# ON THE NORMAL PARAMETERIZATION OF CURVES AND SURFACES*

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## ABSTRACT

A set of parametric equations of an algebraic curve or surface is called *normal*, if all the points of the curve or the surface can be given by the parametric equations. In this paper, we propose the problem of finding normal parametric equations for rational curves and surfaces if possible. We present a method to decide whether a set of parametric equations is normal. In addition, we give some simple criteria for a set of parametric equations to be normal. As an application, we present a method to find normal parametric equations for conics. We also present a new method to find parametric equations for conicoids, and if the parametric equations found are not normal, we give the missing points. For the parametric equations of conics and conicoids found by our methods, we also give their inversion maps.

**Keywords**: Parametric equations, normal parametric equations, irreducible variety, inversion map, conics, conicoids.

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## 1. Introduction

It is known that in the parametric representation for algebraic curves and surfaces, certain points on the curves or the surfaces may not be given by parametric equations. Two such examples are the following parametric equations for the circle $x^2 + y^2 = 1$

$$x = \frac{t^2 - 1}{t^2 + 1}, \; y = \frac{2t}{t^2 + 1} \tag{1.1}$$

and the following parametric equations for the surface $x^4 - y^2 z = 0$

$$x = uv, \; y = uv^2, \; z = u^2. \tag{1.2}$$

The point $(1, 0)$ of the circle cannot be given by (1.1). The line $(0, c, 0)$ with $c \neq 0$ is on the surface $x^4 - y^2 z = 0$, but cannot be given by (1.2).

In the work of finding the parametric equations for curves and surfaces, e.g., in [AB1-4, CG1, SS1], the problem of missing points are not considered. These missing points may be the critical points of the described figure and cause problem when we try to display the figure by a computer. In [LI1], a method to find these missing points is given based on an algorithm of quantifier elimination for the theory of algebraic closed fields which is presented in [WU2]. The quantifier elimination algorithm is in turn based on Ritt-Wu's decomposition algorithm [WU1]. Finding the missing points is a solution to the problem. A better solution is to find normal parametric equations if possible. This is the purpose of the present paper.

Parametric equations of a curve or a surface are called *normal*, if all the points of the curve or the surface can be given by the parametric equations. For example, (1.1) and (1.2) are not normal. We now ask the question whether we can find normal parametric equations for them. For (1.2), this is easy: the following parametric equations for $x^4 - y^2 z = 0$

$$x = uv, y = v^2, z = u^4 \tag{1.3}$$

are normal (because $u$ and $v$ can be independently determined by $y$ and $z$). But for (1.1), it is not easy. For the simplest normal parametric equations for the circle, see Section 3.

In this paper, we give a method to decide whether parametric equations for a curve or a surface are normal. Also some simple criteria for parametric equations to be normal are given. These criteria are very easy to use. Based on these criteria, we prove that polynomial parametric equations for a curve are always normal.

Methods for parameterization of conics and conicoids have been given by S. Abhyankar and C. Bajaj in [AB1]. But in general, the parametric equations obtained by their method are not normal. In this paper, we propose a new method for parameterization of conics and conicoids. The idea is that by known methods in analytical geometry, we can transform general forms of conics or conicoids to the standard forms. Thus if (normal) parametric equations for these standard forms are given, then (normal) parametric equations for the general forms can be obtained using the coordinate transformations. In this way, we can find normal parametric equations for all conics. It is easy to find normal parametric equations for parabolas and hyperbolas. To find normal parametric equations with real coefficients for ellipses is not trivial. We have proved that an ellipse cannot have parametric equations of odd degree. We have also proved that quadratic parametric equations for an ellipse are not normal. As a consequence, all the parametric equations with real coefficients for ellipses found using the method in [AB1] are not normal. The simplest normal parametric equations for an ellipse are at least of degree four and we find such one. For conicoids, we give quadratic parametric equations for all kinds of standard forms. Some of them are normal. For those which are not normal, the missing points are given. Also, for all the parametric equations of conics and conicoids, we give their inversion maps, i.e., functions which give the parametric values corresponding to the points on the curves or surfaces.

