MECHANICAL THEOREM PROVING IN DIFFERENTIAL GEOMETRY
II. BERTRAND CURVES

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Abstract This paper reports the study of the properties of the curve pairs of the Bertrand types using our automated reasoning program for differential geometry. A complete list of about 50 results of Bertrand curves in metric and affine geometries has been obtained. The list includes most of the known results of various Bertrand curves which are among the most eminent results in the local theory of space curves. Some of the properties found by our program are unknown to us, and to the best of our knowledge, are not in textbooks for differential geometry. Computer plays a major role in obtaining such a list with very few human interactions. The basis of our automated reasoning system is the Ritt–Wu method for differential polynomials.

Keywords Mechanical theorem proving, Mechanical formula derivation, metric differential geometry, affine differential geometry, Bertrand's theorem, Mannheim's theorem.

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1. Introduction to the Problem

This is a collection of the results about various Bertrand curves obtained using a computer program based on an improved version of Ritt-Wu's zero decomposition algorithm presented in part I of this paper [CG1]. We adopt two approaches to treating the problems. First, we use Formulation II to prove known results under some explicitly given non-degenerate conditions. Second, we derive "unknown" relations among certain variables using Ritt-Wu's characteristic method and then prove them using Formulation II. In this way, we have proved or derived most of the known results for various Bertrand curves mechanically. We have also derived some results which we have not found in textbooks of differential geometry or relevant papers.

The Bertrand curves problem was first studied using a computer by Wu in [WU1]. This paper is a further study of the same problem, but contains more results than those of Wu's: totally 18 types of Bertrand curves in metric and affine differential geometries are studied and a complete list of about 50 results are given. Also our study here follows a different approach: we use the complete decomposition algorithm to derive or prove certain results under some explicitly given conditions. Also, the proving procedure for the known or derived results is automatically carried out by our program without any human assistance.

Theorems on various Bertrand curves are among the most eminent results in the local theory of space curves. The success of our method in dealing with these problems shows that our program based on the Ritt-Wu's decomposition algorithm can be used to solve quite difficult problems in elementary differential geometry, or even discover new results.

A pair of space curves having their principal normals in common are said to be associate Bertrand curves [BE1]. Here following Wu [WU1], we shall further consider more general problems. Given two space curves C_1 and C_2 in an one to one correspondence, let us attach *moving triads* $(C_1, e_{11}, e_{12}, e_{13})$ and $(C_2, e_{21}, e_{22}, e_{23})$ to C_1 and C_2 at the corresponding points of C_1 and C_2 respectively. We denote the arcs, curvature and torsions of C_1 and C_2 by s_1, k_1, t_1 and s_2, k_2, t_2 respectively. Then all the quantities introduced above can be looked as functions of s_1. Let $r = \frac{ds_2}{ds_1}$, and let

$$C_2 = C_1 + a_1 E_{11} + a_2 E_{12} + a_3 E_{13} \tag{1.1}$$

$$\begin{aligned} e_{21} &= u_{11} e_{11} + u_{12} e_{12} + u_{13} e_{13} \\ e_{22} &= u_{21} e_{11} + u_{22} e_{12} + u_{23} e_{13} \\ e_{23} &= u_{31} e_{11} + u_{32} e_{12} + u_{33} e_{13} \end{aligned} \tag{1.2}$$

where a_i are variables and (u_{ij}) is a matrix of variables satisfying certain relations which will be given in the following sections. For the original Bertrand Curves, (1.1) and (1.2) become

$$C_2 = C_1 + a_2 E_{12}$$

$$\begin{aligned} e_{21} &= u_{11} e_{11} + u_{13} e_{13} \\ e_{22} &= e_{12} \\ e_{23} &= -u_{13} e_{11} + u_{11} e_{13} \end{aligned}$$

where $u_{11}^2 + u_{13}^2 = 1$.

Roughly speaking, the problem is to find under what conditions for the curve pairs (C_1 and C_2) their moving triads will satisfy some given relations. For example, the original Bertrand curve problem is to ask under what conditions (of C_1 or C_2) C_1 and C_2 will have identical principal normals at the corresponding points, i.e. $E_{22} = E_{12}$ at the correponding points.

In this paper, we mainly consider the following three groups of problems.

MI_{ij} $(1 \le i \le j \le 3)$ means that e_{2j} is identical with e_{1i} in metric differential geometry.

MP_{ij} $(1 \le i \le j \le 3)$ means that e_{2j} is parallel to e_{1i} in metric differential geometry.

AI_{ij} $(1 \le i \le j \le 3)$ means that e_{2j} has the same direction with e_{1i} in affine differential geometry.

So totally 18 kinds of Bertrand curves are studied.

