Qualitative Subdivision Algebra: moving towards the Quantitative

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Abstract

Qualitative Reasoning has achieved significant successes in providing abstractions for processes and functionality. However many important questions (e.g modeling shape) are not resolvable at purely qualitative levels, and traditional hybrid models have resorted to quantitative data when it came to these issues. This requires that different sets of data be maintained and updated continuously, and does not provide a mechanism for obtaining the "adequate" level of approximation. An alternative approach for hybrid qualitative-quantitative reasoning is that of subdividing the qualitative regions, where the desired discretization is determined by the needs of the application. We define a class of such subdivision algebras, called *k-proper*, which limit the uncertainty involved in a transitive inference to at most k regions. We construct such algebras for linear and circular domains (−,0,+ and Front/Left/Behind/Right), and construct models for 2D shapes and of 3D positional information with Qualitative Vector Algebra. We use this algebra to build a 3D spatial reasoning system, and a 2D shape modeler.

KEYWORDS: Qualitative Reasoning; Spatial Reasoning; Shape; Approximation; Hybrid Qualitative Reasoning

1 Introduction

Artificial Intelligence has long been concerned with the interaction between abstraction and detail ([4], [14]). At one end of the abstraction spectrum are the traditional numerical approaches, often called quantitative, and at the other end are a number of knowledge representation paradigms, many of which

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claim to be known as "qualitative". One of the key questions confronting AI today is that of finding "adequate" models for representing tasks, and also of transferring information between different levels of granularity adequate for different problem domains.

The question of granularity crops up implicitly in a wide range of problems. Consider, e.g., a robot that needs to open a door based on this instruction:

> "There is a lock on the door in your front, on the upper right side of your hand. The key to the lock is on a shelf built in the wall, to the left of the lock and a little above it."

This is typical of the type of information humans use to communicate spatial information. There are two objects in the scene: the key and the keyhole. The shape of the objects (key, keyhole) need to be known to help locate these. The task is clearly to open the door which may be part of a larger task. This information can be called "qualitative" in the sense that

(a). reference frames are local,

(b). the relative position is specified non-quantitatively,

(c). the description is under-determined, i.e. there may be many configurations in practice that would match this description, and

(d). descriptions are intrinsic to the task.

These are some of the attributes typically associated with qualitative reasoning, hence it is proper to seek a qualitative framework for representing this information. However, as of now, little work has been done to relate 3D-spatial frames in qualitative reasoning, and even less in the matter of modeling shape. Indeed it would be a foolish robot that tried to control all motions based on purely qualitative information. Part of the reason lies in what Forbus, Nielsen and Faltings call the "Poverty conjecture":

> "Without some metric information as to the relative sizes and positions of the parts of a compound surface, the rolling problem cannot be solved. Consider for example two wheels, one with a bump on it and the other with a notch carved out of it. Without more detail, we cannot say how smoothly they will travel across each other: Both perturbations of the shape could be trivial, or the notch might include sharp corners that cause the bump to catch." – ([3])

In other words, too much information is lost in the purely qualitative, non-metric model. If the problem is still useful for single variables or in one dimension, it gets much worse when we enter two, and then three dimensions. The solution proposed traditionally involves a type of "hybrid" model, where quantitative and qualitative data co-exist ([3],[9]). However, this requires that different sets of data be maintained and updated continuously. Moreover, it defeats the objective of trying to find representations that are just "adequate", since the user has to either use a purely qualitative model or a purely quantitative one.

We propose in this work an alternative approach based on hybrid qualitative-quantitative reasoning: that of subdividing the qualitative regions, where the desired discretization is determined by the needs of the application. For example, in a relation like "between A and B" one would be interested in saying "closer to A than to B": clearly this type of query is well-served by a subdivision of the interval between A and B, into two parts "closer-to-A" and "closer-to-B". Gapp [6], for example, investigates cognitive models for the closer-to discretizations in multi-dimensional space. The principal benefit of our model is that the finer-grained subdivisions allow us to talk about issues that were previously not accessible to qualitative reasoners, such as shapes (section 5), and it lets us reason about 3D space with more powerful transitive inference. We start with point-interval algebra: in the qualitative version, a point divides the real-line into three regions. "$-$", "0", and "+". *Qualitative Subdivision Algebra (QSA)* suggests that we subdivide the "$-$" and "+" regions into finer categories, the nature of which it is the purpose of this paper to explore. This type of a discretization also helps another important aspect; it makes explicit a weakness of the traditional ($-$,0,+) formalism: the "0" relation, traditionally taken to be an ideal point, is in practice invariably a toleranced interval $(-\epsilon, \epsilon)$. This results in improper results when adding regions, e.g. "$-$" and "0" can now add up to "0" as well as "$-$" (for a way out in such toleranced models, see [11]).

We first develop QSA and show how some desirable properties can be achieved, e.g. that the results of operations (such as addition and multiplication) be restricted to at most k discretization intervals. Next, we see how this algebra can be used in spatial reasoning to reduce the uncertainties obtained by composing relations in the traditional qualitative algebra. Finally, we also show some results from the modeling of two-dimensional shapes using this algebra, a problem that is entirely beyond the reach of qualitative modelers.

