now combining Corollary 3.6 and Corollary 3.7 with the Equality Soundness Corollary 4.7 and the Equality Completeness Corollary 4.19, we conclude that all equality checks lead to an implementation of CWA for function-free restricted programs. Moreover, a depth first interpreter augmented by any of the equality checks based on resultants yields an implementation of query processing for these programs.

5. Subsumption checks

As already stated, there are eight subsumption checks. We shall define them by means of two parametrized definitions, again trusting that the reader is willing to understand our notation. The inclusion relation between goals regarded as lists is denoted by $\subseteq_L$; similarly $\subseteq_M$ for multisets. Note: $L_1 \subseteq_L L_2$ if all elements of L_1 occur in the same order in L_2; they need not to occur on adjacent positions. For example, $(a,c) \subseteq_L (a,b,c)$.

5.1. Definitions

DEFINITION 5.1 (Subsumption checks for Goals).
For Type $\in$ {L,M}, the *Subsumes Variant/Instance of Goal$_{Type}$* check is the set of SLD-derivations

$$\mathrm{SVG/SIG}_{Type} = \mathrm{RemSub}(\{ D \mid D = (G_0 \Rightarrow_{C_1,\theta_1} G_1 \Rightarrow \ldots \Rightarrow G_{k-1} \Rightarrow_{C_k,\theta_k} G_k) \text{ such that for some } i,\ 0 \leq i < k, \text{ there is a renaming/substitution } \tau \text{ with } G_k \supseteq_{Type} G_i\tau \}).$$ □

DEFINITION 5.2 (Subsumption checks for Resultants).
For Type $\in$ {L,M}, the *Subsumes Variant/Instance of Resultant$_{Type}$* check is the set of SLD-derivations

$$\mathrm{SVR/SIR}_{\mathrm{Type}} = \mathrm{RemSub}(\{\ D \mid D = (\ G_0 \Rightarrow_{C_1,\theta_1} G_1 \Rightarrow \ldots \Rightarrow G_{k-1} \Rightarrow_{C_k,\theta_k} G_k) \text{ such that for some } i,\ 0 \le i < k, \text{ there is a renaming/ substitution } \tau \text{ with } G_k \sqsupseteq_{\mathrm{Type}} G_i\tau \text{ and } G_0\theta_1\ldots\theta_k = G_0\theta_1\ldots\theta_i\tau\ \}).$$ □

LEMMA 5.3. *All subsumption checks are simple loop checks.*
PROOF. Straightforward. □

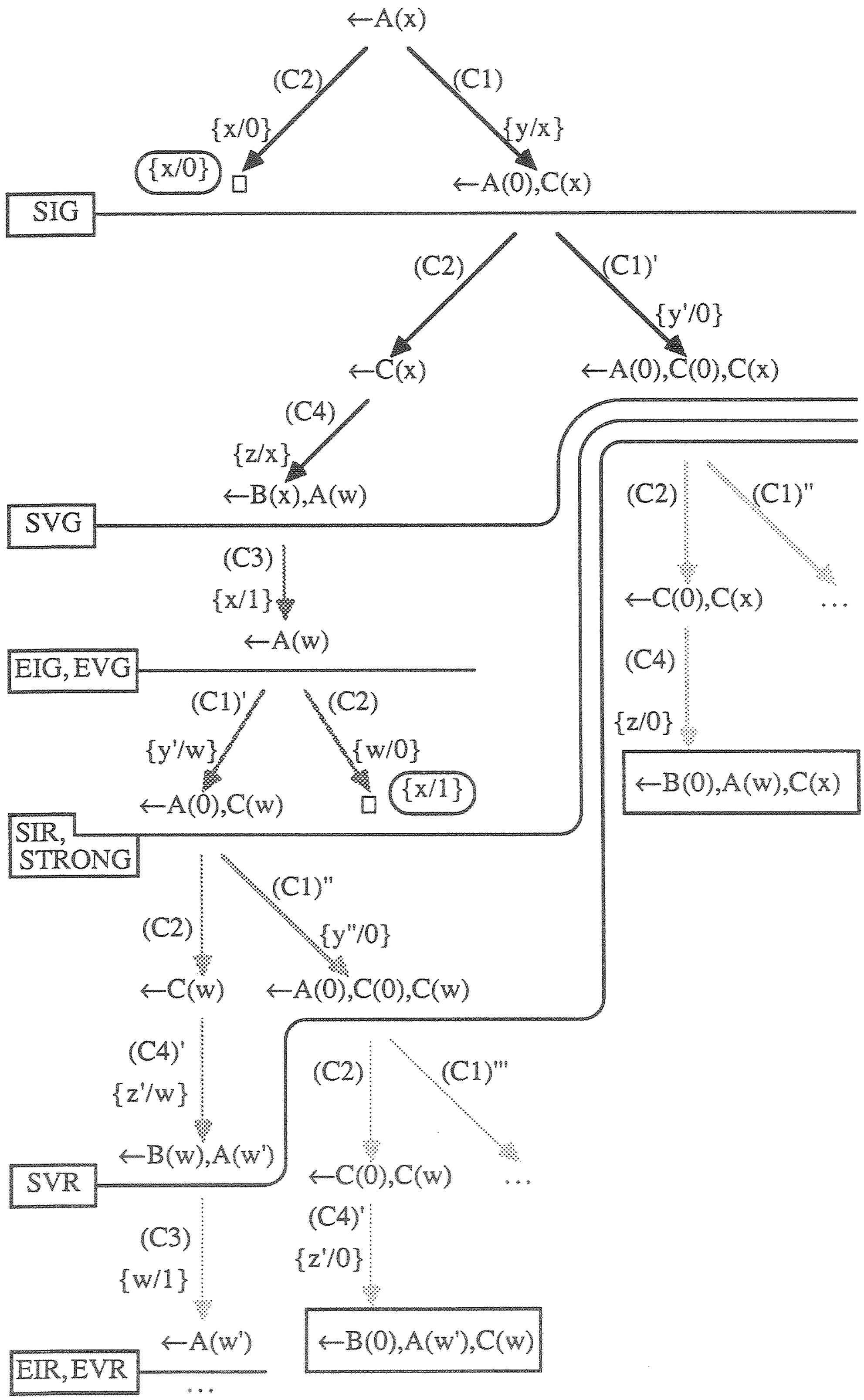
The following example shows the differences between the behaviour of various subsumption checks and the equality checks.

EXAMPLE 5.4.
Let P = { A(y) ← A(0),C(y) (C1),
A(0) ← (C2),
B(1) ← (C3),
C(z) ← B(z),A(w) (C4) },
and let G = ←A(x).

Figure 4.3 shows an SLD-tree of P∪{G} using the leftmost selection rule. It also shows how this tree is pruned by different loop checks. First we explain the behaviour of the loop checks with respect to this tree. Then we shall make some generalizing comments on this behaviour. In this example, the distinction between list versus multiset based loop checks does not play a role.

Starting at the root, the first loop check that prunes the tree is the SIG check. It prunes the goal ←A(0),C(x), because it contains A(0), an instance of A(x). Following the leftmost infinite branch two steps down, the SVG check prunes the goal ←B(x),A(w), because it contains A(w), a variant of A(x). One step later, the atom B(x) is resolved, so the EIG and EVG checks prune the goal ←A(w) for the same reason.

However, the loop checks based on resultants do not yet prune the tree. The computed answer substitution built up so far maps x to x after the first three steps and to 1 later on. This is clearly different from the substitutions {x/0} and {x/w}, which are used to show that A(0) resp. A(w) are an instance resp. a variant of A(x).



←A(x)
(C2)
(C1)
{x/0}
{y/x}
{x/0}
←A(0),C(x)
SIG
(C2)
(C1)'
{y'/0}
←C(x)
←A(0),C(0),C(x)
(C4)
{z/x}
←B(x),A(w)
SVG
(C2)
(C1)"
(C3)
{x/1}
←C(0),C(x)
...
←A(w)
(C4)
EIG, EVG
(C1)'
(C2)
{z/0}
{y'/w}
{w/0}
←A(0),C(w)
{x/1}
←B(0),A(w),C(x)
SIR, STRONG
(C1)"
(C2)
{y"/0}
←C(w)
←A(0),C(0),C(w)
(C4)'
(C2)
(C1)'''
{z'/w}
←B(w),A(w')
←C(0),C(w)
...
SVR
(C4)'
(C3)
{z'/0}
{w/1}
←A(w')
←B(0),A(w'),C(w)
EIR, EVR
...