In this paper, we assume the reader has already known the Ritt-Wu's decomposition algorithm and Wu's method of mechanical theorem proving in the differential case. A detailed description of the algorithm can be found in [WU2] or part I of this paper [CG1].

In section 2, we will discuss the following two cases in metric differential geometry. In section 3, we will discuss the case in affine differential geometry.

2. Bertrand Curves In Metric Space

In metric differential geometry, let (e_{11}, e_{12}, e_{13}) and (e_{21}, e_{22}, e_{23}) be the Frenet triads of C_1 and C_2 at their corresponding points respectively, then we have the following Frenet formulas.

$$
\begin{aligned}
e'_{11} &= k_1 e_{12} \\
e'_{12} &= -k_1 e_{11} + t_1 e_{12} \\
e'_{13} &= -t_1 e_{12}
\end{aligned}
\tag{2.1}
$$

$$
\begin{aligned}
e'_{21} &= r k_2 e_{22} \\
e'_{22} &= -r k_2 e_{21} + r t_2 e_{22} \\
e'_{23} &= -r t_2 e_{22}
\end{aligned}
\tag{2.2}
$$

where $r = \frac{ds_2}{ds_1}$ and the differentiations here and in the following of this paper are all wrpt s_1.

Differentiate (1.1) and (1.2); eliminate $e'_{11}, e'_{12}, e'_{13}, e'_{21}, e'_{22}$ and e'_{23} using (2.1) and (2.2); eliminate e_{21}, e_{22}, and e_{23} using (1.2); at last, comparing coefficients for the vectors e_{11}, e_{12}, and e_{13}, we have:

$$
\begin{aligned}
& a_2 t_1 - r u_{13} + a'_3 = 0 \\
& a_3 t_1 - a_1 k_1 + r u_{12} - a'_2 = 0 \\
& a_2 k_1 + r u_{11} - a'_1 - 1 = 0 \\
& r u_{23} k_2 - u_{12} t_1 - u'_{13} = 0 \\
& r u_{22} k_2 + u_{13} t_1 - u_{11} k_1 - u'_{12} = 0 \\
& r u_{21} k_2 + u_{12} k_1 - u'_{11} = 0
\end{aligned}
$$

$$
\begin{aligned}
&ru_{33}t_2 - ru_{13}k_2 - u_{22}t_1 - u'_{23} = 0 \\
&ru_{32}t_2 - ru_{12}k_2 + u_{23}t_1 - u_{21}k_1 - u'_{22} = 0 \\
&ru_{31}t_2 - ru_{11}k_2 + u_{22}k_1 - u'_{21} = 0 \\
&ru_{23}t_2 + u_{32}t_1 + u'_{33} = 0 \\
&ru_{22}t_2 - u_{33}t_1 + u_{31}k_1 + u'_{32} = 0 \\
&ru_{21}t_2 - u_{32}k_1 + u'_{31} = 0
\end{aligned} \tag{2.3}
$$

To transform a right-handed orthogonal system $\{e_{11}, e_{12}, e_{13}\}$ to another right handed orthogonal system $\{e_{21}, e_{22}, e_{23}\}$, (u_{ij}) must satisfy

$$
\begin{aligned}
&u_{13}^2 + u_{12}^2 + u_{11}^2 - 1 = 0 \\
&u_{23}^2 + u_{22}^2 + u_{21}^2 - 1 = 0 \\
&u_{33}^2 + u_{32}^2 + u_{31}^2 - 1 = 0 \\
&u_{13}u_{23} + u_{12}u_{22} + u_{11}u_{21} = 0 \\
&u_{13}u_{33} + u_{12}u_{32} + u_{11}u_{31} = 0 \\
&u_{23}u_{33} + u_{22}u_{32} + u_{21}u_{31} = 0 \\
&(u_{11}u_{22} - u_{12}u_{21})u_{33} + (-u_{11}u_{23} + u_{13}u_{21})u_{32} + (u_{12}u_{23} - u_{13}u_{22})u_{31} - 1 = 0
\end{aligned} \tag{2.4}
$$

(2.3) and (2.4) were first given by Wu in [WU1] except the last equation in (2.4) which is added by us to preerve the right-handness of the moving triads.

2.1. The Identical Case

Let MI_{ij} be the case for which e_{2j} is identical with e_{1i} at the corresponding points. Since (e_{11}, e_{12}, e_{13}) and (e_{21}, e_{22}, e_{23}) are orthogonal systems, at case MI_{ij} we have:

$$
\begin{aligned}
&a_m = 0 \quad m \neq i \\
&u_{ji} - 1 = 0 \\
&u_{jk_1} = 0 \quad k_1 \neq i \\
&u_{ki} = 0 \quad k \neq j
\end{aligned} \tag{2.5}
$$

For each concrete case $MI_{i_0 j_0}$, apply Ritt-Wu's decomposition theorem to (2.3), (2.4), and (2.5) under the following variable order $r < a_1 < a_2 < a_3 < u_{11} < u_{12} < u_{13} < u_{21} < u_{22} < u_{23} < u_{31} < u_{32} < u_{33} < k_1 < t_1 < k_2 < t_2$. Once the decomposition is furnished, we can prove or derive formulas from the given asc chains in the decomposition.