2 Qualitative models and discretizations

The question we must address now is: How to decide on a "good" hybrid discretization? What constitutes some measures of a good vs a bad discretization? Let us look at this issue through a case study.

2.1 Case study: Modeling 2D Angles

Angles are a measure of 2D orientation, and are also important in most models of 3D orientations. Typically, this constitutes a cyclic space, $\theta + 2\pi = \theta$. The discretizations with two orthogonal lines divides up this space into four directions (exact) and four quadrants (qualitative): The lines have three discretizations each: $\{-,0,+\}$, and the plane has a total of nine discretizations, of which one is a point[1], four are exact (the four directions), and four are

[1] Note that the discretization 00 represents a point and not a direction.

qualitative (the quadrants): $+0$ ($0°$), $++$ (quad. I), $0+$ ($90°$), $-+$ (quad. II), -0 ($180°$), $--$ (quad. III), $0-$ ($270°$), and $+-$ (quad. IV). This is the same model as adopted by [3] and several others, but it is by no means the only logical model; for example, [8] uses eight half-quadrants of 45 degrees, and does not model exact directions.

Now, in constructing a hybrid qualitative-quantitative model, we shall be dividing up the non-exact zones into smaller regions. In order to decide on a good modality for doing this, we must first decide on a set of operations that are to be performed in the space. For example, in the angular domain, typically one adds or subtracts angles, therefore the operations may be thought of as the unary negative operation and the binary addition (figure 1).

0	+	-
0	-	+

(a)

	0	+	-
0	0	+	-
+	+	+	?
-	-	?	-

(b)

	0	+	-
0	0	0	0
+	0	+	-
-	0	-	+

(c)

Figure 1: The tables for unary negation (a), addition (b), and multiplication (c) in traditional point-interval algebra.

One of the criteria which can be used to evaluate a hybrid discretization is the extent to which the result of a given operation *spreads* over several discretization intervals. i.e. is the result a single interval (as is possible for negation, in general), or in case spreads over several zones, can the number of such zones be bounded? In section 4.2 we quantify a measure called *k-properness* for this and show that uniform discretization results in exact 2-properness for addition, and that furthermore, discretizations better than 2-proper cannot be constructed for additions in a cyclic space. Thus, for angles, in order to create a hybrid model, we need to choose a suitable resolution $\frac{1}{n}$ and use n equal zones for each qualitative region.

3 Qualitative Subdivision Algebra

An alternate Hybrid Approach

Different hybrid models differ in the degree of coarseness and the uniform/non-uniform character of the zones of discretization. Thus the issue in developing any hybrid model is the discretization of the domain into various zones.

To address this issue more clearly, let us consider the problem of discrete models in general. All such models are a mapping from a continuous domain to a discrete domain. Clearly, no such model can be one-to-one, but certainly, the mapping can be unique in the sense that a given element of the continuous model should correspond to a single element of the discrete model.

"Uniqueness" requires the following property for any discretization:

Property (1a): A discretization should comprise of non-overlapping zones.

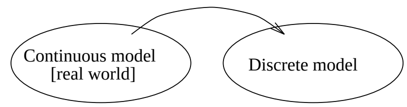

Figure 2: Mapping from a continuous to discrete domain should be unique.

This is necessary for the mapping to be unique. For example, in one dimension, if we have two overlapping zones, $z_1 = [2,5)$ and $z_2 = [3,7)$, then the zone by which the number 4 should be represented is ambiguous. Representing it by the intersection of the two zones would imply redundancy.

In addition to uniqueness, we require that a mapping exist for all points in the continuous domain:

Property (1b): A discretization must span each element of the domain.

We call a discretization that satisfies both the above properties, a *Unique Discretization*. From properties (1a) and (1b), it follows that *the zones of a* unique *discretization* Δ *have common boundaries.* Figure 3 shows an example of a unique discretization of the real line.

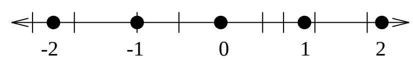

Figure 3: An example of a unique discretization of the real line.

Other desirable properties include that the endpoints form a group over the operations:

Property 2: *Exactness:* A discretization should be closed with respect to its endpoints. In other words, any permissible operation on one or more zones in the model should result in one or more number of zones belonging to the same discretization.

This property is desirable since it ensures that there is no loss of information while using any of the operators (permitted in the model) on Δ. For example, consider the discretization $\Delta = \{ [1,3), [3,5), [5,7), \ldots \}$ of the space of real numbers ≥ 1. If we define the operator $\oplus$ as $[a,b) \oplus [c,d) = [a+b,c+d)$ where $+$ is the addition operator on real numbers, then Δ is not closed w.r.t. $\oplus$, since $[1,3) \oplus [3,5) = [4,8) \not\in \Delta$. Note that representing [4,8) by the union of the zones [3,5), [5,7) and [7,9) would mean a loss of information since by doing so, we are losing the information that the result of the operation ≥ 4 and < 8.

