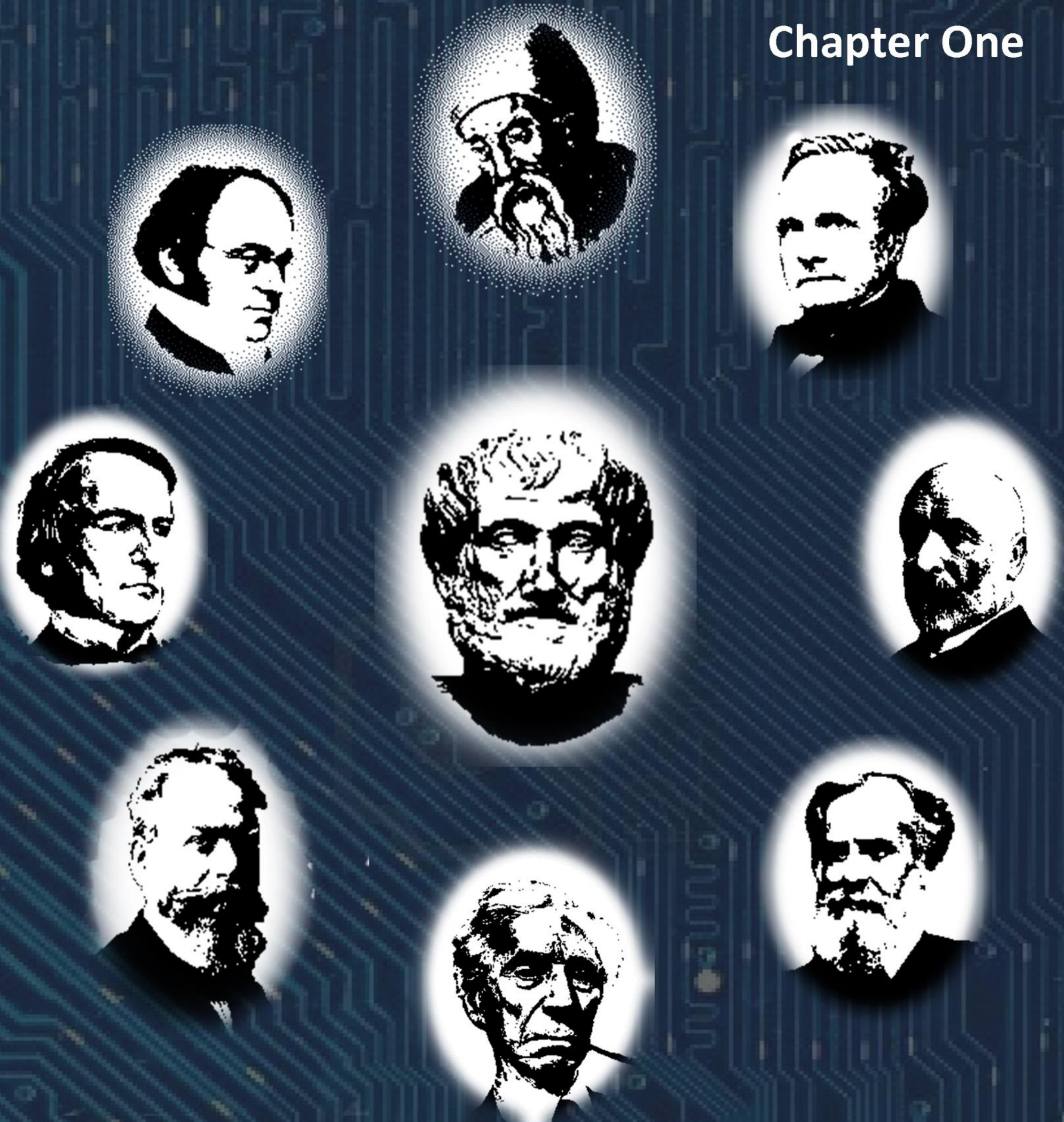
Chapter One



REASONING

elaine rich
alan kaylor cline

The Logicians on our cover are:

Euclid (? - ?)

Augustus De Morgan (1806 – 1871)

Charles Babbage (1791 – 1871)

George Boole (1815 – 1864)

Aristotle (384 BCE – 322 BCE)

George Cantor (1845 – 1918)

Gottlob Frege (1848 – 1925)

John Venn (1834 – 1923)

Bertand Russell (1872 – 1970)

REASONING

AN INTRODUCTION TO LOGIC, SETS, AND FUNCTIONS

CHAPTER 1
INTRODUCTION

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Introduction to Logical Thought

Introduction

Logic as the Gold Standard of Thought

In everyday conversation, we say things like:

> That's logical.

When we say that, we mean that what we've just heard makes sense in some way. It generally means that we accept the conclusion.

We also say things like:

> Where's the logic in that?

When we say that, we mean that we're not convinced by what we've just heard.

In the second episode of season 4 of Downton Abbey, Branson, who was born to a working class family, but married into an aristocratic one, is baffled when some fancy-folks rule is explained to him by the dowager countess. He asks where the logic is in the rules. She replies, "If I were to search for logic, I'd not forage among the English upper class."

We may not have formalized what we mean by the term, "logic", but we know that it is the basis for arguments that convince others and that sensibly serve as the basis for action.

Deduction at its Literary Best

At the heart of logic is ***deduction***: a process for reasoning from premises and observations to conclusions that follow from them.

At least in literature, the gold standard for powers of deduction is ***Sherlock Holmes***. There are, for example, websites that offer to teach us how to develop Sherlock-level deduction capabilities.

"Dr. Watson, Mr. Sherlock Holmes," said Stamford, introducing us.

"How are you?" he said cordially, gripping my hand with a strength for which I should hardly have given him credit. "You have been in Afghanistan, I perceive."

"How on earth did you know that?" I asked in astonishment. ... "You were told, no doubt."