The following non-degenerate conditions are often used:

$k_1 \neq 0$ means curve C_1 is not a straight line.

$k_2 \neq 0$ means curve C_2 is not a straight line.

$r \neq 0$ means the arc length of C_2 as a function of the arc length of C_1 is not a constant, i.e., C_2 is not a fixed point.

At first, we list some of the known or derived results.

Case MI_{11}. Under the non-degenerate condition $rk_1k_2 \neq 0$, C_1 and C_2 must be identical, i.e. $C_1 = C_2$. For other variables, we have two cases:

(i). $r = 1$, $e_{11} = e_{21}, e_{12} = e_{22}, e_{13} = e_{23}$ $k_1 = k_2$, and $t_1 = t_2$.

(ii). $r = 1$, $e_{11} = e_{21}, e_{12} = -e_{22}, e_{13} = -e_{23}$ $k_1 = -k_2$, and $t_1 = t_2$.

Note that (ii) is impossible as we assume $k_1 > 0$ and $k_2 > 0$.

Case MI_{12}. Under the non-degenerate condition $r \neq 0$, we have

a. C_2 and C_1 are both plane curves ($t_1 = t_2 = 0$).

b. $C_2 = C_1 + a_1 e_{11}$.

c. There are two cases:

$$\begin{aligned} &e_{21} = -e_{12}, e_{22} = e_{11}, e_{23} = e_{13} \\ &a_1' = -1, a_1 k_2 = -1 \\ &r = -a_1 k_1 \end{aligned} \tag{2.6}$$

$$\begin{aligned} &e_{21} = e_{12}, e_{22} = e_{11}, e_{23} = -e_{13} \\ &a_1' = -1, a_1 k_2 = -1 \\ &r = a_1 k_1 \end{aligned} \tag{2.7}$$

The geometric meaning of the above results can be stated as follows.

If C_2 is the involute of C_1 in the strong sense that the principal normals of C_2 are identical with the tangent lines of C_1, then both curve must be plane curves, and

(i) $C_2 = C_1 + (c_0 - s)e_{11}$ where c_0 is a constant.

(ii) $C_1 = C_2 + \frac{1}{k_2}e_{22}$, i.e C_1 is the locus of the curvature center of C_2.

(iii) The arc length of C_1 between two points equal to the difference of the reciprocal of the curvature of C_2 at the corresponding points.

Case MI_{13}. There exist no curves satisfying $e_{11} = e_{23}$ under the condition $r \neq 0$.

Case MI_{22}. Under the non-degenerate conditions $r \neq 0$ and $a_2 \neq 0$ ($C_2 \neq C_2$), we have

a. The distance from C_1 to C_2 is a constant.

b. The angle formed by the tangent lines at C_1 and C_2 respectively is a constant.

c. (Bertrand) There exists a linear relation between k_1 and t_1 with constant coefficients.

d. (Schell) The production of t_1 and t_2 is a constant.

We actually have the concrete expressions for results c and d in (2.10) and (2.11).

Case MI_{23}. Under the non-degenerate condition $rk_1 \neq 0$, we have

a. The distance from C_1 to C_2 is a constant.

b. (Mannheim) $\quad k_1^2 + t_1^2 = c_1 k_1$

c. $t_1 t_2^2 = c_2(t_1 - t_2)$

where c_1 and c_2 are constants.

We actually have the concrete expressions for results b and c in (2.12).

Case MI_{33}. Under the non-degenerate condition $rk_1k_2 \neq 0$, we have either

a. $C_1 = C_2$; or

b. C_1 and C_2 are both plane curves and $e_{11} = e_{21}, e_{12} = e_{22}, e_{13} = e_{23}\ a_3' = 0, r = 1, k_1 = k_2$; or

c. C_1 and C_2 are both plane curves and $e_{11} = -e_{21}, e_{12} = -e_{22}, e_{13} = e_{23}\ a_3' = 0, r = -1, k_1 = -k_2$.

In this case, we have either C_1 and C_2 are identical or both curves are plane curves and C_2 is translation of C_1 with a constant distance along the binormal of C_1.