Property 3: *Scaling:* A discretization over an infinite or semi-infinite domain is $well-scaled$ if the length of the interval is proportional to the mean value in the interval.

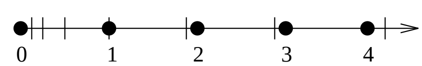

Figure 4: A well-scaled discretization over a linear space.

Definition: $k-Properness$: A discretization Δ on a given domain is $k-Proper$ on a set of operations Σ iff for all operations $\sigma \in \Sigma$ on discretization intervals $\delta_i \in \Delta$, the result of the operation $\sigma(\delta_1, ...\delta_n)$ can be expressed as the union of at most k contiguous intervals in Δ.

Definition: $Exact\ k-Properness$: A discretization Δ is exactly proper on a set of operations Σ if the result of any operation σ is exactly equal to the union of k contiguous intervals.

Definition: $Disjoint\ k-Properness$: A discretization Δ is exactly proper on a set of operations Σ if the result of any operation σ is the the union of at most k intervals, which need not be contiguous.

In general, a k-proper discretization seeks to minimize losses in composition by restricting the results of each operation to at most k intervals. Exact k-properness is desirable in the sense of the previous paragraph; we avoid the information loss associated with partial intervals. Disjoint k-properness is a property that comes up not too frequently, and causes computational problems in identifying the consistency of a set of relations; in this paper however, we restrict ourselves to the simpler cases of contiguous proper operations; the disjoint k-proper relation, which is needed to model operations such as A $\neq$ B, is the subject of ongoing work on the theoretical properties of this algebra.

The following example would make the distinction between exact and inexact properness clearer: consider two discretizations of the space of positive reals greater than 1, i.e., the space [1,∞) - Δ_1 = { [1,2), [2,3), [3,4), [4,5), ... }, and Δ_2 = { [1,3), [3,5), [5,7), [7,9), ... }. Define the operation $\oplus$ on a pair on intervals as: $[i_1, i_2) \oplus [j_1, j_2) = [i_1 + j_1, i_2 + j_2)$, where + is the standard addition operation on reals. Δ_1 is exactly (2-)proper on $\oplus$ whereas Δ_2 is not. This is because for any two intervals, [i,i+1) and [j,j+1) $\in \Delta_1$, [i,i+1) $\oplus$ [j,j+1) = [i+j,i+j+1) $\cup$ [i+j+1,i+j+2) = $z_1 \cup z_2$, where z_1 and z_2 are intervals $\in \Delta_1$. However, in case of Δ_2, for example, [1,3) $\oplus$ [3,5) = [4,8), cannot be exactly equal to the union of any number of intervals in Δ_2 since 4 is not the starting point (as also 8 is not the end point) of any interval $\in \Delta_2$.

Recalling Property 2 in light of the above definition, we can immediately see that it states that "exact"-properness is a desirable property of any discretization.

3.1 Operations on intervals

In the following discussion, the domain of interest is that of real numbers (R). By symmetry, we can simplify the task of discretizing the whole real line to the space of positive reals (R^+). Moreover, from a discretization for [0,1), we can obtain a corresponding discretization for [1,∞), and vice-versa, by inverting the endpoints of each interval (note that while inverting, we keep the intervals 'closed' at the left (smaller) end point and 'open' at the right end point, so that the point "1" is not repeated, and we have an open interval at ∞). Hence, we concentrate on discretizing the domain [1,∞) only. Also, we consider only *unique* discretizations.

We consider two operations on intervals, the Addition operation ($\oplus$) which has already been described above, and the Multiplication ($\otimes$) operation, which is defined as follows: $[i_1, i_2) \otimes [j_1, j_2) = [i_1 * j_1, i_2 * j_2)$, where $*$ is the standard multiplication operation on reals.

Lemma 1: No unique discretization, Δ, of $[1,\infty)$, is exactly *1-Proper* on $\otimes$.
Proof: Let the first interval of Δ be $[1,e)$ $(e > 1)$. Since $[1,e) * [1,e) = [1,e^2) \notin \Delta$ (two discretization intervals do not overlap as per the uniqueness property), the lemma follows. $\square$

Lemma 2: A unique discretization Δ of $[1,\infty)$ is exactly *2-Proper* on $\otimes$ iff the sizes of the intervals are in geometric proportion.
Proof: Let *Delta* be a unique discretization of $[1,\infty)$ which is *2-Proper* on $(\otimes)$. Let the first two intervals of Δ be $[1,e_1)$ and $[e_1,e_2)$. $[1,e_1) * [1,e_1) = [1,e_1^2) = [1,e_1) \cup [e_1,e_1^2)$. Since Δ is *2-Proper*, we must have $[e_1,e_1^2) \in \Delta$. But $[e_1,e_2) \in \Delta$. Hence, $e_2 = e_1^2$ (since Δ is unique). It can easily be shown by induction that Δ must be - $\{ [1,e), [e,e^2), [e_2,e^3), [e_3,e^4), \ldots \}$. The converse, that this discretization is indeed exactly *2-Proper*, is simple. $\square$