FIGURE 4.3

Now the derivation repeats itself, but with x replaced by w. Therefore the loop checks based on resultants prune the tree during this second phase, exactly there where the corresponding loop checks based on goals pruned during the first phase.

The side branch that is obtained by repeatedly applying the first clause (and corresponding side branches later on) is pruned by the subsumption checks at the goal $\leftarrow$A(0),C(0),C(x). This goal contains the previous goal $\leftarrow$A(0),C(x). Therefore both the resultant based and the goal based loop checks prune this goal. In contrast, the equality checks do not prune this infinite branch because the goals in it become longer in every derivation step (analogously to Example 4.8).

The loop checks based on goals all prune the solution {x/1}, so they are not sound. Among these loop checks, the SIG check prunes as soon as possible for a weakly sound loop check. Conversely, the SIR check prunes this tree as soon as possible for a shortening loop check. So on this tree, it behaves exactly like STRONG, which exhibits such a behaviour by definition. □

Another example shows that there can be a non-trivial difference between the behaviour of subsumption checks based on list subsumption and those based on multiset subsumption.

EXAMPLE 5.5.
Let P = { A(x) $\leftarrow$ A(y),S(x),T(y) }. (Note the similarity between this clause and the clause A(x) $\leftarrow$ A(y),S(x,y) in Theorem 3.10.)
Let G = $\leftarrow$A(x_0),B(x_0).
An SLD-derivation (and SLD-tree) of P$\cup${G} via the leftmost selection rule is depicted in Figure 4.4. This infinite SLD-derivation is pruned by the SVR_M check at the goal $\leftarrow$A(x_2),S(x_1),T(x_2),S(x_0),T(x_1),B(x_0), since a variant of an earlier goal, namely ($\leftarrow$A(x_1),S(x_0),T(x_1),B(x_0)){x_1/x_2}, is 'multiset-contained' in it.

$$\leftarrow A(x_0),B(x_0)$$
$$\Downarrow$$
$$\leftarrow A(x_1),S(x_0),T(x_1),B(x_0)$$
$$\Downarrow$$
$$\leftarrow A(x_2),S(x_1),T(x_2),S(x_0),T(x_1),B(x_0)$$
$$\Downarrow$$
$$\leftarrow A(x_3),S(x_2),T(x_3),S(x_1),T(x_2),S(x_0),T(x_1),B(x_0)$$
$$\Downarrow$$
$$\ldots$$

FIGURE 4.4

However, this derivation is *not* pruned by the SVR_L check, nor by the stronger SIG_L check. For, assume that the SIG_L check prunes this derivation at the goal $G_k = \leftarrow A(x_k),S(x_{k-1}),T(x_k),S(x_{k-2}),T(x_{k-1}),\ldots,S(x_0),T(x_1),B(x_0)$, because $G_i\tau = (\leftarrow A(x_i),S(x_{i-1}),T(x_i),S(x_{i-2}),T(x_{i-1}),\ldots,S(x_0),T(x_1),B(x_0))\tau$, an instance of an earlier goal G_i, is list-contained in it.

Clearly, the presence of the B-atoms in $G_i\tau$ and G_k requires $x_0\tau = x_0$. So the atom $S(x_0)\tau$ in $G_i\tau$ corresponds to the atom $S(x_0)$ in G_k. Then, because $G_i\tau$ is *list*-contained in G_k, $T(x_1)\tau$ can only correspond to $T(x_1)$, the only atom between $S(x_0)$ and $B(x_0)$. Therefore $x_1\tau = x_1$. Using induction, we can derive $x_2\tau = x_2, \ldots, x_i\tau = x_i$. However, the presence of the A-atoms in $G_i\tau$ and G_k requires $x_i\tau = x_k$. Since $i < k$, this is a contradiction. So the assumption that the SIG_L check prunes the derivation is refuted. □

The above examples suggest some 'stronger than' relationships (although an example can only prove the absence of such a relationship). Figure 4.5 shows the relationships between the subsumption checks, the equality checks, VA and IA. The arrows between the 'cubes' mean that every subsumption check is stronger than the corresponding equality check in the other 'cube'. So the structure of 'stronger than' relations between equality checks and subsumption checks is a four-dimensional hypercube. Again, proving these 'stronger than' relations is straightforward.

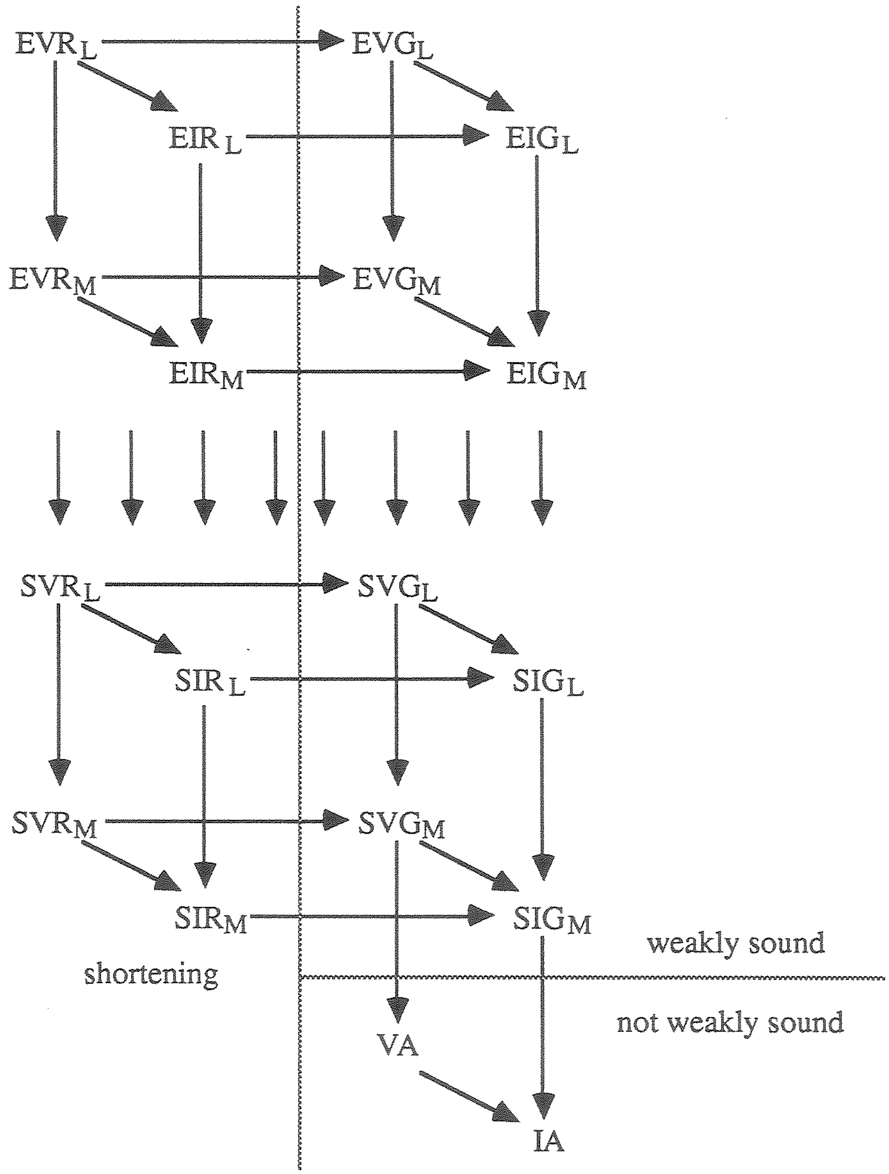


FIGURE 4.5

5.2. Soundness

To prove the desired soundness results, we prove that the SIR_M check is shortening and that the SIG_M check is weakly sound, since these are the strongest loop checks based on resultants, respectively goals, in our scheme. First we need the following lemma.

LEMMA 5.6. *Let P be a program. Let G_1 and G_2 be goals such that $G_2 \subseteq_M G_1$. Suppose D is an SLD-derivation of $P \cup \{G_1\}$ with computed answer substitution σ. Then every SLD-tree of $P \cup \{G_2\}$ contains a successful branch of length $\leq |D|$ with a computed answer substitution more general than σ.*

PROOF. Let $D = (G_1 \Rightarrow_{C_1,\theta_1} \cdots \Rightarrow_{C_n,\theta_n} \Box)$ and let $C_{n_1},\ldots,C_{n_m}$ be those clauses from $C_1,\ldots,C_n$ that are used (directly or indirectly) to resolve atoms belonging to G_2, with $1 \leq n_1 < \ldots < n_m \leq n$. Then there exists an unrestricted (in the sense of [15]) SLD-derivation $G_2\theta_1\ldots\theta_{n_1-1} \Rightarrow_{C_{n_1},\theta_{n_1}\ldots\theta_{n_2-1}} \cdots \Rightarrow_{C_{n_m},\theta_{n_m}\ldots\theta_n} \Box$. Now apply the Mgu Lemma, the Lifting Lemma of [15] and the independence of the computation rule. $\Box$

We can now prove the desired theorem.