"Nothing of the sort. I knew you came from Afghanistan. From long habit the train of thoughts ran so swiftly through my mind that I arrived at the conclusion without being conscious of intermediate steps. There were such steps, however. The train of reasoning ran, 'Here is a gentleman of a medical type, but with the air of a military man. Clearly an army doctor, then. He has just come from the tropics, for his face is dark, and that is not the natural tint of his skin, for his wrists are fair. He has undergone hardship and sickness, as his haggard face says clearly. His left arm has been injured. He holds it in a stiff and unnatural manner. Where in the tropics could an English army doctor have seen much hardship and got his arm wounded? Clearly in Afghanistan.' The whole ***train of thought*** did not occupy a second. I then remarked that you came from Afghanistan, and you were astonished."

Doyle, Arthur Conan "A Study in Scarlet", in *Sherlock Holmes: The Complete Novels and Stories*: Volumes I and II: Random House, Inc.

A finely crafted website, *http://www.thescienceofdeduction.co.uk/*, for a modern fictional Sherlock advertises:

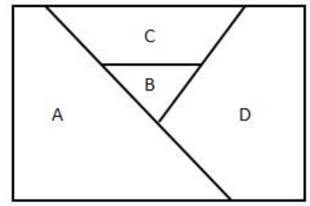
> I'm Sherlock Holmes, the world's only consulting detective.
>
> I'm not going to go into detail about how I do what I do because chances are you wouldn't understand. If you've got a problem that you want me to solve, then contact me. Interesting cases only please. This is what I do:
>
> 1. I observe everything.
> 2. From what I observe, ***I deduce everything.***
> 3. When I've eliminated the impossible, whatever remains, no matter how mad it might seem, must be the truth.
>
> If you need assistance, contact me and we'll discuss its potential.

We can't claim that if you finish this course you'll be Sherlock Holmes. But we hope that you will be a more acute reasoner, both about everyday events and about mathematics.

Logical Doesn't Mean Long or Complicated

Some problems are hard and solutions to them are long and complicated. This can happen even when the problem appears simple.

Consider any plane (flat surface). Divide it into regions. Call this a map. Now color the map's regions, following one rule: Any pair of regions that touch at more than a single point cannot be the same color.



What's the largest number of colors that might be *required* to color a map?

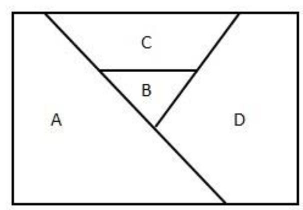
The ***Four Color Theorem*** asserts that it is always possible to color a map in no more than four colors. The theorem is known to have been conjectured as early as 1852. But the first generally accepted proof of it wasn't published until 1977. And it took the help of a computer to construct.

Yet other problems have simple and elegant solutions, even if those solutions aren't immediately obvious.

A ***single elimination tournament*** determines a winner on the basis of matches. In each match there is exactly one winner and one loser. Any team losing a match immediately leaves the tournament. The winner is the single team that remains after all others have lost. (Notice that there is *no* assumption about the structure of the games. There could be seedings, a ladder, a balanced tree, or other structures employed.) Suppose that there are 17 teams. How many matches must be played in order to determine a winner?

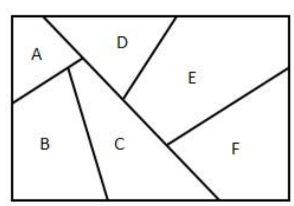
Problems

1. Consider this map:



How many colors are required to color it? ___

2. Consider this map:



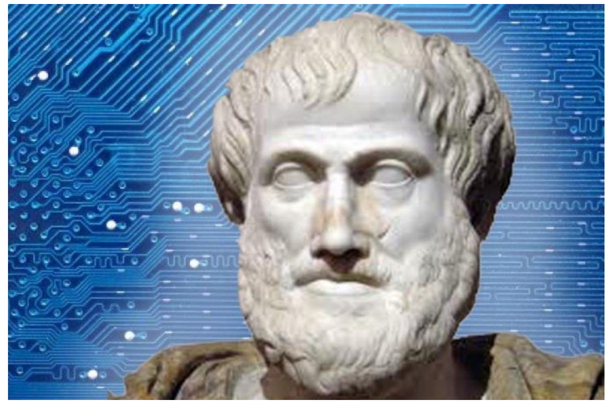
How many colors are required to color it? ___

3. How many matches must be played in a single elimination tournament with 17 teams? ___

Goals

The goal of classical logic, since before Aristotle (Greece, 384 BC - 322 BC), has been to provide a framework within which we make reasoned arguments that lead us to truth.

Since we still care about truth, (particularly in our modern, complex world), one of the goals of this course is to show you how to use this framework effectively.



But we also have some newer problems to solve – ones that Aristotle could never have dreamed of:

- How can we write formal specifications, requirements, and queries that are clear and precise enough that computers can work with them? For example, how can I tell ebay what I'm looking for? Or how can I write my company's personnel policies in a way that lets a program check that every employee record meets my requirements. Or suppose that I have the best idea ever for a phone app. But I can't code. How shall I write the app's specifications so that someone else (a programmer someplace) knows what I want, can write the code, and then can check that her program does what it's supposed to do?

- How can we describe the circuitry of all of our electronic devices, from watches to phones to supercomputers in the cloud?

Over 2,000 years separate the ancient Greek philosophers from modern computer designers and users. What, if anything, do their goals have in common? The answer is that the same logical tools allow us to accomplish all of them.

Those are the tools that we'll study in this course.

Statements and Truth Values

Truth Values

We are going to study two-valued logic. This means that every logical statement is either:

- True, or
- False

Here are some examples of true statements:

[1] 2 + 2 = 4.
[2] The Earth revolves around the sun.
[3] Paris is the capital of France.
[4] It never snows in Austin in August.
[5] The moon is in synchronous rotation with the Earth.
[6] If it's raining, the sidewalks will be wet.