The paper is organized as follows. In section 2, we give a method for deciding whether parametric equations are normal and give some simple criteria for a set of parametric equations to be normal. In section 3, we present a method to give normal parametric equations of conics. In section 4, we present a method to give parametric equations of conicoids.

## 2. Parametric Equations and Normal Parametric Equations

Let $K$ be a computable field of characteristic zero. We use $K[y_1,...,y_n]$ or $K[y]$ to denote the ring of polynomials in the indeterminates $y_1,...,y_n$. Unless explicitly mentioned otherwise, all polynomials in this paper are in $K[y]$. Let $E$ be a *universal extension* of $K$, i.e., an algebraic closed extension of $K$ which contains sufficiently many independent indeterminates over $K$ (Vol2, [HP1]). For a polynomial set $PS$, let

$$Zero(PS) = \{x = (x_1,...,x_n) \in E^n \mid \forall P \in PS, P(x) = 0\}.$$

For two polynomial sets $PS$ and $DS$, we define a quasi zero set $QZero(PS, DS)$ to be $Zero(PS) - Zero(DS)$.

We have mentioned that various methods of parameterization for curves and surfaces have been given. But the exact definition of a set of parametric equations to be the parametric representation for a curve or a surface has not been given yet. The following example suggests that this definition is not obvious. At first sight, one may think that the parametric equations

$$x = u + v,\ y = u^2 + v^2 + 2uv - 1,\ z = u^3 + v^3 + 3u^2v + 3v^2u + 1$$

represent a space surface. Actually, it represents a space curve, because let $t = u + v$, then the above parametric equations become

$$x = t,\ y = t^2 - 1,\ z = t^3 + 1.$$

In the following, we shall give a precise definition for a set of parametric equations representing an irreducible variety with dimension $d$.

Let $t_1,...,t_m$ be indeterminates in $E$. For nonzero polynomials $P_1,...,P_n,Q_1,...,Q_n$ in $K[t_1,...,t_m]$, we call

$$y_1 = \frac{P_1}{Q_1},\ ...,\ y_n = \frac{P_n}{Q_n} \tag{2.1}$$

*a set of (rational) parametric equations.* We assume that $gcd(P_i, Q_i) = 1$. The maximum of the degrees of $P_i$ and $Q_j$ is called the *degree* of (2.1). The image of (2.1) in $E^n$ is defined as

$$IM(P_1,...,P_n,Q_1,...,Q_n) = \{(y_1,...,y_n) \mid \exists t \in E^m (y_i = P_i(t)/Q_i(t))\}.$$

Generally speaking, $IM(P,Q)$ is not an algebraic set. By [LI1], we know that $IM(P,Q)$ is a quasi variety, i.e., we can find polynomial sets $PS_i$ and $DS_i$ such that

$$IM(P,Q) = \cup_{i=1}^{t} QZero(PS_i, DS_i). \tag{2.2}$$

**Definition 2.3.** Let $V$ be an irreducible variety of dimension $d > 0$ in $E^n$. Parametric equations of the form (2.1) are called parameter equations of $V$ if

(1) $IM(P,Q) \subset V$; and

(2) $V - IM(P,Q)$ is contained in an algebraic set with dimension less than $d$.

**Theorem 2.4.** Parametric equations of the form (2.1) are the parametric equations of an irreducible variety whose dimension equals to the transendental degree of $K(P_1/Q_1, ..., P_n/Q_n)$ over $K$.