THEOREM 5.7. *i) The SIR_M check is shortening.*

ii) The SIG_M check is weakly sound.

PROOF. Let P be a program and let G_0 be a goal.

i) Let D be an SLD-refutation of $P \cup \{G_0\}$ with a computed answer substitution σ. If D is pruned by SIR_M, then we must find in every SLD-tree containing D an SLD-refutation D' of $P \cup \{G_0\}$ with computed answer substitution σ' such that $\sigma' \leq \sigma$, $|D'| < |D|$ and D' is not pruned by SIR_M. We prove that D' exists by induction on the length l of the refutation D.

We have $l \geq 1$. For $l = 1$, D cannot be pruned by SIR_M. Now suppose the theorem is true for every refutation of $P \cup \{G_0\}$ of length $\leq l$. Let D be a refutation of length $l+1$. Suppose that D is pruned by SIR_M. Then we have $D = (G_0 \Rightarrow_{C_1,\theta_1} G_1 \Rightarrow \ldots \Rightarrow G_{i-1} \Rightarrow_{C_i,\theta_i} G_i \Rightarrow_{C_{i+1},\theta_{i+1}} G_{i+1} \Rightarrow \ldots \Rightarrow G_{k-1} \Rightarrow_{C_k,\theta_k} G_k \Rightarrow_{C_{k+1},\theta_{k+1}} G_{k+1} \Rightarrow \ldots \Rightarrow \Box)$, with $G_k \supseteq_M G_i\tau$ and $G_0\theta_1\ldots\theta_k = G_0\theta_1\ldots\theta_i\tau$ for some substitution τ.

We now proceed as follows.

1. Thus there exists an SLD-refutation of $P \cup \{G_k\}$ with the computed answer substitution $\theta_{k+1}\ldots\theta_l$, of length l-k+1. $G_i\tau \subseteq_M G_k$, so we conclude using Lemma 5.6 that there exists an SLD-refutation of $P \cup \{G_i\tau\}$ with a computed answer substitution $\sigma_1 \leq \theta_{k+1}\ldots\theta_l$, of length at most l-k+1.

2. Replacing $G_{i-1} \Rightarrow_{C_i,\theta_i} G_i$ by $G_{i-1} \Rightarrow_{C_i,\theta_i\tau} G_i\tau$, we get an unrestricted SLD-derivation $D_1 = (G_{i-1} \Rightarrow \ldots \Rightarrow \Box)$ with a computed answer substitution $\theta_i\tau\sigma_1 \leq \theta_i\tau\theta_{k+1}\ldots\theta_l$.

3. Using the Mgu Lemma, this yields an SLD-derivation $D_2 = (G_{i-1} \Rightarrow \ldots \Rightarrow \Box)$ with a computed answer substitution $\sigma_2 \leq \theta_i\tau\sigma_1 \leq \theta_i\tau\theta_{k+1}\ldots\theta_l$ and $|D_2| \leq l\text{-}k\text{+}2 \leq l\text{-}i\text{+}1$ (since $k > i$).

4. There exists an SLD-derivation $D_3 = (G_0 \Rightarrow \ldots \Rightarrow \Box)$ with a computed answer substitution $\sigma_3 \leq \theta_1\ldots\theta_{i-1}\sigma_2 \leq \theta_1\ldots\theta_{i-1}\theta_i\tau\theta_{k+1}\ldots\theta_l$ and $|D_2| \leq l$ (combining $G_0 \Rightarrow \ldots \Rightarrow G_{i-1}$ of D with D_2). Indeed, Lemma 4.5, Lemma 5.6 and the proofs of the Mgu Lemma and the Lifting Lemma show that for every SLD-tree containing D, such a derivation D_3 can be constructed.

5. By induction hypothesis, either D_3 is not pruned by SIR_M or an SLD-refutation D_4 of $P \cup \{G_0\}$ exists with a computed answer substitution σ_4 such that $G_0\sigma_4 \leq G_0\sigma_3$, $|D_4| < |D_3|$ and D_4 is not pruned by SIR_M. In the first case, we take $D' = D_3$, in the second case $D' = D_4$.

6. $G_0\sigma' \leq G_0\sigma_3 \leq G_0\theta_1\ldots\theta_i\tau\theta_{k+1}\ldots\theta_l = G_0\theta_1\ldots\theta_l = G_0\sigma$, so $\sigma' \leq \sigma$.

ii) Note that the additional condition $G_0\theta_1\ldots\theta_k = G_0\theta_1\ldots\theta_i\tau$ was only used in step 6. □

COROLLARY 5.8 (Subsumption Soundness).

i) All subsumption checks based on resultants are shortening. A fortiori they are sound.

ii) All subsumption checks based on goals are weakly sound.

PROOF. By Theorem 5.7 and the Relative Strength Theorem 3.9. □

5.3 Completeness

We now shift our attention to completeness issues. From the results of the previous section we can immediately deduce the following result.

COROLLARY 5.9 (Subsumption Completeness 1). *All subsumption checks are complete w.r.t. the leftmost selection rule for function-free restricted programs.*

PROOF. By the Equality Completeness Corollary 4.19 and the Relative Strength Theorem 3.9. □

However, the subsumption checks are stronger than the corresponding equality checks. So we can try to find other classes of programs for which the subsumption checks are complete. We know that the subsumption checks are not complete for all programs, not even for all function-free programs. For P = {A(x)←A(y),S(y,x)}, a derivation of P∪{←A(x),B(x)} is not pruned by any of the subsumption checks, as was shown in Theorem 3.10.

A close analysis of the proof of this theorem shows that the problem is caused by three 'events' occurring simultaneously, namely:

1. A new variable, y, is introduced by a 'recursive' atom, A(y).
2. There is a relation between this new variable, y, and an old variable, x, namely via the atom S(y,x).
3. The 'recursive' atom A(y) is selected before the 'relating' atom S(y,x).

It appears that, in order to obtain the completeness of the subsumption checks, it is enough to prevent any of these events. Clearly, the use of restricted programs and the leftmost selection rule prevents the third event. We now introduce two new classes of programs, preventing the first and the second event, respectively.

DEFINITION 5.10 (Nvi Program).
A clause C is *non-variable introducing* (in short *nvi*) if every variable that appears in the body of C also appears in the head of C.
A program P is *nvi* if every clause in P is nvi. □

DEFINITION 5.11 (Svo Program).
A clause C has the *single variable occurrence* property (in short *is svo*) if in the body of C, no variable occurs more than once.
A program P *is svo* if every clause in P is svo. □

Clearly, in nvi programs the first event cannot occur, whereas in svo programs the second event is prevented. We would rather have used the terminology *right-linear* instead of svo, which is common in the area of term rewriting systems. However, in the area of deductive databases this term is already in use for a completely different notion.

EXAMPLE 5.12.

The following program is an nvi program and a svo program, but not a restricted program. It computes in the relation 'add' the sum of two two-digit binary numbers (the first four arguments of 'add'); this sum is a three-digit binary number, stored in the last three arguments of 'add'.

$$
\begin{array}{lllllll}
\mathrm{ADD} = \{ & \mathrm{add}(\ 0,0, & A,B, & 0,A,B\ ) & \leftarrow ; & & \\
& \mathrm{add}(\ A,B, & 0,0, & 0,A,B\ ) & \leftarrow ; & & \\
& \mathrm{add}(\ A,B, & A,B, & A,B,0\ ) & \leftarrow ; & & \\
& \mathrm{add}(\ A_1,B_1, & A_2,B_2, & C,A_3,B_3\ ) & \leftarrow & \mathrm{add}(\ 0,B_1,\ 0,B_2,\ 0,0,B_3\ ), & \\
& & & & & \mathrm{add}(\ 0,A_1,\ 0,A_2,\ 0,C,A_3\ )\ ; & \\
& \mathrm{add}(\ A_1,1, & A_2,1, & 1,0,0) & \leftarrow & \mathrm{add}(\ 0,A_1,\ 0,A_2,\ 0,0,1) & \}.
\end{array}
$$

The first three clauses are evidently correct; every addition of the form 0X + 0Y is taken care of by them. The fourth clause deals with the case where adding the last digits of both numbers does not give a carry (ensured by the first atom in the body). The fifth clause deals with the case where there is such a carry. Only the case $A_1 \neq A_2$ (or equivalently, $A_1 + A_2 = 1$) has to be considered there: if $A_1 = A_2$ then the third clause applies.