Lemma 3: No unique discretization, Δ, of $[1,\infty)$, is exactly *1-Proper* on $\oplus$.
Proof: Let such a Δ exist, and let the first interval of Δ be $[1,e)$. Since this interval may be added to itself any number of times, Δ must contain, in order to be *1-Proper*, all of the following intervals: $[1,e)$, $[2,2e)$, $[3,3e)$, $[4,4e)$, ... Since Δ is unique, $e \leq \frac{n}{n-1}$ $\vee$ n. As $n \to \infty$, $\frac{n}{n-1} \to 1$, hence $e \leq 1$, which is absurd. Hence, the assumption is contradicted. $\square$

Lemma 4: A unique discretization, Δ, of $[1,\infty)$, is exactly *2-Proper* on $\oplus$ iff the set of the end-points of the intervals in Δ is $\{$ i + $k\delta$ $|$ i $\in$ N, $\delta = \frac{1}{n}$ for some n $\in$ N, $0 \leq k \leq$ min(i,n-1) $\}$.
Proof: *'only if':* (proof by contradiction) Consider such a discretization Δ.

Claim: Δ must include the set of points $\{$ n, n + δ, n + 2δ, ... , n + nδ $|$ n $\in$ N, and δ is some real number satisfying $0 < \delta \leq 1\}$. (Note that there might be repetition of points in this list.)
Proof: Let the first interval of Δ be $[1,e)$. Since $[1,e)$ + $[1,e)$ = $[2,2e)$, and Δ is unique, we must have $1 < e \leq 2$. Therefore, let e be $1 + \delta$ for some $0 < \delta \leq 1$. For exact-properness, the addition operation on any two end-points must result in a third end-point belonging to Δ. From this condition, it follows that Δ must (at least) have all points of the form: 1, 2, 3, ... , 1+δ, 2+δ, 3+δ, ... , 2+2δ, 3+2δ, 4+2δ, ... , 3+3δ, 4+3δ, 5+3δ, ... , ... Hence the claim.

Now suppose $\delta = \frac{p}{n}$, where $1 < p < n$, and $\gcd(p,n) = 1$. Let $m = \lfloor \frac{n}{p} \rfloor$. Consider the result of the addition operation on the following two intervals: z_1 = (i, i + $\frac{p}{n}$), and z_2 = (i + $k\frac{p}{n}$, i + $(k+1)\frac{p}{n}$), where $0 < k <$ min(i,m) (This condition on k ensures that z_1 and z_2 are indeed zones $\in \Delta$ in accordance with the above claim). For exact 2-properness, the sum $z_1 \oplus z_2$ = (2i + $k\frac{p}{n}$, i + $(k+2)\frac{p}{n}$), call it s, must be exactly equal to the union of two intervals $\in \Delta$. Now consider an i such that i $> m$ so that min(i,m) = m. Then $0 < k < m$. Choose k to be $(m-1)$. Then $(k+2)\frac{p}{n} = (\lfloor \frac{n}{p} \rfloor + 1)\frac{p}{n} > 1$. Hence, $s \supset P \cup Q$, where $P = (2\text{i} + (m-1)\frac{p}{n}, 2\text{i} + m\frac{p}{n})$ and $Q = (2\text{i} + m\frac{p}{n}, 2\text{i} + 1)$ are zones $\in \Delta$ (by the above claim). This contradicts the assumption that Δ is exactly *2-Proper*.

'if': Classify the intervals in Δ into the following three kinds: (i) $(\text{i}, \text{i} + \frac{1}{n})$, (ii) $(\text{i} + \frac{k}{n}, \text{i} + \frac{k+1}{n})$, and (iii) $(\text{i} + \frac{k}{n}, \text{i} + 1)$. Then, by exhaustively finding the result of the addition operation on each of the six possible kinds of pairs of intervals in Δ, it can be shown that the addition operation is indeed exactly *2-Proper* on Δ. $\square$

Note that although lemma 4 gives states that many different discretizations are possible (each with a different value of δ (or n)), only the discretizations with $\delta = 1/2$ or 1 are useful, since in the rest, the intervals become finer and finer as we move toward ∞, whereas the converse is desirable.

Multiple Operations

Lemma 5: A unique discretization, Δ, of $[1,\infty)$ (or equivalently, of $[1,\infty)$), which is exactly *2-Proper* on both $\otimes$ and $\oplus$ does not exist.