Every second grader agrees that [1] is true. [2] is pretty much universally agreed nowadays to be true, but Galileo [1564 – 1642] spent the last years of his life under house arrest after being tried by the Inquisition for arguing for it. [3] is true. [4] has solid empirical evidence behind it. [5] is also now known to be true. [6] corresponds to an if/then observation that we've all made about the world.

Here are some examples of false statements:

[1] 2 + 2 = 5.
[2] The sun revolves around the Earth.
[3] London is the capital of France.
[4] It never snows in Fairbanks in January.
[5] The moon is made of green cheese.

There's not much controversy about any of these nowadays.

Notice that, to be a statement, a sentence must have a truth value. We don't require that we happen to know what that truth value is.

Here are some examples of statements, each of which is true or false, but I, at least, don't know which:

[1] The social security number of the President of the United States is 224-78-5742.
[2] Every even integer greater than 2 is the sum of two prime numbers.
[3] $P \neq NP$
[4] There was life on Earth 4 billion years ago.

Someone knows whether [1] is true. Just not me and, I'm pretty sure, not you. [2] is called Goldbach's conjecture. It's a famous claim in mathematics. No one knows for sure whether or not it is true. No one has yet been able to prove that it's true or to find a counterexample that shows that it isn't. [3] is a million dollar problem in computer science theory, in which P and NP are well-defined classes of problems. (No kidding – if you can prove either that it is true or that it is false, you'll get $1,000,000.) To learn more about it, google "P=NP". [4] might be true. It's widely acknowledged that there was life on Earth 3.5 billion years ago and some studies suggest that it started more than 4 billion years ago.

Problems

For each of the following claims, indicate what we know about its truth value. (Go ahead, Google if you like.)

(Part 1) There were people in North America 16,000 years ago.

a) It is true and known to be so.
b) It is false and known to be so.
c) It is either true or false but no one (at least that the rest of us is aware of) knows which.

(Part 2). Before World War II, a "computer" was a person.

a) It is true and known to be so.
b) It is false and known to be so.
c) It is either true or false but no one (at least that the rest of us is aware of) knows which.

(Part 3). The Greeks were the only ancient people to study logic.

a) It is true and known to be so.
b) It is false and known to be so.
c) It is either true or false but no one (at least that the rest of us is aware of) knows which.

(Part 4) In the late 1930s, Alan Turing proved that the halting problem is undecidable.

a) It is true and known to be so.
b) It is false and known to be so.
c) It is either true or false but no one (at least that the rest of us is aware of) knows which.

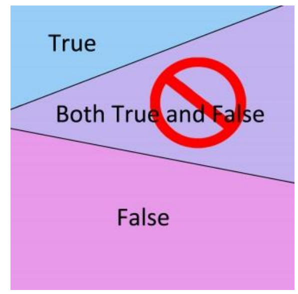
Exactly Two, No More No Less

Recall that we've just said that we'll work with exactly two truth values, True and False. We'll also assume two important laws of classical logic (the logic of our friend Aristotle):

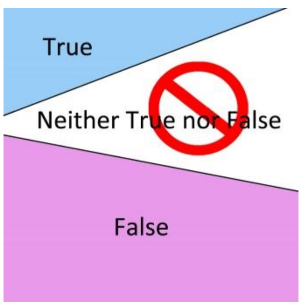
Law of Noncontradiction

No statement (with a given meaning) can be simultaneously true and false.

No double dipping on truth.



The Law of the Excluded Middle:



Every statement must have a truth value: true or false. It can't be neither. Another way to say this is that, for any statement, either it or its negation must be true.

There's no hole in the middle of the truth planet.

There's a reason that, over the last couple of thousand years, most logicians have chosen to work in systems that assume these two laws (although there is some controversy about the second one). They are consistent with the kinds of reasoning that we want to do.

And there's another, more modern reason for us to choose a two-valued logical system. It's easier, cheaper, and more reliable to build computer circuits with two values than with three or more. So the fundamental unit of information is the bit (binary digit). Even quantum computing is based on a two-valued logic. Its fundamental unit of information is the qubit (quantum bit).

Nifty Aside

It is possible (although very rare) to build computers using something other than a two-valued logical system.

The Setun (shown here), built in 1958 at Moscow State University, used ternary (three-valued) logic. So did the TERNAC, built in 1973 in the U.S.

But we should mention that there are logical frameworks that don't accept these laws. For example, constructivist logic doesn't assume the Law of the Excluded Middle.

Nifty Aside

In constructivist logic, the only way to assert that there exists some object that has some interesting property is to exhibit it (or describe a way to construct it). Absent that, the claim that there exists such an object is neither true nor false. And this is so even if we'd be able to derive a contradiction by assuming that no such object exists.

Problems

1. In each of the following problems, assume that it is clear who Riley and Terry are. Indicate whether the truth value of the statement (which could be either true or false) is guaranteed by the Principle of Noncontradiction, the Law of the Excluded Middle, or neither.

a) Riley and Terry are cousins or they are not.
b) Riley and Terry are cousins and they are not.
c) Riley and Terry are cousins or they are brothers.

Statements: The Basic Building Blocks

A logical ***statement*** is an expression that has truth value. In other words, if we assume a particular world, it is either true or false.

> Here are some statements:
>
> [1] The Earth revolves around the sun.
> [2] Paris is the capital of France.
> [3] The moon is made of green cheese.
>
> Let's assume the world we live in. (We can consider the logical possibility of other worlds later.) Then [1] and [2] are true. [3] is false.

But not all English sentences correspond to logical statements.

> Here are some fine English sentences that don't correspond to logical statements:
>
> [4] What does the Earth revolve around?
> [5] Get off my grass!
> [6] Why is grass green?
>
> [4] and [6] are questions. Particular answers to them will be statements, but the questions themselves are neither true nor false. [5] is a command and is neither true nor false.

Typically only declarative statements (not questions or commands) correspond to logical statements.