Note that this program yields infinite derivations that are not pruned by any of the equality checks. Indeed, starting with the goal $\leftarrow$ add(0,B_1,0,B_2,0,0,B_3), the first recursive clause applies, giving the goal $\leftarrow$ add(0,B_1,0,B_2,0,0,B_3), add(0,0,0,0,0,0,0). Repeatedly selecting add(0,B_1,0,B_2,0,0,B_3) and applying the first recursive clause yields an infinite derivation containing goals of increasing length., which is not pruned by any of the equality checks. □

We now prove that the weakest of the subsumption checks, the $\mathrm{SVR_L}$ check, is complete for function-free nvi programs. To this end we use the following (weakened) version of Kruskal's Tree Theorem, called Higman's Lemma. (See [12]; for a formulation of the full version of Kruskal's Tree Theorem, see [9] or [13].)

LEMMA 5.13 (Higman's Lemma). *Let $w_0, w_1, w_2, \ldots$ be an infinite sequence of (finite) words over a finite alphabet Σ. Then for some i and $k > i$, $w_i \subseteq_L w_k$.* □

In order to prove that the $\mathrm{SVR_L}$ check is complete for function-free nvi programs, we prove that, in the absence of function symbols, infinite derivations in which no new variables are introduced are pruned by the SVR_L check. Then we prove that every derivation of a function-free nvi program (and an arbitrary goal) has a variant that indeed does not introduce new variables.

DEFINITION 5.14.
An SLD-derivation $D = (G_0 \Rightarrow_{C_1,\theta_1} G_1 \Rightarrow \ldots)$ is *non-variable introducing* (in short *nvi*) if $var(G_0) \supseteq var(G_1) \supseteq var(G_2) \supseteq \ldots$. □

LEMMA 5.15. *In the absence of function symbols, every infinite nvi SLD-derivation is pruned by SVR_L.*

PROOF. Let $D = (G_0 \Rightarrow_{C_1,\theta_1} G_1 \Rightarrow \ldots)$ be an infinite nvi SLD-derivation.
We take for Σ the set of equivalence classes of $\sim_{var(G_0),G_0,1}$ as defined in Definition 4.15. By Lemma 4.17, Σ is finite. To apply Higman's Lemma 5.13 we represent for $j \geq 0$ a goal $G_j = \leftarrow A_{1j},\ldots,A_{n_jj}$ (or rather the corresponding resultant $G_0\theta_1\ldots\theta_j \leftarrow G_j$) as the word $[G_0\theta_1\ldots\theta_j \leftarrow A_{1j}],\ldots,[G_0\theta_1\ldots\theta_j \leftarrow A_{n_jj}]$ over Σ. (Recall that for a resultant R, [R] denotes its equivalence class.) The sequence of representations of G_0, G_1, G_2, ... yields an infinite sequence of words w_0, w_1, w_2, ... over Σ.

Now by Higman's Lemma 5.13, for some j and $k > j$: $[G_0\theta_1\ldots\theta_j \leftarrow A_{1j}],\ldots, [G_0\theta_1\ldots\theta_j \leftarrow A_{n_jj}] \subseteq_L [G_0\theta_1\ldots\theta_k \leftarrow A_{1k}],\ldots,[G_0\theta_1\ldots\theta_k \leftarrow A_{n_kk}]$. So by the definition of $\sim_{var(G_0),G_0,1}$, there exist renamings $\rho_1,\ldots,\rho_{n_j}$ which do not act on the variables of G_0 such that $(G_0\theta_1\ldots\theta_j \leftarrow A_{1j})\rho_1,\ldots,(G_0\theta_1\ldots\theta_j \leftarrow A_{n_jj})\rho_{n_j} \subseteq_L (G_0\theta_1\ldots\theta_k \leftarrow A_{1k}),\ldots,(G_0\theta_1\ldots\theta_k \leftarrow A_{n_kk})$.

However, D is nvi, so $var(G_j) \subseteq var(G_0)$ and therefore the renamings ρ_h do not act on the atoms A_{ij} of G_j ($1 \leq h,i \leq n_j$). Thus $G_j = G_j\rho_1 \subseteq_L G_k$ and $G_0\theta_1\ldots\theta_j\rho_1 = G_0\theta_1\ldots\theta_k$. So D is pruned by SVR_L. □

LEMMA 5.16. *Let P be a function-free nvi program and let G_0 be a goal in L_P. Let D be an infinite SLD-derivation of $P \cup \{G_0\}$. Then a variant D' of D is an infinite nvi derivation.*

PROOF. Suppose that $D = (G_0 \Rightarrow_{C_1,\theta_1} G_1 \Rightarrow_{C_2,\theta_2} G_2 \Rightarrow \ldots)$. We show that there exists an infinite nvi derivation $D' = (G_0' \Rightarrow_{C_1,\theta_1'} G_1' \Rightarrow_{C_2,\theta_2'} G_2' \Rightarrow \ldots)$ that is a variant of D. Note that D' uses the same input clauses as D.

We give an inductive construction of D'. By definition, $G_0' = G_0$. Suppose we have constructed D' up to a goal G_{i-1}' $(i > 0)$. G_{i-1}' and G_{i-1} are variants, say $G_{i-1} = G_{i-1}'\rho$. $G_0' = G_0$ and the clauses $C_1, \ldots, C_{i-1}$ are the same as in D, so C_i is well standardized apart and we may assume that $C_i\rho = C_i$. Therefore $\rho\theta_i\rho^{-1}$ is an applicable (idempotent) mgu.

Now we obtain θ_i' by replacing every pure variable binding x/y within $\rho\theta_i\rho^{-1}$ by y/x whenever $x \in var(G_{i-1}')$ and $y \in var(C_i)$, and replacing for such x and y every other binding z/y within $\rho\theta_i\rho^{-1}$ by z/x.

Since no function symbols appear in P, this yields that for every variable $x \in var(G_{i-1}')$ either $x\theta_i' \in var(G_{i-1}')$ or $x\theta_i'$ is a constant. Hence $var(G_{i-1}'\theta_i') \subseteq var(G_{i-1}')$. Now let A be the selected atom in G_{i-1}', let R be the rest of G_{i-1}' and let $x \in var(G_i')$. Two cases arise.

- x is introduced by C_i, that is $x \in var(body(C_i)\theta_i')$. Then, since P is an nvi program, $x \in var(head(C_i)\theta_i')$. θ_i' is a unifier of $head(C_i)$ and A, so $x \in var(A\theta_i') \subseteq var(G_{i-1}\theta_i') \subseteq var(G_{i-1}')$.
- x is introduced by G_{i-1}', that is $x \in var(R\theta_i')$. Then $x \in var(G_{i-1}\theta_i') \subseteq var(G_{i-1}')$.

This proves the induction hypothesis for D' up to the goal G_i'. □

THEOREM 5.17. *The SVR_L loop check is complete for function-free nvi programs.*

PROOF. By Lemma 5.3, 5.15 and 5.16. □

COROLLARY 5.18 (Subsumption Completeness 2). *All subsumption checks are complete for function-free nvi programs.*

PROOF. By Theorem 5.17 and the Relative Strength Theorem 3.9. □

We now prove that the SVR_L check (and hence all subsumption checks) are complete for function-free svo programs. By a construction similar to the one used in the proof of Lemma 5.16, we may assume that in an SLD-derivation $D = (G_0 \Rightarrow_{C_1,\theta_1} G_1 \Rightarrow \ldots)$, $var(G_0\theta_i) \subseteq var(G_0)$ for $i > 0$. (Note that for this construction, only the absence of function symbols was needed, and not the nvi property.) Under this assumption we can prove the following lemma.

LEMMA 5.19. *Let P be a function-free svo program and let G_0 be a goal in L_P. Let $D = (G_0 \Rightarrow_{C_1,\theta_1} G_1 \Rightarrow_{C_2,\theta_2} G_2 \Rightarrow \ldots)$ be an SLD-derivation of $P \cup \{G_0\}$. Then for every goal G_i ($i \geq 0$): if x occurs more than once in G_i, then $x \in var(G_0)$.*

PROOF. By induction. For i=0, the claim is trivial.