*Proof.* Let $I = \{F \in K[y] \mid F(P_1/Q_1, ..., P_n/Q_n) = 0\}$, then $I$ is a prime ideal and it is clear that $IM(P,Q) \subset Zero(I)$. We need to prove $Zero(I) - IM(P,Q)$ is contained in an algebraic set of less dimension than the dimension of $I$. By (2.2), $IM(P,Q)$ is a quasi variety, i.e., $IM(P,Q) = \cup_{i=1}^{l} QZero(PS_i, DS_i)$ where $PS_i$ and $DS_i$ are polynomial sets. Further more we can assume that each $PS_i$ is a prime ideal and $DS_i$ is not contained in $PS_i$ by the decomposition theorem in algebraic geometry. Since $\eta = (P_1/Q_1, ..., P_n/Q_n) \in IM(P,Q)$, $\eta$ must be in some components, say in $QZero(PS_1, DS_1)$. Note that $\eta$ is a generic point for $I$ and $Zero(PS_1) \subset Zero(I)$, then $PS_1 = I$. Hence $Zero(I) - IM(P,Q) = Zero(I \cup DS_1) - \cup_{i=2}^{l} Zero(PS_i/DS_i)$. Thus $Zero(I) - IM(P,Q)$ is contained in $Zero(I \cup DS_1)$ the dimension of which is less than the dimension of $I$ as $DS_1$ is not contained in $I = PS_1$. The dimension of $I$ is obviously equal to the transendental degree of $K(P_1/Q_1, ..., P_n/Q_n)$ over $K$. The proof is completed. ∎

**Remark.** If we use Ritt-Wu's decomposition algorithm [WU1] to realize the decomposition theorem, the above proof is actually a constructive one, i.e., for parametric equations of the form (2.1), we can find a finite polynomial set $PS$ such that the ideal generated by $PS$ is a prime ideal and a quasi variety $W = \cup_{i=1}^{l} QZero(PS_i, DS_i)$ such that $IM(P,Q) = Zero(PS) - W$.

**Definition 2.5.** (2.1) is called a set of *normal parametric equations* if $IM(P,Q)$ is an irreducible variety.

**Theorem 2.6.** We can decide in a finite number of steps whether parametric equations of the form (2.1) are normal parametric equations.

*Proof.* As mentioned in the above remark, we can find a finite polynomial set $PS$ such that the ideal generated by $PS$ is a prime ideal and a quasi variety $W = \cup_{i=1}^{l} QZero(PS_i, DS_i)$ such that $IM(P,Q) = Zero(PS) - W$. Then (1.2) is normal if and only if $IM(P,Q) = Zero(PS)$, or equivalently $Zero(PS)$ and $W$ have no common points. Without loss of generality, we only need to show how to decide whether $W' = Zero(PS) \cap QZero(PS_1, DS_1)$ is empty. Note that $W' = QZero(PS \cup PS_1, DS_1)$, then we can decide whether $W'$ is empty using Ritt-Wu's decomposition algorithm [WU1] or the Gröbner basis method. ∎

The method in theorem 2.6, though complete, usually needs extensive computations. In what follows, we give some simple criteria for normal parameterization which can be used without any computational costs.

**Lemma 2.7.** If the image $IM(P,Q)$ of (2.1) is an algebraic set, then (2.1) are normal parametric equations.

*Proof.* Let $IM(P,Q) = Zero(PS)$, and let (2.1) be parameter equations of the irreducible variety $V$. By (2) of Definition 2.3, a generic point of $V$ is in $Zero(PS)$. Thus $V \subset Zero(PS)$. By (1) of Definition 2.3, we have $Zero(PS) \subset V$, and hence $Zero(PS) = V$. ∎

**Theorem 2.8.** Let $y_1 = u_1(t)/v_1(t), ..., y_n = u_n(t)/v_n(t)$ be parametric equations of an algebraic curve. If $deg(u_i) > deg(v_i)$ for some $i$, they are normal parametric equations.