Proof: The proof simply follows from lemmas 2 and 4, which state that such a Δ, if it exists, must either (a) be { [1,2), [2,3), [3,4), [4,5), ... }, with the sizes of intervals in geometric progression, which is impossible, or (b) have 1, 2, 3, ... , $1+\delta$, $2+\delta$, $3+\delta$, ... , $2+2\delta$, $3+2\delta$, $4+2\delta$, ... , $3+3\delta$, $4+3\delta$, $5+3\delta$, ... , ... (where $0 < \delta < 1$) as the end-points of its intervals, which must also be in geometric progression, which is again impossible for any choice of δ. $\square$

3.2 The domain

The domain in which these operations are carried out can be one of the following two types: (a) linear/infinite, and (b) cyclic/finite. By linear spaces we refer to a topology homomorphic to the real line. An example of linear space is "the domain of object dimensions". An example of cyclic space is the angular domain, i.e., the domain of angles on a plane ($\theta + 2\pi = \theta$), or $cos\theta$ where θ can take any real value. Obviously, any discretization of a linear space will have an infinite number of zones (since the continuum model is uncountably infinite, whereas any discretization, by definition, is countable). On the other hand, any discretization of a cyclic space will consist of a finite number of zones.

4 A hybrid model for 3D spatial reasoning

A recent synthesis of spatial reasoning models [15] claims that all spatial models can ultimately be shown to be based on multiple instances of one-dimensional interval logic. However, the loss of information in higher dimensions is so critical that the qualitative information becomes much less useful (the poverty conjecture cited above came up in trying to do spatial reasoning). Thus QSA, which is an approximation model for point-interval logic, can be useful in the spatial reasoning domain, which appears to be subject to very high levels of uncertainty.

4.1 The homogeneous coordinates model

Various qualitative models have been proposed for 3D spatial reasoning. One of the popular ones is based on *homogeneous coordinates* in which a frame of reference is associated with each object. The spatial relation between two objects, represented by a 4×4 matrix, describes the frame of one object w.r.t. that of the other. The numbers in the matrices are replaced by the common $\{-,0,+\}$ discretization. 'Qualitative' multiplication of two values in this discretization results in '0' if either value is '0', '+' if both are either '+' or '−', and '−' if one is '+' and the other '−'. Thus, multiplication 'preserves' information in the sense that the the product is as fine as the finer of the values being multiplied. Addition, however, results in complete uncertainty ('?') whenever we add a '+' and a '−'. Even otherwise, the sum is as coarse as the coarser of the values being added, since the addition of a '0' with '+' results in a '+'. Since the composition of two relations in this model corresponds to matrix multiplication, which involves three scalar additions per element, the resulting relation matrix has a lot of uncertainties (i.e., many elements are '?'s).

To overcome these drawbacks, we propose a hybrid qualitative-quantitative model. Since the elements of the relational matrix are either the direction cosines of the axes, or the (x-, y-, z-) components of the displacement of one frame w.r.t. the other, our task is to discretize the following two domains:

(a). D_c = the range of the function $cos\theta$ (for discretizing the direction cosines of the axes of a frame w.r.t. another frame), which is a *cyclic* space.

(b). D_l = the space of non-negative reals (for discretizing the components of the displacement vector of one frame w.r.t. another) which is a *linear* space, and

The operations that we need to perform on these domains are Addition ($\oplus$) and Multiplication ($\otimes$) (as defined previously).

For D_c, we need to discretize the region [0,1), and for D_l, the space of positive reals, R^+ (since the discns. for (-1,0] and R^- are symmetric). In light of lemma 5, we cannot hope to find *2-proper* discretizations for these domains w.r.t. the operations $\oplus$ and $\otimes$. Hence, we look for suitable *approximations*.

4.2 Discretization of the Cyclic Domain

Most qualitative models have orientations that are indicated by the qualitative vectors (0+, ++, +0 ...) which results in exact alignment at the axes, and a 90° quadrant in between. Here the discretization is in 90° quadrants. In 3D solid angle discretizations, a similar octant solid angle of $4\pi/8$ or $\pi/2$ appears.

In defining this qualitative discretization then, our aim is to discretize the $[0,\pi/2)$ (since this dscretization can be replicated onto each of the other three quadrants) range of angles (θ) in such a way that (a) the angular zones are finer near $\theta = 0°$ and 90°, in conformity with the qualitative discretizations (the four zones and the four directions) used by us, and (b) the discretization, say Δ, implies a corresponding discretization of $cos\theta$ (call it Δ_c) which is a *good* approximation to a *2-Proper* discretization of [0,1) on both $\oplus$ and $\otimes$.

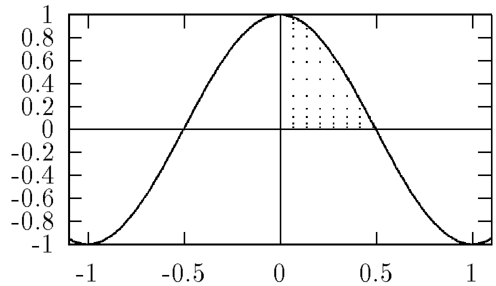

Figure 5: The zones of the discn. Δ_c must be finer near $\theta = 0°$ and $90°$.