And recall (from our previous discussion of the bald king of France and John, the possible wife beater) that figuring out what *logical* statement actually corresponds to a particular *English* declarative sentence is not always obvious.

Problems

1. Consider, "All purple unicorns live in Texas." It:

a) is a statement.
b) is not a statement.

2. Consider, "Do you know any purple unicorns?" It:

a) is a statement.
b) is not a statement.

3. Consider, "Paint all the purple unicorns pink." It:

a) is a statement.
b) is not a statement.

English Sentences versus Logical Statements - Nonsense

The relationship between English declarative sentences and logical statements is not always straightforward. We'll have a lot more to say about this in later chapters. But let's just get a glimpse of the problem now so that we don't get confused and worry that, because English is complicated, our logic must also be.

Issue 1: Some English declarative sentences can't be translated into logical statements because they don't make sense (at least without assuming we're in a cartoon or some other non-everyday world).

> [1] A serving of pie has dragon calories.
> [2] The sister of our textbook is walking into class.
> [3] There are no people in 7.
>
> All of these sentences are nonsense because of what we can call type conflicts. In each of them, we appear to be asserting a claim about some property of some object, but the object isn't the kind of thing that can possess the stated property. [1] is nonsense because dragon isn't a number. [2] is nonsense for a couple of reasons, but we'll start with the fact that textbooks don't have biological relatives. [3] is nonsense because 7 isn't a set, so it doesn't make sense to talk about whether or not something is in it.

Notice the difference between being nonsense and being false.

> Consider:
>
> [4] Pencils are fond of chocolate.
>
> [4] is nonsense. If it were simply false, then it's negation would have to be true (by the Law of the Excluded Middle). But that is:
>
> [4'] Pencils are not fond of chocolate.
>
> But this, too, is nonsense.

Compilers for programming languages are very clear about what it means for there to be a type conflict (as opposed to what it means for something to be false).

> In Python (just to pick one example):
>
> | | |
> |---|---|
> | `7 + 1 > 0` | executes and returns the value true |
> | `7 + 1 < 0` | executes and returns the value false |
> | `"abc" + 1 > 0` | produces the red message: `Type Error` |

As we explore the use of our logical language, we'll see that we can avoid writing nonsense by giving clear definitions for all of our terms and then using those terms in ways that are consistent with those definitions.

Problems

1. For each of the following sentences, indicate whether or not there is a sensible interpretation that can be expressed as a logical statement. Don't stretch to think of some metaphorical or imaginary interpretation that makes sense. Note that we are talking about making a sensible claim. Not necessarily a true one. Sensible claims can be false.

a) Spinach is fattening.
b) Pencils are optimistic.
c) You can access the Internet on the moon.

English Sentences versus Logical Statements - Vagueness

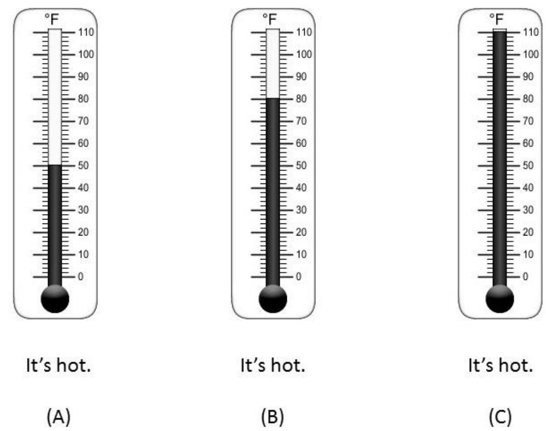
Issue 2: Some English sentences don't seem to have a truth value because they are vague or fuzzy.

Consider the three situations shown here:

There is little disagreement that (A) is false.

There is little disagreement that (C) is true.

But what about (B)? The problem is that our notion of what it means to be "hot" is vague and it may depend on context.



If we want to translate English sentences like (A) – (C) into logical statements, we will first have to provide precise definitions of the predicates (like "hot") that we are going to use.

English Sentences versus Logical Statements - Nonsense

Issue 3: Some seemingly simple English sentences can't easily be translated into logical statements with truth values because they carry unstated assumptions.

Consider:

[4] The king of France is bald.

Is this sentence true or false? It doesn't feel false because there is no king of France with hair. But it doesn't feel true because there is no bald King of France. We could translate this sentence into the following logical claim:

[4′] There is a king of France and that person is bald.

Now we can assign a truth value. This new sentence is clearly false.

English Aside:

It's not uncommon for English sentences to carry what are called presuppositions. A ***presupposition*** is a claim that the speaker/writer assumes is already shared with the audience. So, in the interest of efficient communication, it can simply be skipped. For example, "The king of France is bald," carries the presupposition that there is a king of France. We tend to have difficulty assigning truth values to sentences (like [4]) that carry presuppositions that are false. Logicians and linguists have proposed various ways of solving this problem. One is the technique that we just used (which makes such sentences false). Another is just to say that such English sentences have no truth value.

The bottom line: While logical statements must be either true or false, that's not necessarily so of English sentences. So we'll have to be careful when we translate back and forth between the language of English and the language of logic.

Problems

1. We want to convert each of the following English sentences into a logical statement. But, in each case, there is a problem. Indicate, for each of the sentences, what that problem is.

(Part 1) The daily shuttle between New York and the moon leaves at 8:00 am.

a) The sentence doesn't make sense because there's a type conflict. (In other words, we are trying to assert that some object *X* has some property *P* but things like *X* can't have property *P*.)
b) At least one of the terms used in the sentence is vague. Before we can translate the sentence into a logical statement, we'll have to provide precise definitions of the vague terms.
c) The sentence carries a presupposition that is false. So if we want to convert it to a logical statement with a truth value, we'll have add a clear statement of what that presupposition is.