*Proof.* Let $RS = \{r_1, ..., r_h\}$ ($r_i \in K[y]$) be the resultant system of $h_1(t) = u_1(t) - v_1(t)y_1, ..., h_n(t) = u_n(t) - v_n(t)y_n$ for variable $t$ (see p158 Vol1, [HP1]). Then we have that for any $y_0 = (y_{0,1}, ..., y_{0,n}) \in E^n$, $r_i(y_0) = 0, i = 1, ..., h$ if and only if $h'_1 = u_1 - v_1 y_{0,1} = 0$, ..., $h'_n = u_n - v_n y_{0,n} = 0$ have common solutions for $t$ or the leading coefficients of $h'_1(t), ..., h'_n(t)$ all vanish. The later case is impossible, because there is an $i_0$ such that $degree(u'_{i_0}(t)) > degree(v'_{i_0}(t))$, hence the leading coefficient of $h'_{i_0}(t)$ is a nonzero number in $K$. Therefore, $r_i(y_0) = 0, i = 1, ..., h$ if and only if $h'_1(t) = 0, ..., h'_n(t) = 0$ have a common solution $t_0$. We say $v_i(t_0) \neq 0$ for all $i$, for otherwise $u_i(t_0) = v_i(t_0)y_{0,i} = 0$. Therefore $u_i(t)$ and $v_i(t)$ have common solutions which is contradict to the fact $gcd(u_i, v_i) = 1$. Thus $y_0 = (u_0(t_0)/v_0(t_0), ..., u_n(t_0)/v_n(t_0))$ is in the image $IM(u,v)$ of the parametric equations, i.e., $Zero(RS) \subset IM(u,v)$. It is easy to show that $IM(u,v) \subset Zero(RS)$. We have proved that $IM(u,v) = Zero(RS)$. By Lemma 2.7, theorem has been proved. ∎

**Corollary 2.9.** A set of polynomial parametric equations of a curve is normal.

## 3. Normal Parameterization for Conics

In this section, we present a method to find normal parametric equations for conics. To do this, we first transform general forms of conics to the standard forms by known methods in analytical geometry, then normal parametric equations for the general forms can be obtained using the coordinate transformations if normal parametric equations for these standard forms are given. Thus we only need to find normal parametric equations for the standard forms of conics.

*Inversion maps* for (2.1) are functions

$$t_1 = f_1(y_1, ..., y_n), ..., t_m = f_m(y_1, ..., y_n)$$

such that $y_i = P_i(f_1, ..., f_m)/Q_i(f_1, ..., f_m)$ on $IM(P,Q)$ i.e., functions which give the parameter values corresponding to points on the curves or surfaces. We shall give the inversion maps for the parametric equations of conics and conicoids found by the method in this paper. To do this we only need to give inversion maps for the parametric equations of the standard forms of conics and conicoids.

In this section, we assume that $K$ is the real number field. We consider a conics with real coefficients (not all $a$, $b$ and $c$ are zero)

$$C(x,y) = ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

Let $\delta = b^2 - 4ac$ and

$$\Delta = \begin{vmatrix} 2a & b & d \\ b & 2c & e \\ d & e & 2f \end{vmatrix}.$$

We have the following method to find normal parametric equations for $C = 0$.

**Case 1.** If $\Delta = 0$, then generally speaking $C$ represents two straight lines whose parameter equations can be easily found.

**Case 2.** If $\Delta \neq 0$, $C = 0$ is a nontrivial conics. We consider the following four cases.

**Case 2.1. Parabola.** A nontrivial conics $C = 0$ is a parabola if and only if $\delta = 0$. In this case, $a$ or $c$, say $a$, must not be zero. Let $t = x + \frac{b}{2a}y$, we have the following parametric equations for $C = 0$ (it can be checked directly by computation)

$$x = (abt^2 + 2aet + bf)/(2ae - bd)$$
$$y = -(2a^2t^2 + 2adt + 2af)/(2ae - bd).$$

It is easy to prove that $2ae - bd = 0$ implies $\Delta = 0$ or $a = 0$ which is impossible. By Corollary 2.9, this is a normal parameterization. The inversion map is $t = x + \frac{b}{2a}y$.