Finer angular zones in Δ near $\theta = 0°$ and $\theta = 90°$ necessiate finer intervals in Δ_c near 0 and 1. In addition, we must try to make sure that Δ_c is a *good* approximation. Keeping these requirements in mind, we propose the following discretization of [0,1) as Δ_c:

$$0,\ \frac{1}{sr^l},\ \frac{1}{sr^{l-1}},\ \ldots,\ \frac{1}{sr},\ \frac{1}{s},\ 1-\frac{1}{sr^f},\ 1-\frac{1}{sr^{f+1}},\ \ldots,\ 1-\frac{1}{sr^{m-1}},\ 1-\frac{1}{sr^m},\ 1$$

where $s, r > 1$. The values of s, r and f should be judiciously decided, as described ahead. The values of m and l are chosen depending on the precision or the number of zones wanted for the particular application. In addition, Δ_c has the two precise points, 0 and 1. With proper choice of s, r and f, this discretization is finer near both 0 and 1, and is coarser in the middle. The coarseness increases from either end upto $\frac{1}{s}$. Further, for a majority of pairs of intervals in this discretization, both the operations, $\otimes$ and $\oplus$, yield only two (contiguous) intervals as the resultant.

Δ can then be derived from Δ_c by taking the inverse of the cosine.

4.2.1 Constraints on s, r and f

Below we mention some of the constraints, which if satisfied, result in a "good" approximation to a *2-Proper* discretization of [0,1) on both $\oplus$ and $\otimes$:

$$\begin{aligned} 1-\frac{1}{sr^f} \quad &> \quad \frac{1}{s} \quad \geq \quad 1-\frac{1}{sr^{f-1}} \\ \Rightarrow \quad r^{f-1} \quad &\leq \quad \frac{1}{s-1} \quad < \quad r^f \end{aligned} \qquad (1)$$

$$\begin{aligned} \frac{1}{sr^i}+\frac{1}{sr^{i+1}} \quad &\leq \quad \frac{1}{sr^{i-1}} \qquad \vee\, i \geq 1 \\ \Rightarrow \quad r^2-r-1 \quad &\geq \quad 0 \\ \Rightarrow \quad r \quad &\geq \quad \frac{\sqrt{5}+1}{2} \end{aligned} \qquad (2)$$

$$\begin{aligned} \frac{1}{s} + \frac{1}{sr} &\leq 1 - \frac{1}{sr^f} \\ \Rightarrow \quad (s-1)r^f &\geq r^{f-1} + 1 \end{aligned} \tag{3}$$

Constraints (1) and (2) can be combined together as

$$r^{f-2} + \frac{1}{r} \leq (s-1)r^{f-1} \leq 1$$

Constraint (1) ensures that we do not omit a possible candidate for an end-point between $\frac{1}{s}$ and $1-\frac{1}{sr^f}$. Constraints (2) and (3) ensure that Δ_c is *2-Proper* on $\oplus$ for a fairly large number of interval pairs.

If we choose $s = 2$, then r must be ≥ 4. This is because only then will the intervals of Δ, the corresponding discretization for θ, be such that the last interval, $(1, 1 - \frac{1}{sr^m})$ is smaller than the interval previous to it, $(1 - \frac{1}{sr^m}, 1 - \frac{1}{sr^{m-1}})$, which is essential else the zones for the angular domain won't be progressively finer toward 0°, starting from 60° ($= cos^{-1}(\frac{1}{2})$). On the other hand, a smaller r implies finer intervals and hence, greater precision. Therefore, we choose (with $s =$, $r = 4$). With $s = 2$ and $r = 4$, eqn.(1) $\Rightarrow$ $f = 1$.

4.3 Discretization of the Linear Domain

We have seen earlier how linear space can be divided into three regions based on a point of reference - $\{-, 0, +\}$. This treats 0 as a special point, but gives equal importance to all other points on the number line. However, there might be other points of qualitative interest: for example, while representing relative sizes of objects, the point '1' (signifying "equal" sizes of objects) might be important. Thus, an aspect ratio of 1, representing a square, separates "tall" rectangles from "squat" ones. Also, as per Property 3 (section 3), since the linear space is often infinite, it is desirable that its discretization be well-scaled.

Considering the operations of addition and multiplication, we see that a uniform discretization is 2-proper w.r.t. addition, but is undesirable owing to poor scaling of the discretization. On the other hand, 2-properness property for multiplication guarantees well-scaledness, i.e. if the interval endpoints are in a geometric series, then the discretization is 2-proper and also well-scaled, and is usually recommended for linear spaces (figure 4). However, a problem with such a discretization is that near zero, it has infinite density. Also, it treats only one point, viz. '0', as a special point.

Any discretization of a linear space must satisfy two conditions: (a) the zones should be finer near points of interest (such as '0' or '1'), and (b) the zone size should scale up along with the quantities being modeled. If we invert the discretization Δ_c (defined in the previous section) of [0,1), we indeed get such a discretization for [1,∞). Combining these two discretizations, we get a discretization Δ_l of R^+, which not only satisfies both the above conditions, but is also a good approximation to a 2-Proper discretization on $\oplus$ and $\otimes$.