Now suppose x occurs more than once in G_{i+1} ($i \geq 0$) and $x \notin var(G_0)$.

Let $G_i = (A,S)$, where A is the selected atom (not necessarily the leftmost atom) and let $C_{i+1} = H \leftarrow X$. Then θ_{i+1} is an idempotent mgu of A and H and $G_{i+1} = (X,S)\theta_{i+1}$. There are two ways in which we can obtain a variable x occurring more than once in G_{i+1}.

1. A variable y occurs more than once in (X,S) and $y\theta_{i+1} = x$.

By standardizing apart, $var(S) \cap var(X) = \emptyset$, so y occurs either only in S or only in X. Since C_{i+1} is svo, y does not occur more than once in X. Therefore y occurs more than once in S. Then by the induction hypothesis, $y \in var(G_0)$. But then $x = y\theta_{i+1} \in var(G_0\theta_{i+1}) \subseteq var(G_0)$.

2. There are two variables y_1 and y_2 in (X,S) such that $y_1\theta_{i+1} = y_2\theta_{i+1} = x$ and $y_1 \neq y_2$.

In this case $y_1,y_2 \in var(A,H)$, since $dom(\theta_{i+1}) \subseteq var(A,H)$.

If $y_1 \in var(S)$, then by standardizing apart $y_1 \notin var(H)$, so $y_1 \in var(A)$. Therefore y_1 occurs more than once in G_i (in A and in S), and we can apply the induction hypothesis again. Since the same argument holds for $y_2 \in var(S)$, only the case $y_1,y_2 \in var(X)$ is left. In this case, since $y_1,y_2 \in var(A,H)$, by standardizing apart $y_1,y_2 \in var(H)$.

Since $y_1\theta_{i+1} = y_2\theta_{i+1} = x$, the sets

$Z_1 = \{ z \in var(A) \mid$ z occurs in A at the position of an occurrence of y_1 in H $\}$ and $Z_2 = \{ z \in var(A) \mid$ z occurs in A at the position of an occurrence of y_2 in H $\}$ are not disjoint. (Otherwise, a more general unifier of A and H than θ would exist, mapping y_1 to an element of Z_1 and y_2 to an element of Z_2.) Let $z \in Z_1 \cap Z_2$. z occurs at least twice in A, so $z \in var(G_0)$. Thus $x = z\theta_{i+1} \in var(G_0\theta_{i+1}) \subseteq var(G_0)$. □

We can now prove the desired theorem.

THEOREM 5.20. *The SVR_L loop check is complete for function-free svo programs.*

PROOF. Let P be a function-free svo program and let G_0 be a goal in L_P. Let $D = (G_0 \Rightarrow_{C_1,\theta_1} G_1 \Rightarrow_{C_2,\theta_2} G_2 \Rightarrow \ldots)$ be an infinite SLD-derivation of $P \cup \{G_0\}$.

Again, we take for Σ the set of equivalence classes of $\sim_{var(G_0),G_0,1}$ as defined in Definition 4.15. By Lemma 4.17, Σ is finite. To apply Higman's Lemma 5.13 we represent a goal $G_j = A_{1j},\ldots,A_{n_jj}$ in D as the word $w_j = [G_0\theta_1\ldots\theta_j \leftarrow A_{1j}],\ldots,[G_0\theta_1\ldots\theta_j \leftarrow A_{n_jj}]$ over Σ. The sequence of representations of $G_0, G_1, G_2, \ldots$ yields an infinite sequence of words $w_0, w_1, w_2, \ldots$ over Σ.

Now by Higman's Lemma 5.13, for some j and k > j: $[G_0\theta_1\ldots\theta_j \leftarrow A_{1j}],\ldots, [G_0\theta_1\ldots\theta_j \leftarrow A_{n_jj}] \subseteq_L [G_0\theta_1\ldots\theta_k \leftarrow A_{1k}],\ldots,[G_0\theta_1\ldots\theta_k \leftarrow A_{n_kk}]$. So there are renamings $\rho_1,\ldots,\rho_{n_j}$ such that $(G_0\theta_1\ldots\theta_j \leftarrow A_{1j})\rho_1,\ldots,(G_0\theta_1\ldots\theta_j \leftarrow A_{n_jj})\rho_{n_j} \subseteq_L (G_0\theta_1\ldots\theta_k \leftarrow A_{1k}),\ldots,(G_0\theta_1\ldots\theta_k \leftarrow A_{n_kk})$.

Now we can define the renaming ρ by

$$x\rho = \begin{cases} x & \text{if } x \in var(G_0) \text{ or } x \notin var(G_j), \\ x\rho_k & \text{if } x \in var(A_{kj}), \end{cases}$$

since by Lemma 5.19, if $x \notin var(G_0)$, x occurs at most once in G_j, and therefore k is uniquely defined. Applying Lemma 5.19 on G_k yields that ρ is again a renaming.

We have assumed that $var(G_0\theta_1\ldots\theta_j) \subseteq var(G_0\theta_2\ldots\theta_j) \subseteq \ldots \subseteq var(G_0\theta_j) \subseteq var(G_0)$, and since by the definition of $\sim_{var(G_0),G_0,1}$, the renamings ρ_i do not act on variables in $var(G_0)$, we have $(\,(G_0\theta_1\ldots\theta_j \leftarrow A_{1j}),\ldots,(G_0\theta_1\ldots\theta_j \leftarrow A_{n_jj})\,)\rho \subseteq_L (G_0\theta_1\ldots\theta_k \leftarrow A_{1k}),\ldots, (G_0\theta_1\ldots\theta_k \leftarrow A_{n_kk})$. So $G_j\rho \subseteq_L G_k$ and $G_0\theta_1\ldots\theta_j\rho = G_0\theta_1\ldots\theta_k$, and D is pruned by SVR_L. □

COROLLARY 5.21 (Subsumption Completeness 3). *All subsumption checks are complete for function-free svo programs.*

PROOF. By Theorem 5.20 and the Relative Strength Theorem 3.9. □

Now combining Corollary 3.6 and Corollary 3.7 with the Subsumption Soundness Corollary 5.8 and the Subsumption Completeness Corollaries 5.9, 5.18 and 5.21, we conclude that all subsumption checks lead to an implementation of CWA for restricted programs, nvi programs and svo programs without function symbols. Moreover, the subsumption checks based on resultants also lead to an implementation of query processing for these programs.

6. Context checks

The problem with the Instance of Atom check is that it does not take into account the context of the atom. This is incorrect: whereas solving $\leftarrow A(x)$ or $\leftarrow A(y)$ makes no difference, solving $\leftarrow A(x),B(x)$ is essentially more difficult than solving $\leftarrow A(y),B(x)$. To remedy this problem we should keep track of the links between the variables in the atom and those in the rest of the goal.

Roughly speaking, the IA check prunes a derivation as soon as a goal G_k occurs that contains an instance $A\tau$ of an atom A that occurred in an earlier goal G_i. But when a variable occurs both inside and outside of A in G_i, we should not prune the derivation if this link has been altered. Such a variable x in G_i is substituted by $x\theta_{i+1}\ldots\theta_k$ when G_k is reached. Therefore τ and $\theta_{i+1}\ldots\theta_k$ should agree on x. This leads us to a loop check introduced by Besnard [3].

6.1. Definitions

DEFINITION 6.1 (Context checks for Goals).
The *Variant/Instance Context check on Goals* is the set of SLD-derivations

$CVG/CIG = RemSub(\{ D \mid D = (G_0 \Rightarrow_{C_1,\theta_1} G_1 \Rightarrow \ldots \Rightarrow G_{k-1} \Rightarrow_{C_k,\theta_k} G_k)$
such that for some i and j, $0 \le i \le j < k$, there is a renaming/substitution τ such that for some atom A in G_i: $A\tau$ appears in G_k as the result of resolving $A\theta_{i+1}\ldots\theta_j$ in G_j and for every variable x that occurs both inside and outside of A in G_i, $x\theta_{i+1}\ldots\theta_k = x\tau$ $\})$. □

Besnard describes the condition on the substitutions as follows: 'When $A\tau$ is substituted for $A\theta_{i+1}\ldots\theta_k$ in $G_i\theta_{i+1}\ldots\theta_k$, this should give an instance of G_i.' We show that this formulation is equivalent to ours. Let $G_i = (A,S)$, that is A occurs in G_i and S is the list of other atoms in G_i. Then $(A\tau, S\theta_{i+1}\ldots\theta_k)$ should be an instance of (A,S), say $(A\sigma,S\sigma)$.

$$\text{Clearly, } x\sigma = \begin{cases} x\tau & \text{for } x \in var(A) \\ x\theta_{i+1}\ldots\theta_k & \text{for } x \in var(S), \end{cases}$$

so for $x \in var(A) \cap var(S)$, $x\tau = x\theta_{i+1}\ldots\theta_k$.