(Part 2) The square root of Paris is Pygmalion.

a) The sentence doesn't make sense because there's a type conflict. (In other words, we are trying to assert that some object *X* has some property *P* but things like *X* can't have property *P*.)
b) At least one of the terms used in the sentence is vague. Before we can translate the sentence into a logical statement, we'll have to provide precise definitions of the vague terms.
c) The sentence carries a presupposition that is false. So if we want to convert it to a logical statement with a truth value, we'll have add a clear statement of what that presupposition is.

(Part 3) Austin is a big city.

a) The sentence doesn't make sense because there's a type conflict. (In other words, we are trying to assert that some object *X* has some property *P* but things like *X* can't have property *P*.)
b) At least one of the terms used in the sentence is vague. Before we can translate the sentence into a logical statement, we'll have to provide precise definitions of the vague terms.
c) The sentence carries a presupposition that is false. So if we want to convert it to a logical statement with a truth value, we'll have add a clear statement of what that presupposition is.

Paradoxes

Some English declarative sentences don't correspond to *any* logical statement (i.e., an expression with a truth value). The best examples of this are some of the classical ***paradoxes*** (self-contradictory claims).

> Consider the sentence, "This sentence is false." Can we assign it a truth value?
>
> - Suppose we say that it is true. But that cannot be since it would then contradict itself.
> - So it must be false. But then it is telling the truth, which it cannot do since it is false.

The problem in the last example is that the sentence is self-referential. It describes itself.

> Self reference is also the root of the problem in the ***barber paradox***: Imagine a (very) small town with exactly one male barber. And we're told that the barber shaves all and only those men in town who do not shave themselves. Then consider the sentence, "The barber shaves himself." Can we assign it a truth value?
>
> - Suppose we say that it is true. But that cannot be. We've been told that the barber only shaves the men who do not shave themselves.
> - So it must be false. But it cannot be false since, if the barber doesn't shave himself, then we're told that he must do exactly that.

Paradoxes such as these are interesting and have led to more sophisticated systems for reasoning, for example, with sets. But, fun as they are to think about, they're rare. We'll find that, for the most part, English declarative sentences correspond to statements with truth value.

A more serious problem for us will be that not all statements can be straightforwardly and usefully *represented* in the logical frameworks that we're going to study. We'll see why that is as we develop our formal systems.

Problems

1. Consider the following two-part claim:

> "The next statement is true. The previous statement is false."

Which of these sentences is true about our claim:

a) The only way to avoid a contradiction is for it to be true.
b) The only way to avoid a contradiction is for it to be false.
c) There is no way to avoid a contradiction.

2. Try this one only if you like paradoxes. It's called the Grelling–Nelson paradox. Assume that a short word is one with fewer than 8 letters. Anything else is long. We'll say that:

- An adjective is autological ("auto" means "same") just in case it describes itself. For example, "short" is autological since it is a short word.

- An adjective is heterological just in case it does not describe itself. For example, "long" is heterological since it's not a long word.

Consider the sentence, "The word "heterological" is heterological." Is it a statement? In other words, can we assign it a truth value?

a) Yes, it is a statement.
b) No, it isn't a statement.

What Can Logical Statements Represent?

Introduction

We'll use logical statements to represent:

- ***premises***: claims that we will assume to be true before we start to reason,
- ***conclusions***: claims that we want to prove to be true, and
- intermediate results that we may derive along the way from premises to conclusions.

We are going to define two specific logical statement languages. (We'll call one Boolean logic and one Predicate Logic.) While there are limits to what these languages can straightforwardly represent, they are powerful enough to represent many useful kinds of claims.

Claims about the World

When we reason about the everyday world, we don't use exactly the formal process that we are about to study. But logical everyday arguments must have the same structure as the formal arguments we'll be making. And they can do so because claims about our everyday world can be represented as logical statements.

We've already seen some examples of statements that are true in the world.

[1]	The Earth revolves around the sun.
[2]	Paris is the capital of France.
[3]	It never snows in Austin in August.
[4]	The moon is in synchronous rotation with the Earth.
[5]	If it's raining, the sidewalks will be wet.

We've also seen some examples of statements that are false in the world.

[6]	The sun revolves around the Earth.
[7]	London is the capital of France.
[8]	It never snows in Fairbanks in January.
[9]	The moon is made of green cheese.

And we've considered examples of statements whose truth value I, at least, don't know.

[10]	The social security number of the President of the United States is 224-78-5742.
[11]	There was life on Earth 4 billion years ago.

Problems

What is the truth value (in the world in which we live) of each of the following statements:

(Part 1) People born in 2000 are older than people born in 2010.

a) True
b) False
c) People who otherwise seem reasonable disagree on this.

(Part 2) Most professional basketball players are taller than 5′ 10″.

a) True
b) False
c) People who otherwise seem reasonable disagree on this.

(Part 3) The only way to create jobs is to reduce taxes.

a) True.
b) False.
c) People who otherwise seem reasonable disagree on this.

Definitions

Before we can make claims, in English or in logic, we need to define basic sets of terms.

In our arguments, definitions will have the same status as premises. We assert, up front, that they are true. We generally craft our definitions so that they capture properties that we want to reason about. Of course, if two people write different definitions for the same term, they are likely to be able to prove different things about the objects to which the definitions apply.

Mathematicians are particularly careful when they define the terms that they'll use.

> Consider two possible definitions for "prime number":
>
> [1] A ***prime number*** is a <u>positive integer</u> that has, as positive divisors, only itself and 1.
>
> [2] A ***prime number*** is an <u>integer greater than 1</u> that has, as positive divisors, only itself and 1.
>
> Now consider the sentence:
>
> [3] The integer 1 is a prime number.
>
> Clearly [3] is true if we use definition [1]. It is false if we use definition [2].
>
> Note: [2] is the standard definition (and the one that we will use).