An algorithm for composing two relational matrices using the above discretizations has been implemented. In the next section, we apply this hybrid model to an example and illustrate the difference in the uncertainty in information between this and the pure qualitative model.

4.4 Example: Robot's task

Consider the problem posed in the introduction. A robot assistant is to locate a key and open a door based on qualitative instructions regarding the pose of the key w.r.t. the lock, and the pose of the lock w.r.t. itself. From these two pairs of spatial relations, viz., the frame of the key w.r.t. that of the lock, F(K,L), and the frame of the lock w.r.t. that of the robot, F(L,R), the robot must infer the frame of the key w.r.t. its own frame, F(L,R) (figure 6). If this knowledge is entirely qualitative, then the information obtained through transitivity is much weaker than through a QSA approach; furthermore, by selecting the resolution of the QSA discretization used, the model can attempt to match the resulting accuracy to functional determinants such as the view-angle of its sensors.

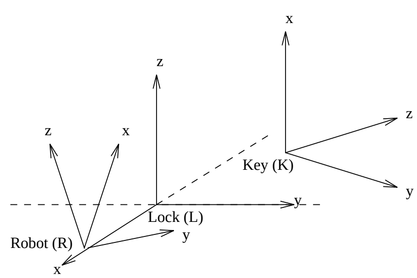

Figure 6: A situation depicting the frames of the robot, the lock and the key.

First we illustrate the qualitative model. Following are the matrices describing the relations F(L,R), F(K,L) and F(K,R) respectively, where F(K,R) is obtained by the matrix multiplication, F(L,R) $\times$ F(K,L),

$$\begin{bmatrix} + & + & + & - \\ - & + & 0 & + \\ - & - & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \quad \begin{bmatrix} 0 & + & - & - \\ 0 & + & + & + \\ + & 0 & 0 & + \\ 0 & 0 & 0 & + \end{bmatrix} \quad \begin{bmatrix} + & + & ? & ? \\ 0 & ? & + & + \\ + & - & ? & ? \\ 0 & 0 & 0 & + \end{bmatrix}$$

Note that there is complete loss of information in five of the twelve (since the entries in the last row are not parameters) elements of the resulting matrix F(K,R). The '?' entries imply that there is an uncertainty of two octants in the orientation of the y-axis and an uncertainty of four octants in the orientation of the z-axis and the position of K w.r.t. the frame of R.

Now suppose that we are provided with more information, so that we can use the hybrid model. Specifically, let the frames F(L,R) and F(K,L) be given as follows: [2]

$$F(L,R) = \begin{bmatrix} 4 & 5 & 6 & -22 \\ -7 & 5 & 1 & 24 \\ 5 & -6 & 6 & 23 \\ 0^* & 0^* & 0^* & 1^* \end{bmatrix} \quad F(K,L) = \begin{bmatrix} 1 & 8 & -4 & -9 \\ 1 & 4 & 8 & 22 \\ 13 & 1 & 1 & 21 \\ 0^* & 0^* & 0^* & 1^* \end{bmatrix}$$

where the numbers denote the zone numbers of the discretization Δ_l mentioned in the previous section. The zone numbers increase from 1 for the interval $(0, 1/(2 \times 4^4)]$ to 13 for $(1 - 1/(2 \times 4^7), 1)$, and from 14 for $(1, \frac{1}{1-1/(2\times4^7)}]$ to 26 for $(512, \infty)$. The negative zone numbers are the counterparts of these for the region $(-\infty, -1) \cup (-1, 0)$. Note that the entries 0^* and 1^*, in the last rows, denote the exact numbers 0 and 1, and are part not of the the discretization but of the homogeneous coordinates model. The hybrid model then gives:

$$F(K,R) = \begin{bmatrix} (5,7) & (3,6) & (1,6) & (-22,22) \\ (-1,2) & (-7,-4) & (4,7) & (24,25) \\ (4,6) & (-13,-3) & (-6,1) & (22,24) \\ 0^* & 0^* & 0^* & 1^* \end{bmatrix}$$

where (i,j) represents union of all zones starting from the i^{th} to the j^{th}.

Observe that in place of the '?' entries in the composed matrix obtained from the qualitative model, we have here, disjunctions of zones. Also, where the qualitative model predicted a '+' (or '−'), the hybrid model predicts the union of some number of zones which spans a much smaller region of the number line than does the '+' (or '−').