The following example clarifies the use of the context checks.

EXAMPLE 6.2.
We use the program P and the goal G of Variant of Atom check Example 2.5 and apply the CIG check on two SLD-trees of $P \cup \{G\}$, via the leftmost and rightmost selection rule, respectively. This yields the trees in Figure 4.6.

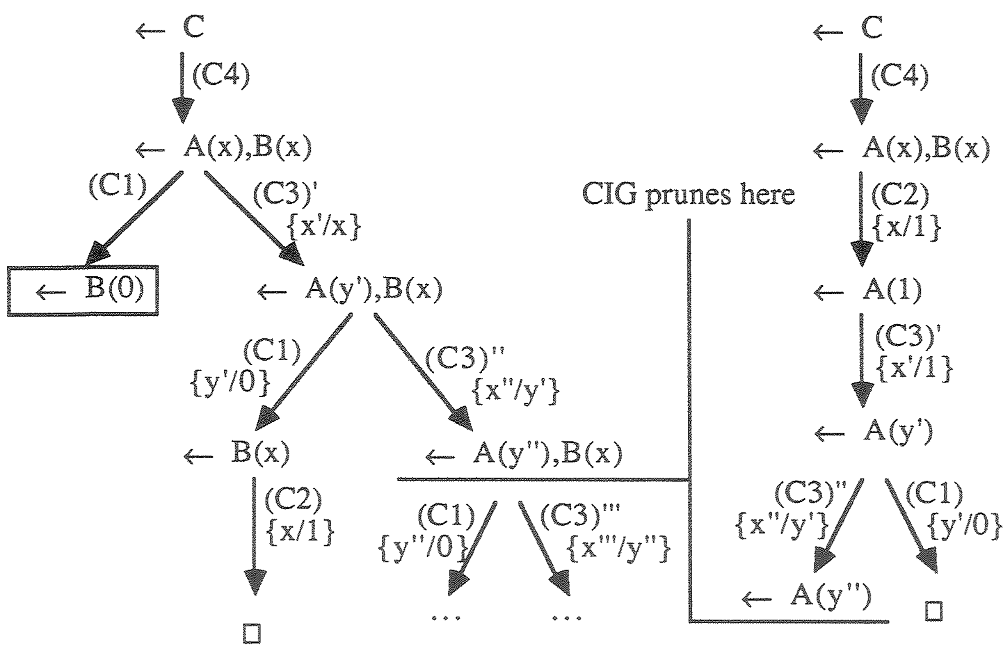


FIGURE 4.6

The goal $G_3 = \leftarrow A(y')$ in the rightmost tree that was incorrectly pruned by the VA check, is not pruned by the CIG check. Certainly, A(y') is the result of resolving A(1) in G_2, the further instantiated version of A(x) in G_1. But replacing $A(x)\theta_2\theta_3$ by A(y') in $G_1\theta_2\theta_3$ yields $\leftarrow$A(y'),B(1), which is *not* an instance of $\leftarrow$A(x),B(x). □

CLAIM 6.3. *CVG and CIG are weakly sound simple loop checks.*
Proving that CVG and CIG are simple loop checks is straightforward. Besnard claims that CIG is weakly sound. From this it follows that the weaker CVG check is also weakly sound. See also Theorem 6.6. □

In Example 4.3, the context checks act exactly in the same way as the corresponding equality checks. This shows that CVG and CIG are not sound. Again we can obtain sound, even shortening, versions by using resultants instead of goals.

DEFINITION 6.4 (Context checks for Resultants).
The *Variant/Instance Context check on Resultants* is the set of SLD-derivations

CVR/CIR = RemSub({ D | D = ($G_0 \Rightarrow_{C_1,\theta_1} G_1 \Rightarrow \ldots \Rightarrow G_{k-1} \Rightarrow_{C_k,\theta_k} G_k$) such that for some i and j, $0 \leq i \leq j < k$, there is a renaming/substitution τ such that $G_0\theta_1 \ldots \theta_k = G_0\theta_1 \ldots \theta_i\tau$ and for some atom A in G_i: $A\tau$ appears in G_k as the result of resolving $A\theta_{i+1} \ldots \theta_j$ in G_j and for every variable x that occurs both inside and outside of A in G_i: $x\theta_{i+1} \ldots \theta_k = x\tau$ }). □

Using Besnard's phrasing, the conditions on the substitutions can be summarized as: 'When $A\tau$ is substituted for $A\theta_{i+1} \ldots \theta_k$ in the resultant $R_i\theta_{i+1} \ldots \theta_k$, this should give an instance of R_i.'

LEMMA 6.5. *CVR and CIR are simple loop checks.*
PROOF. Straightforward. □

6.2. Soundness

Now we shall prove that the CIR check is shortening. From this it follows that the weaker loop check CVR is also shortening.

THEOREM 6.6. *The CIR check is shortening.*
PROOF. Let P be a program and let G_0 be a goal. Let D be an SLD-refutation of $P \cup \{G_0\}$ with a computed answer substitution σ. If D is pruned by CIR, then we must find in every SLD-tree containing D an SLD-refutation D' of $P \cup \{G_0\}$ with computed answer substitution σ' such that $\sigma' \leq \sigma$, $|D'| < |D|$ and D' is not pruned by CIR. We prove the existence of such a refutation by induction on the length l of the refutation D.

We have $l \geq 1$. For $l = 1$, D cannot be pruned by CIR. Now suppose the theorem is true for every refutation of $P \cup \{G_0\}$ of length $\leq l$. Let D be a refutation of length $l+1$. Suppose that D is pruned by CIR. Then we have

$D = (G_0 \Rightarrow_{C_1,\theta_1} G_1 \Rightarrow \ldots \Rightarrow G_{i-1} \Rightarrow_{C_i,\theta_i} G_i \Rightarrow_{C_{i+1},\theta_{i+1}} G_{i+1} \Rightarrow \ldots \Rightarrow G_{k-1} \Rightarrow_{C_k,\theta_k} G_k \Rightarrow \ldots \Rightarrow \square)$, with for some τ: $G_i = (A,S_i)$, $G_k = (A\tau,S_k)$, $S_i\theta_{i+1}\ldots\theta_k = S_i\tau$ and $G_0\theta_1\ldots\theta_k = G_0\theta_1\ldots\theta_i\tau$.

Therefore $S_i \subseteq_M G_i$ and $\{A\tau\} \subseteq_M G_k$. By Lemma 5.6 we have SLD-refutations D_1 of $P\cup\{\leftarrow S_i\}$ and D_2 of $P\cup\{\leftarrow A\tau\}$ via every selection rule. An inspection of the proof of Lemma 5.6 shows that every derivation step in D_1 and D_2 has a corresponding derivation step in the tail $(G_i \Rightarrow \ldots \Rightarrow \square)$ of D. This tail consists of l-i derivation steps. On the other hand, at least one step in this tail has no corresponding step in D_1 or D_2: the step in which $A\theta_{i+1}\ldots\theta_j$ is selected. Therefore $|D_1|+|D_2| < l-i$.

D_1 has a computed answer substitution more general than $\theta_{i+1}\ldots\theta_k\theta_{k+1}\ldots\theta_l$. So there exists an SLD-refutation D_1' of $P\cup\{\leftarrow S_i\theta_{i+1}\ldots\theta_k\}$ with $|D_1'| = |D_1|$ and a computed answer substitution more general than $\theta_{k+1}\ldots\theta_l$. The computed answer substitution of D_2 is also more general than $\theta_{k+1}\ldots\theta_l$. So we can combine D_1' and D_2 into an SLD-refutation D_3 of $P\cup\{\leftarrow(A,S_i)\tau\}$.

Finally we combine $G_0 \Rightarrow \ldots \Rightarrow G_{i-1}$ from D, the unrestricted derivation step $G_{i-1} \Rightarrow_{C_i,\theta_i\tau} G_i\tau$ and D_3 into an unrestricted SLD-refutation D_4 of $P\cup\{G_0\}$. $|D_4| = i+1+|D_1|+|D_2| < i+1+l-i = l+1$.