> Let's consider the all-important question, "Is a tomato a fruit or a vegetable"? The answer is that it depends on the definitions that we start with.
>
> [1] Biologists' definition: A fruit is a plant part that develops in the ovary of the plant's flower and contains the plant's seeds.
>
> [2] Cooks' definition: A fruit is a fleshy plant part, with seeds, used in sweet (rather than savory) dishes. If used in savory dishes, such plants are called "vegetables".
>
> [3] The U.S. Supreme Court's definition (in Nix v. Hedden, 1893): A tomato is a vegetable.
>
> Now consider the sentence:
>
> [4] A tomato is a fruit.
>
> We see that [4] is true if we use definition [1]. It is false if we use definition [2] or [3].

The key for us is that logic can't resolve definitional issues. We must choose definitions and write them down. Then we can reason with them.

Problems

1. Consider the following proposed definitions:

[1] A food is ***healthy*** if it has no chemical additives.
[2] A food is ***healthy*** if it contains fewer than 10 grams of carbohydrates per serving.
[2] A food is ***healthy*** if it gets fewer than half its calories from fat.
[3] A food is ***healthy*** if it is a fruit or vegetable.

Now consider the claim: A baked potato is healthy.

(Part 1) Using definition [1]:
a) This claim is true.
b) This claim is false.
c) This claim could be either true or false depending on one or more other definitions.

(Part 2) Using definition [2]:
a) This claim is true.
b) This claim is false.
c) This claim could be either true or false depending on one or more other definitions.

(Part 3) Using definition [3]:
a) This claim is true.
b) This claim is false.
c) This claim could be either true or false depending on one or more other definitions.

(Part 4) Using definition [4]:
a) This claim is true.
b) This claim is false.
c) This claim could be either true or false depending on one or more other definitions.

2. Consider the following definitions:

[1] A ***natural number*** is a whole number greater than or equal to 0.
[2] A ***natural number*** is a whole number greater than or equal to 1.

Note: We will use definition [1]. But we should point out that both of these definitions are in fairly common use.

Now consider the claim: The ratio of any two natural numbers is a rational number.

(Part 1) Using definition [1]:

a) This claim is true.
b) This claim is false.

(Part 2) Using definition [2]:

a) This claim is true.
b) This claim is false.

Mathematical Claims

Formal logic is the bedrock of mathematics. Mathematicians use statements to represent:

- Premises (often called axioms in mathematics): claims that serve as the starting point for building a logically consistent theory that we may hope is useful (or maybe it's just elegant).
- Claims that we would like to try to prove.
- Theorems: claims that we can prove must logically follow from the axioms.

Over the centuries, mathematicians have proved many useful theorems.

An example of a theorem that can be proved from the standard axioms of arithmetic:

> If x and y are even integers, so is $x + y$.

And here's one that can be proved from the standard axioms of set theory:

> The maximum number of elements in the intersection of two sets is the number of elements in the smaller of the two original sets.

There still remain, however, claims that can easily be stated but that have yet to be either proved or disproved.

Goldbach's Conjecture remains an unsolved problem:

> Every even integer greater than 2 is the sum of two prime numbers.

Keep in mind that definitions play a key role in determining the truth of mathematical claims such as these.

Consider the claim:

> The product of any two primes cannot be prime.

Recall that we considered two possible definitions for prime number: one of them excluded 1; the other didn't. If we allow 1 as a prime, then our claim is false since $1 \cdot 1 = 1$ (which is, by definition, prime). If, however, we don't allow 1, this claim must be true since any number that is the product of two primes greater than 1 must have both of those numbers as factors (and thus not be prime).

Specifications

Sometimes we want to start with claims that we are determined to *make true*. We call these claims ***specifications***. Specifications play a critical role in all kinds of design tasks, from computer software and hardware to the Internet to corporate databases.

Problem domain experts write specifications for software (and/or hardware) systems. Then other people (engineers with the necessary skills) build systems to meet those specifications. Disasters occur when the specifications are fuzzy or unclear. The use of a precise logical specification language can prevent such disasters. It can also make it possible to prove that a particular implementation does in fact meet the specifications.

So we'll represent specifications as logical statements.

Software Specifications

A University course scheduling program: No student is scheduled for back to back classes in buildings more than 2 blocks apart.

A newsletter mailing program: Before the mailing labels are printed, the addresses are sorted by zip code.

A loan underwriting program for a bank: Loans are issued only to people who have been customers of our bank for at least a year and whose credit score is at least 650.

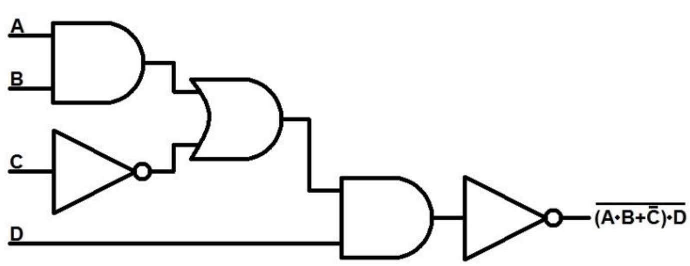
Circuit Design Specifications

Hardware too is first specified, then designed and built.

We might write this specification for a four-input, one-output circuit. It should compute:

NOT ((A AND B) OR NOT C) AND D)

Here's a circuit that meets this specification:



Problems

1. Consider the following procedure:

```
return(average(inputlist))
```

Now consider each of the following specifications. For each, indicate whether our procedure satisfies the specification.

a) Given a list of positive integers, return a positive integer no larger than the largest on the list.
b) Given a list of positive integers, return some integer on the list.
c) Given a list of positive integers, return some even integer on the list.

Claims about Programs and Their Performance

The logical tools that we are about to study are the basis for reasoning about both the correctness and the performance of the programs that we write. So it's critical that we be able to represent correctness and performance claims as logical statements.

Consider the following simple program expressed in pseudocode:

```
def threen(value):
    while value is not equal to 1 do:
        if value is even then set value to value/2
                              else set value to 3 * value + 1
```

The program `threen` takes a positive integer as its input. Call it `value`. The program loops until `value` becomes 1. When that happens, it halts (stops). At each step, if `value` is even, it is divided by 2. If it's odd, we multiply it by 3 and add 1.