Take MI_{22}, the original case of Bertrand as an example. Other cases can be proved similarly. Using Ritt-Wu's decomposition algorithm to (2.3) $\cup$ (2.4) $\cup$ (2.5) and $r \neq 0, a_2 \neq 0$, we get 3 components the ascending chains representing the components are:

$ASC_1 =$	$ASC_2 =$	$ASC_3 =$
a_1	a_1	a_1
a_2'	a_2'	a_2'
a_3	a_3	a_3
$u_{11} - 1$	$u_{11} + 1$	u_{11}'
u_{12}	u_{12}	u_{12}
u_{13}	u_{13}	$u_{13}^2 + u_{11}^2 - 1$
u_{21}	u_{21}	u_{21}
$u_{22} - 1$	$u_{22} - 1$	$u_{22} - 1$
u_{23}	u_{23}	u_{23}
u_{31}	u_{31}	$u_{31} + u_{13}$
u_{32}	u_{32}	u_{32}
$u_{33} - 1$	$u_{33} + 1$	$u_{33} - u_{11}$
$a_2k_1 + r - 1$	$a_2k_1 - r - 1$	$a_2k_1 + ru_{11} - 1$
t_1	t_1	$a_2t_1 - ru_{13}$
$ra_2k_2 + r - 1$	$ra_2k_2 + r + 1$	$ra_2k_2 - u_{11} + r$
t_2	t_2	$ra_2t_2 - u_{13}$

By the method of eliminating constants in [CG1], the four conclusions of MI_{22} are equivalent to

$$\begin{aligned} &a_2' = 0 \\ &u_{11}' = 0 \\ &DLR(1, k_1, t_1) = k_1''t_1' - t_1''k_1' = 0 \\ &(t_1t_2)' = 0 \end{aligned} \tag{2.9}$$

respectively. The pseudo remainders of the differential polynomials (ab. d-pols) in (2.9) wrpt ASC_1, ASC_2, and ASC_3 are zero which proves the result.

On the other hand, we can obtain our results from ASC_3, which represents the main component [CG1], directly. The differential equations representing results a ($a_2' = 0$) and b ($u_{11}' = 0$) are already in ASC_3. Eliminate r from the last four equations of ASC_3, we have:

$$\begin{aligned} &a_2 u_{11} t_1 + a_2 u_{13} k_1 - u_{13} = 0 \\ &a_2^2 t_1 t_2 - u_{13}^2 \\ &a_2^2 t_1 k_2 + a_2 t_1 - u_{11} u_{13} = 0 \end{aligned} \tag{2.10}$$

As a_2, u_{11}, (and hence $u_{13} = \sqrt{1 - u_{11}^2}$) are constants, the first two formulas of (2.10) actually give the concrete expression for Bertrand's theorem and Schell's theorem. From (2.10) we can find formulas between k_1, k_2; k_1, t_2; and k_2, t_2 respectively as follows.

$$\begin{aligned} &(1 - a_2 k_1)(1 + a_2 k_2) - u_{11}^2 = 0 \\ &a_2^2 k_1 t_2 - a_2 t_2 + u_{11} u_{13} = 0 \\ &a_2 u_{11} t_2 - a_2 u_{13} k_2 - u_{13} = 0 \end{aligned} \tag{2.11}$$

The conclusions in (2.10) and (2.11) are correct at the nondegenerate condition $k_1 k_2 r \neq 0$.

For MI_{23}, we can find the following concrete expressions for (b) and (c) of MI_{23} similarly:

$$\begin{aligned} &a_2 t_1^2 + a_2 k_1^2 - k_1 = 0 \\ &a_2^2 t_1 t_2^2 - t_2 + t_1 = 0 \end{aligned} \tag{2.12}$$

where a_2 is a constant. We can also find the following algebraic relations among k_1, k_2, and t_2:

$$\begin{aligned} &a_2 t_1 t_2 - k_1 = 0 \\ &k_1^2 + t_1^2 - t_1 t_2 = 0 \\ &(a_2^2 k_1 - a_2) t_2^2 + k_1 = 0 \end{aligned}$$

For r, we have:

$$\begin{aligned} &r^2 = t_1^2/(t_1^2 + k_1^2) \\ &r^2 = t_1/t_2 \\ &r = u_{11} \end{aligned}$$

Note that k_2 does not occurred in the above expressions. There are no algebraic relations among k_2, k_1, t_1, t_2, and a_2. We have the following formulas for k_2:

$$\begin{aligned} &2 t_1 k_2 + dk_1/ds' = 0 \\ &a_2 t_2 k_2 - r'/r^2 = 0 \end{aligned}$$

All the above results are true under the nondegenerate condition $k_1 k_2 r \neq 0$.

2.2. The Parallel Case

Let MP_{ij} be the case for which vector e_{2j} is parallel to vector e_{1i} at the corresponding points. Then at case MP_{ij}, we have

$$\begin{aligned} &u_{jk} = 0 \quad k \neq i \\ &u_{ki} = 0 \quad k \neq j \end{aligned} \tag{2.13}$$

For each concrete case $MP_{i_0j_0}$, apply Ritt-Wu's decomposition algorithm to (2.3), (2.4) and (2.13) under the non-degenerate condition $k_1k_2r \neq 0$. The following results can be derived and then proved automatically under the non-degenerated condition $k_1k_2r \neq 0$.