**Case 2.2. Hyperbola.** If $\delta > 0$, by an appropriate coordinate transformation, $C = 0$ can be transformed to the following standard form

$$y^2/b^2 - x^2/a^2 = 1.$$

A set of parameter equations is

$$x = \frac{a(t^2 - 1)}{2t}, y = \frac{b(t^2 + 1)}{2t}$$

which is normal by Theorem 2.8. The inversion map is $t = \frac{ab}{ay - bx}$.

**Case 2.3. Ellipse.** If $\delta < 0$ and $(a + c)\Delta < 0$, by an appropriate coordinate transformation, $C = 0$ can be transformed to the following standard form

$$y^2/b^2 + x^2/a^2 = 1. \qquad (2.3.1)$$

If we allow complex coefficients in the parameter equations, then we have the following normal parameter equations for (2.3.1).

$$x = \frac{a(t^2 - 1)}{2it}, y = \frac{b(t^2 + 1)}{2t}$$

where $i = \sqrt{-1}$. The following commonly used parametric equations for an ellipse are not normal.

$$x = \frac{a(t^2 - 1)}{t^2 + 1}, y = \frac{2bt}{t^2 + 1}$$

By the method in Theorem 2.4, the missing point is $(a, 0)$. The inversion map is $t = \frac{ay}{b(a-x)}$. To obtain normal parametric equations for (2.3.1), we first give two general results.

**Theorem 3.2.** If $x = v(t)/w(t), y = u(t)/w(t)$ are real coefficients parametric equations of (2.3.1) with $gcd(u(t), v(t), w(t)) = 1$, then we have

(1) the degree of $w$ equals to the maximal of the degree of $u$ and the degree of $v$; and

(2) $w = 0$ has no real root.

*Proof.* Let $v = a(v_k t^k + ... + v_0)$, $u = b(u_k t^k + ... + u_0)$, and $w = w_k t^k + ... + w_0$, then we have $v^2/a^2 + u^2/b^2 = w^2$. Comparing coefficients of the $t$, we have $v_k^2 + u_k^2 = w_k^2$. Since $u_k, v_k$, and $w_k$ are real numbers, then $w_k = 0$ implies $u_k = v_k = 0$, i.e. the degree of $w$ must be the same as the maximal of the degree of $u$ and the degree of $v$. For (2), let us assume that $w = 0$ has a real root $t_0$. By the assumption $gcd(u(t), v(t), w(t)) = 1$, $t_0$ cannot be a root for both $u$ and $v$. We assume that $t_0$ is not a root of $u$. Then when $t$ is near $t_0$ the value of $u/w$ will become infinitely large. But on the other hand, we have $u^2/w^2 = 1 - v^2/w^2$, i.e. $-1 \le u/w \le 1$. This is a contradiction. ∎

As an obvious consequence of Theorem 3.2, we have

**Corollary 3.2.1** We assume the same notations and conditions as Theorem 3.2. Then the degree of $w$ cannot be an odd integer.

We know that there exist quadratic parametric equations for (2.3.1). But the following result shows that all of such parametric equations are not normal.

**Theorem 3.3.** Parameter equations $x = v(t)/w(t), y = u(t)/w(t)$ with degree two for (2.3.1) are not normal.

*Proof.* Let $v(t) = a(a_1t^2 + b_1t + c_1)$, $u(t) = b(a_2t^2 + b_2t + c_2)$, and $w(t) = a_3t^2 + b_3t + c_3$, then we have $v^2/a^2 + u^2/b^2 = w^2$. Comparing coefficients of $t$, we have

$$\begin{aligned}
&a_3^2 - a_2^2 - a_1^2 = 0\\
&a_3b_3 - a_2b_2 - a_1b_1 = 0\\
&2a_3c_3 + b_3^2 - 2a_2c_2 - b_2^2 - 2a_1c_1 - b_1^2 = 0 \qquad (3.3.1)\\
&b_3c_3 - b_2c_2 - b_1c_1 = 0\\
&c_3^2 - c_2^2 - c_1^2 = 0
\end{aligned}$$