It's interesting to watch this program at work, particularly on large numbers. You can do so by going to:

http://www.nitrxgen.net/collatz.php

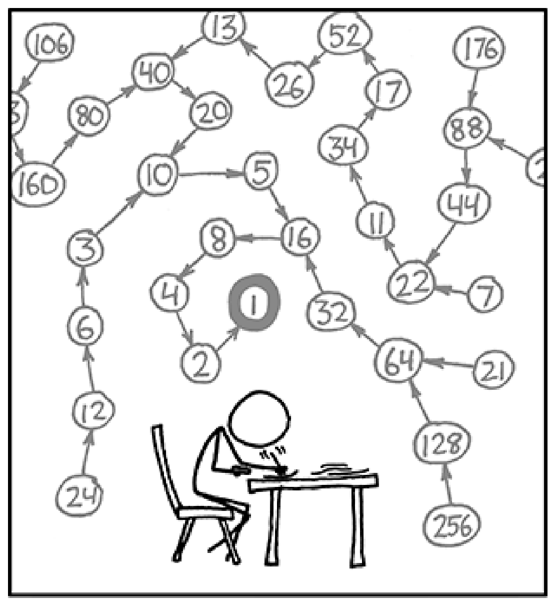
Now consider this claim about `threen`:

[1] Given any positive integer as input, `threen` eventually halts.

Note that this is equivalent to saying that `value` eventually becomes 1.

The claim [1] is called the ***3n+1 conjecture*** or the ***Collatz Conjecture***. It is a logical statement. It is either true or false.

The interesting question is which. It is widely believed that [1] is true. No counterexamples have been found (and many have been tried). But no one has yet proved that `threen` will always halt.



THE COLLATZ CONJECTURE STATES THAT IF YOU PICK A NUMBER, AND IF IT'S EVEN DIVIDE IT BY TWO AND IF IT'S ODD MULTIPLY IT BY THREE AND ADD ONE, AND YOU REPEAT THIS PROCEDURE LONG ENOUGH, EVENTUALLY YOUR FRIENDS WILL STOP CALLING TO SEE IF YOU WANT TO HANG OUT.

Notice that we have just seen another example of a logical statement whose actual truth value we don't happen to know (yet).

Problems

1. Suppose that we invoke threen with the value 78. You can try it by going to:

    http://www.nitrxgen.net/collatz.php

Step 1 will transform 78 into 39. The process will continue from there until it generates 1 and halts. How many total steps will it execute before it halts? ___

2. Consider the following simple program expressed in pseudocode:

```
def greedy(value):
    while value is greater than 0:
        if value is even:
            value = value/2
        else:
            value = value - 1
```

Now consider this claim about greedy:

[1] Given any positive integer as input, greedy eventually halts.

Note that [1] is equivalent to saying that value eventually becomes 0 or less.

Which of the following is true of [1]:

a) It is a true statement.
b) It is a false statement.
c) It is a statement but it's not readily apparent whether it's true or false.
d) It is not a statement.

Database Integrity Constraints

Maintaining the integrity and consistency of data in large databases is hard. People make input errors, for example. So most practical database systems exploit a collection of rules that check to make sure that data have not become corrupted. Like other specifications, these rules are written as statements that must be guaranteed to become and remain true.

Every employee is assigned to at least one project.

Every supervisor supervises at least one employee.

No one may be his/her own supervisor.

Problems

1. Database integrity constraints can serve multiple purposes, including checking for data errors and checking for policy violations.

(Part 1) Assume that our corporate Human Relations (HR) database contains the constraint:

All employees are at least 18 years old.

If an employee record violates this rule:

a) It's almost certainly a data entry error.
b) There has almost certainly been a policy violation.
c) Either a data entry error or a policy violation (or both) could have occurred.

(Part 2) Assume that our corporate (HR) database contains the constraint:

For all employees, hiring date comes after birthdate.

If an employee record violates this rule:

a) It's almost certainly a data entry error.
b) There has almost certainly been a policy violation.
c) Either a data entry error or a policy violation (or both) could have occurred.

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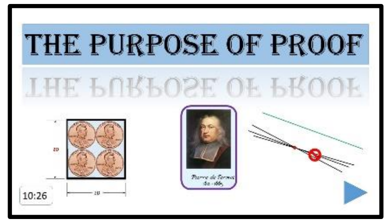
Valid Arguments and Proofs

Introduction

Logic is the study of valid argument (proof) structures (i.e., arguments that preserve truth).

It lets us construct and understand good arguments.



https://www.youtube.com/watch?v=iiCqs6Vi3gw

In his classic novel, *Catch-22*, Joseph Heller clearly describes the Army's rules for getting a mental health discharge. There are two:

[1] To get a discharge, you've got to be crazy and you've got to request the discharge.

[2] If you request a mental health discharge, you're clearly not crazy – any sensible person would make exactly such a request.

From these two rules, we can reason logically and conclude that it's not possible to get a mental health discharge.

Try proving this yourself.

Argument (Proof)

A ***proof*** is an argument that applies one or more:

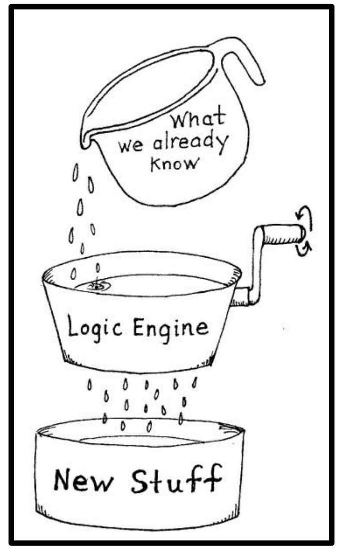
- sound reasoning methods

to a collection of:

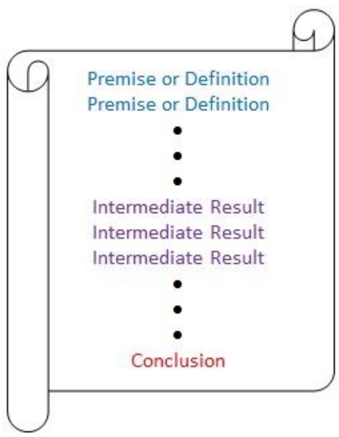
- premises and definitions

to produce a conclusion that must be true whenever the facts are true.