In order to judge the relative merit of the two models, we need some metric that characterizes the degree of uncertainty in the results obtained. We define the *uncertainty metric* (UM) [3] as 1/[the extent of the resulting region]. Thus, a '?' in the pure qualitative model would correspond to a UM of 0.5 for the cyclic space (since $cos\theta$ ranges from -1 to $+1$), and 0 for the linear space. An *exact* value (such as 0 or 1) would correspond to a UM of ∞. The larger the UM the less is the uncertainty, and hence the better is the result. Using this metric, we can see that UM_{hybrid} for $\mathrm{F(K,R)}_{22} \approx 1.0667$, for $\mathrm{F(K,R)}_{13} \approx 1.1429$, and for $\mathrm{F(K,R)}_{31} \approx 1.1403$ (where subscript ij denotes i^{th} row and j^{th} column). On the other hand, $UM_{qual} = 0.5$ for any of these. Similarly, for $\mathrm{F(K,R)}_{14}$, $UM_{hybrid} = 0.0625$, and for $\mathrm{F(K,R)}_{34}$, $UM_{hybrid} \approx 0.0079$, whereas $UM_{qual} = 0$ for both of these.

Thus, the hybrid model results in much less uncertainty than the qualitative model, which is only to be expected, as it involves less degree of abstraction and hence, can preserve more information.

[2] One can verify that the zones allow the sum of the squares of the cosines to be 1.

[3] We do not claim that this is the best metric. The issue of deciding on the best metric is in itself an important issue, but is beyond the scope of this paper.

5 Shape models

Consider once again the task outlined in section 1. The robot is required to locate the items mentioned, e.g. the key, or the lock, by imprecise information about their 3D position. This is to be combined with some notion of their 3D geometry. Now, a full-blown geometric model of the average key may contain over a hundred 3D vertex coordinates. Such detailed knowledge may well prove a hindrance at this level of performance, and is certainly not an "adequate" model. On the other hand, using a purely qualitative model, one is unable to infer any shape aspect at all. One of the poverties of qualitative reasoning is its inability to provide abstractions for shape.

On the other hand, even what direction *can* be provided by shape is often underutilized. [12] provides some simple aspect ratio models of shapes, built up by comparing the x and y dimensions of objects. This provides an extremely rudimentary model of shape. More accuracy in shape may be possible by coding vertex information, in which case one faces questions such as "which vertices to omit?", and "what grid size to use for representing vertex coordinates?" Using other boundary based models such as chain code leads to possible non-closure of closed contours etc. Also, other methods of propagating the closure constraint (e.g. distributing the error evenly) result in poor models of shape that lose the essential characteristics of the geometry, such as parallel edges or perpendicularity, sharpness at the corners, etc. This is also true of rectangular decomposition models, such as the spatial occupancy arrays [7]. We feel that a better model for shape abstraction may be based on the *medial axis transform* or the *line-site voronoi diagram* [5]. Figure 7 (b) and (c) highlights this model for a keyhole shape. In the medial axis model, information loss is accommodated by dropping axis segments that correspond to short segment length, shallow changes of angle, and low angles between the forming boundaries (velocities). The angles between successive axes in the diagram, and the lengths of these axis segments as well as radius information along the axes are modeled in the QSA paradigm, by choosing suitable levels of subdivision appropriate to the accuracies one desires to maintain. See [13] for an application of this paradigm in visualizing design shapes at the concept level. Using this model, shape data can be preserved from coarse to fine resolutions, thus providing a range of approximations over which to choose an "adequate" value.

6 Conclusion

In this work, we have developed an alternative paradigm for building hybrid qualitative-quantitative models based on the notion of subdividing the qualitative zones in a model further. The principal advantage of this paradigm over the traditional approach of resorting to purely quantitative data when needed is that it is capable of providing a graded level of approximations, at some point on which an "adequate" model may be determined. This paper presents

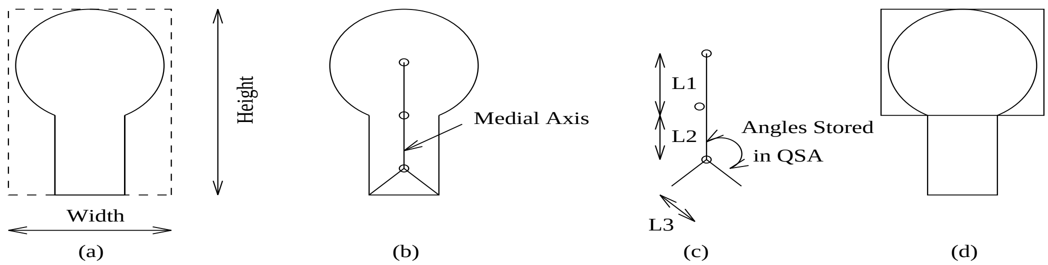

Figure 7: *Shape models for a keyhole.* (a) Simple aspect ratio model. Height > width, so the object is labeled "tall". (b) A medial axis model. The model contains radius information for the maximally inscribed circle. (c) The medial axis model represented in Qualitative Subdivision Algebra. (d) The spatial occupancy array fails to capture the roundness of the hole.

the basic algebra, some theoretical properties of the subdivisions used in the algebra, and some applications from spatial reasoning. The first application reasons about 3D space, where purely qualitative models result in high degree of uncertainty. The second application deals with modeling shape, an aspect that has not been covered in qualitative spatial reasoning. Thus, it appears that the subdivision paradigm holds promise for opening a class of tasks that were previously not accessible from qualitative reasoning.

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