Case MP_{11}. There are four cases:

$$\begin{aligned}
a.\quad & a_3a_3' + a_2a_2' + a_1a_1' + (r+1)a_1 = 0 \\
& r = -k_1/k_2 = -t_1/t_2 \\
& e_{21} = -e_{11}, e_{22} = e_{12}, e_{23} = -e_{13}; \text{or} \\
b.\quad & a_3a_3' + a_2a_2' + a_1a_1' + (r+1)a_1 = 0 \\
& r = k_1/k_2 = -t_1/t_2 \\
& e_{21} = -e_{11}, e_{22} = -e_{12}, e_{23} = e_{13}; \text{or} \\
c.\quad & a_3a_3' + a_2a_2' + a_1a_1' + (-r+1)a_1 = 0 \\
& r = -k_1/k_2 = t_1/t_2 \\
& e_{21} = e_{11}, e_{22} = -e_{12}, e_{23} = -e_{13}; \text{or} \\
d.\quad & a_3a_3' + a_2a_2' + a_1a_1' + (-r+1)a_1 = 0 \\
& r = k_1/k_2 = t_1/t_2 \\
& e_{21} = e_{11}, e_{22} = e_{12}, e_{23} = e_{13}.
\end{aligned}$$

Case MP_{12}. There are two cases:

$$\begin{aligned}
a.\quad & u_{21} - 1 = 0 \\
& a_2k_1 - a_1' - 1 = 0 \\
& r^2 = \frac{k_1^2}{t_2^2 + k_2^2} \\
& k_2/u_{12} = -t_2/u_{13} = -k_1/r; \text{or} \\
b.\quad & u_{21} + 1 = 0 \\
& a_2k_1 - a_1' - 1 = 0 \\
& r^2 = \frac{k_1^2}{t_2^2 + k_2^2} \\
& k_2/u_{12} = t_2/u_{13} = k_1/r.
\end{aligned}$$

Case MP_{13}. There are four cases:

$$\begin{aligned}
a.\quad & e_{21} = -e_{13}, e_{22} = -e_{12}, e_{23} = -e_{11}, \text{or} \\
& r = -t_1/k_2 = -k_1/t_2 \\
b.\quad & e_{21} = e_{13}, e_{22} = e_{12}, e_{23} = -e_{11}, \text{or} \\
& r = -t_1/k_2 = k_1/t_2. \\
c.\quad & e_{21} = -e_{13}, e_{22} = e_{12}, e_{23} = e_{11}, \text{or} \\
& r = t_1/k_2 = -k_1/t_2. \\
d.\quad & e_{21} = e_{13}, e_{22} = -e_{12}, e_{23} = e_{11}. \\
& r = t_1/k_2 = k_1/t_2.
\end{aligned}$$

Case MP_{22} There are two cases

$$\begin{aligned}
a.\quad & u_{22} - 1 = 0 \\
& u'_{11} = 0 \\
& u'_{13} = 0 \\
& u_{13}(t_1 t_2 + k_1 k_2) = u_{11}(k_1 t_2 - t_1 k_2) \\
& r^2 = \frac{t_1^2 + k_1^2}{t_2^2 + k_2^2} \\
& a_3 t_1 - a_1 k_1 - a'_2 = 0 \\
& r^2 k_1 t_2 - r^2 t_1 k_2 - a_2 t_1^3 - a'_3 t_1^2 - a_2 k_1^2 t_1 - a'_3 k_1^2 = 0 \\
& r^2 t_1 t_2 + r^2 k_1 k_2 + (a_2 k_1 - a'_1 - 1) t_1^2 + a_2 k_1^3 + (-a'_1 - 1) k_1^2 = 0; \text{or} \\
b.\quad & u_{22} + 1 = 0 \\
& u'_{11} = 0 \\
& u'_{13} = 0 \\
& u_{13}(t_1 t_2 - k_1 k_2) = u_{11}(k_1 t_2 + t_1 k_2) \\
& r^2 = \frac{t_1^2 + k_1^2}{t_2^2 + k_2^2} \\
& a_3 t_1 - a_1 k_1 - a'_2 = 0 \\
& r^2 k_1 t_2 + r^2 t_1 k_2 - a_2 t_1^3 - a'_3 t_1^2 - a_2 k_1^2 t_1 - a'_3 k_1^2 = 0 \\
& r^2 t_1 t_2 - r^2 k_1 k_2 + (a_2 k_1 - a'_1 - 1) t_1^2 + a_2 k_1^3 + (-a'_1 - 1) k_1^2 = 0
\end{aligned}$$

Note that from the fourth differential polynomial, we know that t_1, k_1, k_2, and t_2 satisfy a homogeneous quadratic equation of constant coefficients.