By the Mgu Lemma, an SLD-refutation D_5 exists with $|D_5| = |D_4|$ and a computed answer substitution $\sigma_5 \le \theta_1\ldots\theta_{i-1}\theta_i\tau\theta_{k+1}\ldots\theta_l$. D_5 is an SLD-refutation of $P\cup\{G_0\}$ which is shorter than D, so by the induction hypothesis there exists an SLD-refutation D' of $P\cup\{G_0\}$ with computed answer substitution σ' such that $G_0\sigma' \le G_0\sigma_5$ and D' is not pruned by CIR. Now we have $G_0\sigma' \le G_0\sigma_5 \le G_0\theta_1\ldots\theta_i\tau\theta_{k+1}\ldots\theta_l = G_0\theta_1\ldots\theta_l = G_0\sigma$, so $\sigma' \le \sigma$. □

COROLLARY 6.7 (Context Soundness).

i) The context checks based on resultants are shortening. A fortiori they are sound.

ii) The context checks based on goals are weakly sound.

PROOF. By Theorem 6.6 and the Relative Strength Theorem 3.9. Note that omitting the considerations about computed answer substitutions from this proof yields a proof for ii), i.e. for Claim 6.3. □

For derivations via certain selection rules (including leftmost and rightmost selection rule), a much easier soundness proof exists. Namely, w.r.t. these selection rules, the subsumption checks are stronger than the corresponding (w.r.t. variant vs. instance and goal vs. resultant distinctions) context checks.

DEFINITION 6.8.
(This definition is equivalent to the definition of local selection functions in [22].) A selection rule **R** is *most recent* if every SLD-derivation $D = (G_0 \Rightarrow_{C_1,\theta_1} G_1 \Rightarrow \ldots)$ via **R** satisfies the following property. If in a goal G_i, an atom A is selected and in a goal G_j (j>i) the further instantiated version $B\theta_{i+1}\ldots\theta_j$ of the atom B in G_i is selected, then A is resolved completely between G_i and G_j. □

LEMMA 6.9. *The subsumption checks are stronger than the corresponding context checks w.r.t. most recent selection rules.*

PROOF. Suppose $D = (G_0 \Rightarrow_{C_1,\theta_1} G_1 \Rightarrow \ldots \Rightarrow_{C_k,\theta_k} G_k)$ is pruned at G_k by a context check. We show that D is pruned by the corresponding subsumption check based on lists at G_k (or earlier).

We have an atom A in G_i, $A\theta_{i+1}\ldots\theta_j$ in G_j as the selected atom and $A\tau$ as the result of resolving $A\theta_{i+1}\ldots\theta_j$. Let $G_i = (A,S,T)$, where S consists of those atoms in G_i that are completely resolved between G_i and G_j. The use of a most recent selection rule yields that $G_j = (A\theta_{i+1}\ldots\theta_j, T\theta_{i+1}\ldots\theta_j)$ and $G_k = (A\tau, U, T\theta_{i+1}\ldots\theta_k)$ (U consists of the other atoms in G_k that are the result of resolving $A\theta_{i+1}\ldots\theta_j$). Finally $(S,T)\theta_{i+1}\ldots\theta_k = (S,T)\tau$.

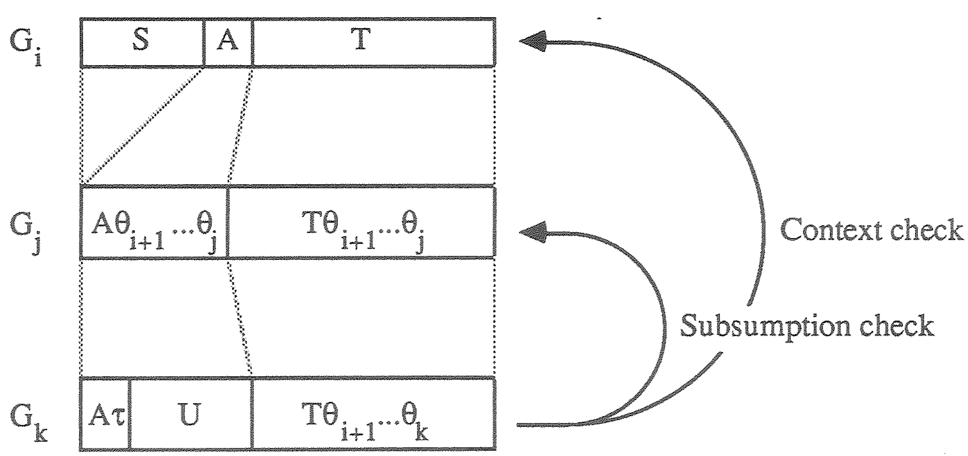


FIGURE 4.7

First we show that for every x,y, if $x\theta_{i+1}\ldots\theta_j = y\theta_{i+1}\ldots\theta_j$ then $x\tau = y\tau$. Suppose $x\theta_{i+1}\ldots\theta_j = y\theta_{i+1}\ldots\theta_j$ and $x \neq y$. Then clearly $x,y \in var(S)$. So $x\tau = x\theta_{i+1}\ldots\theta_k = y\theta_{i+1}\ldots\theta_k = y\tau$.

Now we can define σ on $var(G_j)$ as follows:

- if $x \in var(A\theta_{i+1}\ldots\theta_j)$ then $x\sigma = y\tau$, where $y \in var(A)$ such that $y\theta_{i+1}\ldots\theta_j = x$. (Although y may not be unique, $y\tau$ is.) (case a),
- if $x \in var(T\theta_{i+1}\ldots\theta_j)$ then $x\sigma = x\theta_{j+1}\ldots\theta_k$. (Hence for some $y \in var(T)$ such that $y\theta_{i+1}\ldots\theta_j = x$, we have $x\sigma = y\theta_{i+1}\ldots\theta_k = y\tau$. (case b),

and we show that :

1) $G_j\sigma \subseteq_L G_k$.

We show that $A\theta_{i+1}\ldots\theta_j\sigma = A\tau$ and $T\theta_{i+1}\ldots\theta_j\sigma = T\theta_{i+1}\ldots\theta_k$.

If $y \in var(A)$, then either $y\theta_{i+1}\ldots\theta_j \in var(A\theta_{i+1}\ldots\theta_j)$, so $y\theta_{i+1}\ldots\theta_j\sigma = y\tau$, or $y\theta_{i+1}\ldots\theta_j$ is a constant, so $y\theta_{i+1}\ldots\theta_j\sigma = y\theta_{i+1}\ldots\theta_j = y\tau$, since in this case $y \in var(S)$.

If $y \in var(T)$, then if $y \in var(A)$, $y\theta_{i+1}\ldots\theta_j\sigma = y\tau = y\theta_{i+1}\ldots\theta_k$, otherwise $y\theta_{i+1}\ldots\theta_j\sigma = y\theta_{i+1}\ldots\theta_j\theta_{j+1}\ldots\theta_k = y\theta_{i+1}\ldots\theta_k$.

2) If $G_0\theta_1\ldots\theta_k = G_0\theta_1\ldots\theta_i\tau$ then $G_0\theta_1\ldots\theta_k = G_0\theta_1\ldots\theta_j\sigma$ (goal vs. resultant).

Let $x \in var(G_0\theta_1\ldots\theta_j)$.

In case a, $x = y\theta_{i+1}\ldots\theta_j$ for some $y \in var(G_0\theta_1\ldots\theta_i)$. Then $x\sigma = y\tau = y\theta_{i+1}\ldots\theta_k = x\theta_{j+1}\ldots\theta_k$.

In case b, $x\sigma = x\theta_{j+1}\ldots\theta_k$ by definition.

3) If τ is a renaming, then σ is a renaming (variant vs. instance).

For every $x \in var(G_j)$ there exists a $y \in var(A,T)$ such that $y\theta_{i+1}\ldots\theta_j = x$ and $x\sigma = y\tau$. □

The following example shows that the previous result does not hold for selection rules that are not most recent.

EXAMPLE 6.10 (based on Example 10 in [3]).

Let P = { $A \leftarrow B$ (C1),
$B \leftarrow A$ (C2),
$C \leftarrow D$ (C3) },

and let $G = \leftarrow A,C$.