Since $a_1$, $a_2$, and $a_3$ are real numbers, the first equation in (3.3.1) implies $-1 \le a_1/a_3 \le 1$ and $-1 \le a_2/a_3 \le 1$. If $a_1a_2 \ne 0$, we shall find a point which is on the ellipse but cannot represented by the parameter equations. Note that $(aa_1/a_3, \pm ba_1/a_3)$ are two distinct points on the ellipse. To get the value of $t$ for which $v(t)/w(t) = aa_1/a_3$, we find a linear equation of $t$. So there only one value for $y$ corresponding to $x = aa_1/a_3$, i.e., one of $(aa_1/a_3, \pm ba_1/a_3)$ cannot be given by the parametric equations. If one of $a_1$ or $a_2$, say $a_2$, is zero, we have $a_3 = \pm a_1$. Without loss of generality, we assume $a_1 = a_3$. The second equation of (3.3.1) is $a_3b_3 - a_2b_2 - a_1b_1 = 0$ from which we get $b_3 = b_1$. Now we know that $x = v(t)/w(t) = a(a_1t^2 + b_1t + c_1)/(a_3t^2 + b_3t + c_3) = a$ has no solution for $t$, then the point $(a, 0)$ which is on the ellipse cannot be given by the parameter equations. ∎

According to Theorem 3.2 and 3.3, the simplest normal parametric equations for an ellipse are of at least degree 4. There actually exist such normal parametric equations. One example is

$$x = \frac{a(t^4 - 4t^2 + 1)}{t^4 + 1}, \quad y = \frac{2\sqrt{2}b(-t^3 + t)}{t^4 + 1}. \qquad (3.4)$$

Using the algorithm based on Theorem 2.6, we can prove the above parametric equations are normal. The inversion maps are determined by the following equation when $x \ne a$

$$\sqrt{2}b(x - a)t^2 - 2ayt + \sqrt{2}b(-x + a) = 0. \qquad (3.5)$$

The value of $t$ corresponding to point $(a,0)$ is zero.

**Case 2.4. Imaginary Ellipse.** If $\delta < 0$ and $(a+c)\Delta > 0$, the standard form is $x^2/a^2 + y^2/b^2 = -1$ which is meaningless in the real number case.

Note that the concept of normal parametric equations we defined in actually in an algebraically closed extension of $K$. But in computer graphics only real points can be displayed. We shall show that the parametric equations of conics found by our method are actually normal in the real number field.

**Definition 3.6.** A set of parametric equations of the form (2.1) is called *normal in the real number field* if there is an irreducible variety $V$ in $K^n$ such that (1) for any $t \in K^m$, if $Q_1(t)...Q_n(t) \neq 0$, $(P_1(t)/Q_1(t), ..., P_n(t)/Q_n(t)) \in V$; and (2) for any $p \in V$ there is a $t \in K^m$ such that $p = (P_1(t)/Q_1(t), ..., P_n(t)/Q_n(t))$.

**Theorem 3.7.** We have a method to find normal parametric equations in real number field for conics.

*Proof.* We shall prove the parametric equations found by the method in this section for conics are normal in the real number field. We only need to show the parametric equations for the standard forms are normal in the real number field. For Case 1 it is trivially true. Case 2.1 is also true, because the inversion map $t = x + \frac{b}{2a}y$ gives real value of $t$ for a real point of the parabola. For Case 2.2, note that $ay - bx$ can not be zero on the hyperbola, then the inversion map $t = \frac{ab}{ay-bx}$ gives real values for all real points in the hyperbola. For Case 2.3, we need to show that (3.5) gives real value of $t$ for all real points of the ellipse. Note that the $t$ value correspondent to point $(a,0)$ is zero. For any real coordinate point $(x_0, y_0)$ which is not $(a,0)$ on the ellipse, the discriminate of (3.5) is $\Delta = 4(a^2y^2 + 2b^2(x-a)^2) > 0$. Then (3.5) always has real root for $t$. We have proved that the parametric equations of the standard forms are all normal in the real number field. ∎

## 4. Parameterization for Conicoids

Let us now consider the parameterization of a conicoid with real coefficients, i.e. a surface of degree two whose equation is

$$F = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2lx + 2my + 2nz + d = 0 \tag{4.1}$$

It is a known result [SV1] that for an irreducible conicoid (4.1), we can find a real coefficients coordinate transformation to transform the (4.1) to one of the following standard forms. Thus, we only need to find parametric equations for these special conicoids.