Case MP_{23} We have

$$\begin{aligned}
& r^2 t_2^2 - t_1^2 - k_1^2 = 0 \\
& t_1/u_{11} = k_1/u_{13} = r t_2 \\
& (a_2 a_3^2 + a_1^2 a_2) k_1^2 + (a_1 a_3 a'_3 + (-a'_1 - 1) a_3^2 + 2 a_1 a_2 a'_2) k_1 + a'_2 a_3 a'_3 + a_2 a_2'^2 = 0 \\
& a_3 t_1 - a_1 k_1 - a'_2 = 0
\end{aligned}$$

Case MP_{33}. We have the same results as MP_{11}.

Take MP_{11} as an example. Using Ritt-Wu's decomposition algorithm to (2.3), (2.4), and $\{u_{12} = 0, u_{13} = 0, u_{21} = 0, u_{31} = 0\}$ under the following variable order: $k_1 < t_1 < k_2 < t_2 < r < a_1 < a_2 < a_3 < u_{11} < u_{12} < u_{13} < u_{21} < u_{22} < u_{23} < u_{31} < u_{32} < u_{33}$, we find four main components which give the four results respectively.

Some of the results obtained in this section cannot be found in textbooks of differential geometry.

3. Bertrand Curves in Affine Space

In affine differential geometry, let $e_{11} = \frac{dC_1}{ds_1}, e_{12} = \frac{de_{11}}{ds_1}, e_{13} = \frac{de_{12}}{ds_1}$ and $e_{21} = \frac{dC_2}{ds_2}, e_{22} = \frac{de_{21}}{ds_2}, e_{23} = \frac{de_{22}}{ds_2}$ be the moving triads of C_1 and C_2 at their corresponding points respectively, where s_i are the arc length of curves C_i for $i = 1, 2$. Then we have the following Frenet formulas.

$$\begin{aligned} e'_{11} &= e_{12} \\ e'_{12} &= e_{13} \\ e'_{13} &= -k_1 e_{12} + t_1 e_{11} \end{aligned} \tag{3.1}$$

$$\begin{aligned} e'_{21} &= r e_{22} \\ e'_{22} &= r e_{23} \\ e'_{23} &= -r k_2 e_{22} + r t_2 e_{21} \end{aligned} \tag{3.2}$$

where $r = \frac{ds_2}{ds_1}$. We also have

$$(e_{11}, e_{12}, e_{13}) = 1 \tag{3.3}$$

$$(e_{21}, e_{22}, e_{23}) = 1 \tag{3.4}$$

Similar as section 2, we can get the following d-pol equations.

$$\begin{aligned} & r u_{13} - a'_3 - a_2 = 0 \\ & a_3 k_1 + r u_{12} - a'_2 - a_1 = 0 \\ & a_3 t_1 + r u_{11} - a'_1 - 1 = 0 \\ & r u_{23} - u'_{13} - u_{12} = 0 \\ & u_{13} k_1 + r u_{22} - u'_{12} - u_{11} = 0 \\ & u_{13} t_1 + r u_{21} - u'_{11} = 0 \\ & r u_{33} - u'_{23} - u_{22} = 0 \\ & u_{23} k_1 + r u_{32} - u'_{22} - u_{21} = 0 \\ & u_{23} t_1 + r u_{31} - u'_{21} = 0 \\ & r u_{13} t_2 + r u_{23} k_2 + u'_{33} + u_{32} = 0 \\ & r u_{12} t_2 + r u_{22} k_2 - u_{33} k_1 + u'_{32} + u_{31} = 0 \\ & r u_{11} t_2 + r u_{21} k_2 - u_{33} t_1 + u'_{31} = 0 \end{aligned} \tag{3.5}$$

From (3.3), (3.4), and (1.2) we have that the determinant of the transformation matrix (u_{ij}) is the unit, i.e.

$$(u_{11}u_{22} - u_{12}u_{21})u_{33} + (-u_{11}u_{23} + u_{13}u_{21})u_{32} + (u_{12}u_{23} - u_{13}u_{22})u_{31} - 1 = 0 \tag{3.6}$$

Let AI_{ij} be the case such that e_{2j} has the same direction* as e_{1i} at the corresponding points. At case AI_{ij}, we have:

$$\begin{aligned} & a_k = 0 \quad k \neq i \\ & u_{jk_1} = 0 \quad k_1 \neq i \end{aligned} \tag{3.7}$$

* *In affine case the vectors in the moving triads are not unit vectors. Then there is no identical case as section 2.1*

We first apply Ritt-Wu's decomposition theorem to (3.5), (3.6), and (3.7) under the following variable order $r < a_1 < a_2 < a_3 < u_{11} < u_{12} < u_{13} < u_{21} < u_{22} < u_{23} < u_{31} < u_{32} < u_{33} < k_1 < t_1 < k_2 < t_2$. Once the decomposition is furnished, we may prove or derive results from the given asc chains in the decomposition.