Then the derivation $\leftarrow \underline{A},C \Rightarrow_{(C1)} \leftarrow B,\underline{C} \Rightarrow_{(C3)} \leftarrow \underline{B},D \Rightarrow_{(C2)} \leftarrow A,D$ (in which the selected atoms are underlined) is pruned by the context checks (the A in the fourth goal is the result of resolving the A in the first goal), but not by the subsumption checks. □

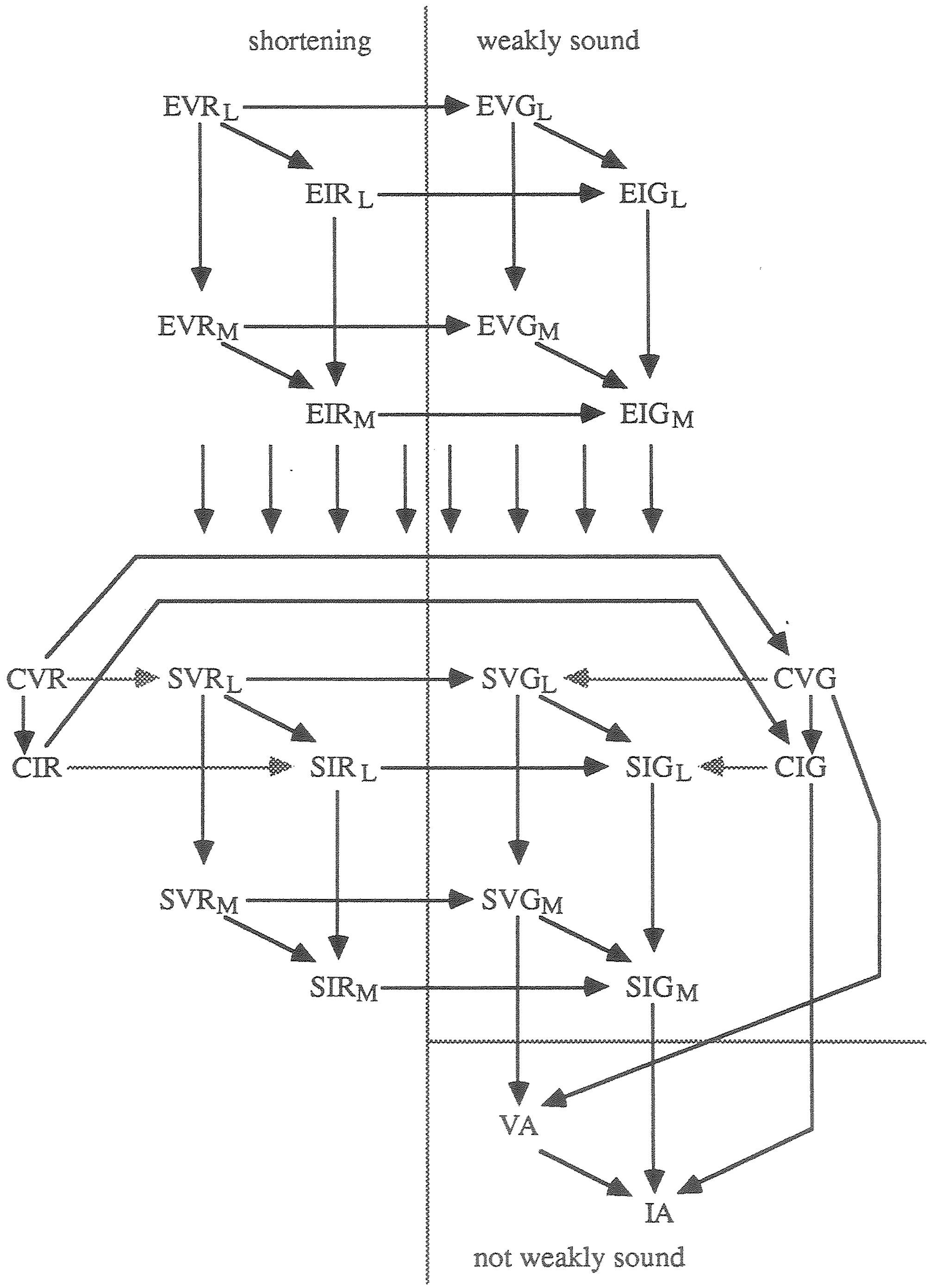


FIGURE 4.8

Now we can add the context checks to our 'stronger than' scheme. The dotted arrows are only valid for most recent selection rules.

6.3. Completeness

Again we shift our attention to completeness issues. Besnard [3] claims without a proof that his loop check is complete for function-free nvi programs.

CLAIM 6.11 (Context Completeness 1).
The CIG check is complete for function-free nvi programs. □

We conjecture that even the weakest of the four context checks, CVR, is complete for function-free nvi programs. We now prove that, like the equality checks and the subsumption checks, the context checks are complete w.r.t. the leftmost selection rule for function-free restricted programs.

THEOREM 6.12. *The CVR check is complete w.r.t. the leftmost selection rule for function-free restricted programs.*

PROOF. Let P be a function-free restricted program and let G_0 be a goal in L_P. Let k=weight(G_0). Consider an infinite SLD-derivation $D = G_0 \Rightarrow_{C_1,\theta_1} G_1 \Rightarrow \ldots \Rightarrow G_{i-1} \Rightarrow_{C_i,\theta_i} G_i \Rightarrow \ldots$ of $P \cup \{G_0\}$. By Corollary 4.14 for every $i \geq 0$: $|G_i| \leq k$. Every goal G_i is a goal in L_P and hence every resultant $G_0\theta_1\ldots\theta_i \leftarrow G_i$ belongs to an equivalence class of $\sim_{\varnothing,G_0,k}$. L_P satisfies the conditions of Lemma 4.17, so $\sim_{\varnothing,G_0,k}$ has only finitely many equivalence classes. Thus the set $E = \{\underline{e} \mid \underline{e}$ is an equivalence class of $\sim_{\varnothing,G_0,k}$ and for infinitely many resultants R in D: $R \in \underline{e}\}$ is non-empty. For simplicity, we shall say that the goal G_i is in an equivalence class $\underline{e}$, when in fact $(G_0\theta_1\ldots\theta_i \leftarrow G_i) \in \underline{e}$.

For every equivalence class $\underline{e}$ of $\sim_{\varnothing,G_0,k}$, we define the length of $\underline{e}$, denoted by $|\underline{e}|$, as the length of the goals in $\underline{e}$. Since $E \neq \varnothing$, we can define $l = \min\{ |\underline{e}| \mid \underline{e} \in E \}$. Now we choose an equivalence class $e \in E$ with $|e| = l$. According to the choice of e, D contains infinitely many goals in e and a finite number of shorter goals (since the number of equivalence classes of $\sim_{\varnothing,G_0,k}$ is finite).

Let G_i and G_k be (the first) two goals in D that are in e such that no goal lying in D between them is shorter. Since G_i and G_k are in the same equivalence class e, we have $G_k = G_i\tau$ and $G_0\theta_1\ldots\theta_k = G_0\theta_1\ldots\theta_i\tau$ for some renaming τ.

Let A be the leftmost atom in G_i and let S be the rest of G_i. A is selected in G_i. However, A is not completely resolved between G_i and G_k, otherwise a goal shorter than G_i, namely an instance of S, would appear between G_i and G_k in D. Therefore the atom $A\tau$ in G_k is the result of resolving A. Furthermore, no atom of S is selected between G_i and G_k, so $G_k = (A\tau, S\theta_{i+1} \ldots \theta_k)$. Hence $S\theta_{i+1} \ldots \theta_k = S\tau$.

When in the resultant $R_i\theta_{i+1} \ldots \theta_k$, we replace $A\theta_{i+1} \ldots \theta_k$ by $A\tau$, we obtain $(G_0\theta_1 \ldots \theta_k \leftarrow A\tau, S\theta_{i+1} \ldots \theta_k) = (G_0\theta_1 \ldots \theta_i\tau \leftarrow A\tau, S\tau)$, which is a variant of R_i. Therefore D is pruned by the CVR check. □

COROLLARY **6.13 (Context Completeness 2).** *All context checks are complete w.r.t. the leftmost selection rule for function-free restricted programs.*

PROOF. By Theorem 6.12 and the Relative Strength Theorem 3.9. □

Now combining Corollary 3.6 and Corollary 3.7 with the Context Soundness Corollary 6.7 and the Context Completeness Corollary 6.13, we conclude that all context checks lead to an implementation of CWA for restricted programs without function symbols. Moreover, the context checks based on resultants also lead to an implementation of query processing for these programs. Finally, if the Context Completeness Claim 6.11 can be confirmed, the same holds for function-free nvi programs.

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