**Case 4.1. Elliptic paraboloid:** $x^2/a^2 + y^2/b^2 = 2z$.

Taking $x$ and $y$ as parameters, we have normal parameter equations.

**Case 4.2. Hyperbolic paraboloid:** $x^2/a^2 - y^2/b^2 = 2z$.

Taking $x$ and $y$ as parameters, we have normal parameter equations.

**Case 4.3. Cone:** $x^2/a^2 + y^2/b^2 = z^2/c^2$.

We have the following parametric equations

$$x = a(u^2 - v^2),\ y = 2buv,\ z = c(u^2 + v^2)$$

which have been proved to be normal by Theorem 2.6. The inversion maps are

$$u = \pm\sqrt{\frac{acy^2}{2ab^2z - 2b^2cx}},\ v = \frac{y}{abu}$$

**Case 4.4. Ellipsoid:** $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

Use the sterographic projection, we can obtain the following quadratic parametric equations for the ellipsoid

$$x = \frac{2au}{u^2 + v^2 + 1}, y = \frac{2bv}{u^2 + v^2 + 1}, z = \frac{c(u^2 + v^2 - 1)}{u^2 + v^2 + 1}.$$

The missing point is $(0, 0, c)$. The inversion maps are

$$u = \frac{cx}{a(c - z)},\quad v = \frac{cy}{b(c - z)}.$$

**Case 4.5. Hyperboloid with one sheet:** $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$.

we find the following quadratic parametric equations

$$x = \frac{a(u^2 - v^2 + 1)}{u^2 + v^2 - 1}, y = \frac{2bvu}{u^2 + v^2 - 1}, z = \frac{2cu}{u^2 + v^2 - 1}.$$

The missing points are $Zero(z, x^2/a^2 + y^2/b^2 - 1) - Zero(x + a)$. The inversion maps are

$$u = \frac{cx + ac}{az}, v = \frac{cy}{bz}.$$

**Case 4.6. Hyperboloid with two sheets:** $x^2/a^2 + y^2/b^2 - z^2/c^2 = -1$.

We find the following quadratic parametric equations

$$x = \frac{2au}{u^2 + v^2 - 1}, y = \frac{2bv}{u^2 + v^2 - 1}, z = \frac{c(u^2 + v^2 + 1)}{u^2 + v^2 - 1}.$$

The missing point is $(0, 0, c)$. The inversion maps are

$$u = \frac{cx}{az - ac}, v = \frac{cy}{bz - bc}.$$

**Case 4.7. Cylinders:** A conicoid is a cylinder if after an appropriate coordinate transformation, the equation of the conicoid only involves two variables.

We simply assume that $F$ is in two variables $x$ and $y$. Then $F$ can be looked as conics in the $xy$ plane. We can get parametric equations for the cylinder as follows: first use the method in section 3 to obtain parametric equations $x = u(u)/w(u), y = v(u)/w(u)$ for $F(x, y) = 0$, then parametric equations $x = u/w, y = v/w, z = v$ for the cylinder. The parametric equations for the cylinder are normal if and only if $x = u/w, y = v/w$ are normal. Thus, we can find normal parametric equations for cylinders.

**Case 4.7. Imaginary conicoids:** There are other conicoids which are meaningless in the real number case, e.g. the *imaginary elliposoid* $x^2/a^2 + y^2/b^2 + z^2/c^2 = -1$. We do not consider them here.

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