At first, we list some of the known or derived results.

Case AI_{11}. The following results are true under the non-degenerate condition $r \neq 0$

a. $C_2 = C_1$.

b. There are two cases.

$$r = 1$$
$$k_2 = k_1, t_2 = t_1$$
$$e_{21} = e_{11}, e_{22} = e_{12}, e_{23} = e_{13}$$

$$r = -1$$
$$k_2 = k_1, t_2 = -t_1$$
$$e_{21} = -e_{11}, e_{22} = e_{12}, e_{23} = -e_{13}$$

Case AI_{12}. There exist no curves such that e_{11} has the same direction as e_{22}.

Case AI_{13}. The following results are true under the non-degenerate condition $r \neq 0$

$$C_2 = C_1 + a_1 e_{11}$$
$$e_{23} = u_{31} e_{11}$$
$$a_1 = \frac{r}{3} \int r ds_1$$
$$u_{31} = r^3 / a_1^2$$

$$k_2 = 0$$
$$a_1^3 t_2 + r^3 = 0$$
$$a_1 t_2 + u_{31} = 0$$
$$r^3 t_2'^3 + t_2^4 = 0$$
$$r^2 a_1 k_1 + (-2rr'' + 3r'^2) a_1 + 2rr' = 0$$
$$3r^3 a_1^3 t_1 + (-3r^2 r''' + 12 r r' r'' - 9 r'^3) a_1^3 + (4r^2 r'' - 10 r r'^2) a_1^2 + 3r^9 = 0$$
$$18 t_2^2 t_2' t_2''' - 9 t_2^2 t_2''^2 - 66 t_2 t_2'^2 t_2'' + 64 t_2'^4 + 9 k_1 t_2^2 t_2'^2 = 0$$
$$3 r a_1' - 3 r' a_1 + r = 0$$

Case AI_{22}. The following results are true under the non-degenerate conditions $r \neq 0, a_2 \neq 0$.

$$C_2 = C_1 + a_2 e_{12}$$
$$e_{22} = u_{22} e_{11}$$
$$a_2^2 = c u_{22}$$

where c is a constant. For k_1 and t_1, we get a formula for $k_1, k_1', k_1''.t_1, t_1', t_1''$, and t_1''' of 55 terms.

$$
\begin{aligned}
&((12t_1^2 - 4k_1' t_1)t_1')t_1'''^2 \\
&+ ((-15t_1^2 + 5k_1' t_1)t_1''^2 + ((-24t_1 + 4k_1')t_1'^2 + 4k_1'' t_1 t_1' - 90k_1 t_1^3 + 30k_1 k_1' t_1^2)t_1'' \\
&- 20k_1 k_1' t_1 t_1'^2 + (-144t_1^4 + 216k_1' t_1^3 + (20k_1 k_1'' - 56k_1'^2)t_1^2)t_1' - 75k_1^2 t_1^4 + 25k_1^2 k_1' t_1^3)t_1''' \\
&+ (15t_1 t_1' - 5k_1'' t_1)t_1''^3 + (12t_1'^3 - 4k_1'' t_1'^2 + (90k_1 t_1^2 + 25k_1 k_1' t_1)t_1' \\
&- 135t_1^4 + 15k_1' t_1^3 + (-55k_1 k_1'' + 10k_1'^2)t_1^2)t_1''^2 \\
&+ (20k_1 k_1' t_1'^3 + (144t_1^3 - 72k_1' t_1^2 + (-20k_1 k_1'' - 12k_1'^2)t_1)t_1'^2 + ((-144k_1'' + 75k_1^2)t_1^3 \\
&+ (68k_1' k_1'' + 150k_1^2 k_1')t_1^2)t_1' + 90k_1 t_1^5 - 510k_1 k_1' t_1^4 + (-175k_1^2 k_1'' + 160k_1 k_1'^2)t_1^3)t_1'' \\
&- 16k_1'^2 t_1'^4 + 32k_1' k_1'' t_1 t_1'^3 + (240k_1 k_1' t_1^3 + (-16k_1''^2 - 180k_1 k_1'^2)t_1^2)t_1'^2 \\
&+ (432t_1^6 - 1152k_1' t_1^5 + (-240k_1 k_1'' + 924k_1'^2)t_1^4 + (180k_1 k_1' k_1'' - 196k_1'^3 + 125k_1^3 k_1')t_1^3)t_1' \\
&+ 225k_1^2 t_1^6 - 525k_1^2 k_1' t_1^5 + (-125k_1^3 k_1'' + 150k_1^2 k_1'^2)t_1^4 = 0
\end{aligned}
$$

For other variables we have two cases

$$
\begin{aligned}
&u_{22} - r = 0 \\
&2ra_2' - r' a_2 = 0 \\
&t_2 - t_1 = 0 \\
&r^2 a_2 k_2 + a_2 k_1 + 2r^3 - 2 = 0 \\
&r a_2 t_1 + r' = 0 \\
&a_2^2 t_1 + 2a_2' = 0
\end{aligned}
$$

$$
\begin{aligned}
&u_{22} + r = 0 \\
&2ra_2' - r' a_2 = 0 \\
&t_2 + t_1 = 0 \\
&r^2 a_2 k_2 + a_2 k_1 - 2r^3 - 2 = 0 \\
&r a_2 t_1 + r' = 0 \\
&a_2^2 t_1 + 2a_2' = 0
\end{aligned}
$$

Case AI_{23}**.** The following results are true under the non-degenerate condition $r \neq 0$.

$$
\begin{aligned}
&r^4 a_2 t_2 + (2ra_2' - r' a_2)u_{32}^2 + r^4 u_{32} = 0 \\
&r^2 k_2 - u_{32}^2 = 0 \\
&a_2^2 u_{32} t_1 + 2a_2' u_{32} + r^3 = 0 \\
&r^2 a_2 k_1 - 3r^2 a_2'' + 6rr' a_2' + (rr'' - 3r'^2)a_2 - r^2 = 0 \\
&(a_2^4 t_1^2 + 4a_2^2 a_2' t_1 + 4a_2'^2)k_2 - r^4 = 0 \\
&(a_2^4 t_1^2 + 4a_2^2 a_2' t_1 + 4a_2'^2)k_2^3 - u_{32}^4 = 0
\end{aligned}
$$

Case AI_{33}**.** The following results are true under the non-degenerate condition $r \neq 0$.

$$
\begin{aligned}
&r^2 k_2 - u_{33}^2 k_1 = 0 \\
&(r^2 a_3 t_1 - r^2)t_2 + (r a_3 t_1' + (2ra_3' - r' a_3)t_1 + r')k_2 + r^2 u_{33} t_1 = 0
\end{aligned}
$$

All the above results can be proved similar as section 2. For example, let us show how to prove the result $a_2^2 = cu_{22}$ in case AI_{22}. Since c is a constant. We need to prove $DLR(a_2^2, u_{22}) =$

$2a_2a_2'u_{22} - a_2^2u_{22}' = 0$. Using Ritt-Wu's decomposition theorem to (3.5)∪(3.6)∪$\{a_1, a_3, u_{21}, u_{23}\}$ under the non-degenerate condition $r \neq 0$, we get six components. the pseudo remainders of $DLR(a_2^2, u_{22})$ with respect to (ab. wrpt) the six ascending chains representing the six components are all zero which means the result is true. We actually can derive this result automatically. Of the six components, there are two main components whose ascending chains are:

$ASC_1 =$	$ASC_2 =$
a_1	a_1
$2ra_2' - r'a_2$	$2ra_2' - r'a_2$
a_3	a_3
$ru_{11} - 1$	$ru_{11} - 1$
$2r^2u_{12} - r'a_2$	$2r^2u_{12} - r'a_2$
$ru_{13} - a_2$	$ru_{13} - a_2$
u_{21}	u_{21}
$u_{22} + r$	$u_{22} - r$
u_{23}	u_{23}
u_{31}	u_{31}
$ru_{32} + r'$	$ru_{32} - r'$
$u_{33} + 1$	$u_{33} - 1$
$4r^2a_2k_1 + (-2rr'' + 3r'^2)a_2 - 4r^5 - 4r^2$	$4r^2a_2k_1 + (-2rr'' + 3r'^2)a_2 + 4r^5 - 4r^2$
$ra_2t_1 + r'$	$ra_2t_1 + r'$
$4r^4a_2k_2 + (2rr'' - 3r'^2)a_2 - 4r^5 - 4r^2$	$4r^4a_2k_2 + (2rr'' - 3r'^2)a_2 + 4r^5 - 4r^2$
$ra_2t_2 - r'$	$ra_2t_2 + r'$

In ASC_1 and ASC_2, there is a differential equation $2ra_2' - r'a_2 = 0$ whose solution is just $a_2^2 = cr$ for a constant c. All the results of case AI_{22} can be found similarly. In this way, we have rediscovered all the formulas among t_1, t_2, a_2, r, u_{22} in [OG1]. We also found some results which are not in [OG1], e.g. the relations between k_1 t_1 and k_1 k_2.

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