- The reasoning methods are general purpose. They're the major focus of this course.
- The premises and definitions must be chosen so that they model whatever situation(s) we want to reason about.

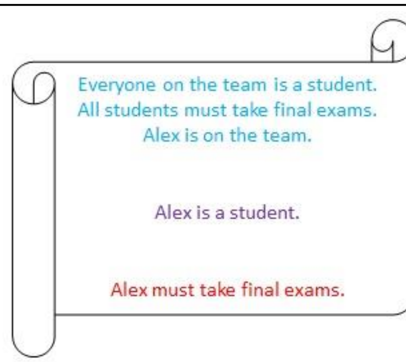


A formal proof typically looks something like this:



Each line of such a proof (other than the premise/definition lines) must follow logically from one or more of the preceding lines.

Here's an example of a simple proof. We reason first that, by virtue of being on the team, Alex is a student. Then, since all students must take final exams, Alex must do so.



In this class, we will spend some time practicing writing formal proofs in this way.

But often, both in everyday reasoning and in mathematics, we write arguments with a less rigid structure. That's okay, as long as it's clear what the premises/definitions are and it's still true that every other claim we make is justified by the application of a sound logical rule to one or more claims that are known already to be true.

Here's a possibly more natural way to state the argument about Alex:

Alex must take final exams. Since he's on the team he must be a student and all students have to take the exams.

Notice that, unlike in the formal structure, we actually wrote the conclusion first. That's common. But don't be fooled. The reasoning is the same. The conclusion follows from the premises.

Big Idea

The tools of logic are premise-neutral. They can be stated in a general way and then applied to any premises we choose. Their job is to preserve truth. They will allow us to derive only claims that must be true whenever all the premises are true.

Problems

1. Consider the following argument:

> Jean is the President of the club. She'll be at the party on Saturday. Everyone in the club is coming.

(Part 1) "Jean is the President of the club," is a:

a) Premise
b) Intermediate Result/Conclusion that could reasonably be derived from the other claims.

(Part 2) "Jean will be at the party on Saturday," is a:

a) Premise
b) Intermediate Result/Conclusion that could reasonably be derived from the other claims.

(Part 3) "Everyone in the club is coming," is a:

a) Premise
b) Intermediate Result/Conclusion that could reasonably be derived from the other claims.

(Part 4) Which of the following is a correct description of our argument:

a) It is correct (i. e., it uses only the given premises and reasons correctly to the conclusion.)
b) It isn't quite right, but it would be if we made explicit one or more premises that are common knowledge.
c) It isn't correct and it can't easily be fixed by adding obvious premises.

Unstated Premises

A logical argument only makes sense with respect to a particular set of premises. However, in stating real arguments (and proofs), we often omit explicit mention of premises that we can assume everyone accepts. Brevity requires this.

Consider:

> Stacey can't be at the beach. I just saw her in her office.

Most of us have no trouble accepting this argument. It does, however, leave out a key premise (without which it isn't valid): It's not possible for someone to be in two different places at once. Thus being in the office precludes being at the beach.

When we write proofs in mathematics, we typically omit explicit mention, as premises, of such things as the facts about standard arithmetic and algebra.

Prove that, for any real x: $x^2 + 1 > 0$.

Proof:

[1]	$x^2 \geq 0$	Since the square of any real number is nonnegative
[2]	$1 > 0$	
[3]	$x^2 + 1 > 0$	Adding [1] and [2]

Notice that we haven't justified our addition of [1] and [2].

But we must be very careful when we rely on unstated premises:

- In the real world, it is possible (and, in fact, happens all the time) that things that seem obvious to me may not be accepted as premises by everyone else.

- In mathematics, it is possible to build very different (but interesting) theories by starting with different sets of premises. So, while few theories are built without the standard rules of algebra, there are, for example, competing theories of geometry.

Consider this dialogue:

A: Your friend Kris will love you for life if you get her Cool Dragon tickets for her birthday.

B: You've got to be kidding. Kris would hate Cool Dragon.

A's argument makes sense to him if he starts with the premise that, of course, everyone loves Cool Dragon. But *B* doesn't share that assumption.

We could have replaced the Cool Dragon example with just about any discussion of modern politics. Of course, sometimes people are irrational. But often, if you examine political disagreements, you'll see that the various people involved have started with different sets of premises.

Problems

1. Consider the following dialogue:

A: We need to vote for Smith for mayor.
B: No. We absolutely have to vote for Jones.
A: But Jones supports the new downtown development plan.

Let's assume that both *A* and *B* are making reasonable logical arguments. But they don't share the same set of premises. Mark each of the following as True if it helps to explain the fact that *A* and *B* have come to different conclusions. Mark False otherwise.

a) *B*, but not *A*, assumes that the development plan will create jobs. ________
b) *B*, but not *A*, assumes that the development plan will cost too much. ________
c) *B*, but not *A*, assumes that the development plan will disturb some wildlife habitats. ________
d) *A*, but not *B*, assumes that the development plan will disturb some wildlife habitats. ________
e) *A*, but not *B*, assumes that the development plan was crafted without citizen input. ________

Other Proof Structures

Some proofs look somewhat different from the ones that we have just described.

For example, some claims can be proved or disproved with just a single example.

Define the function $n!$ (pronounced, "n factorial"), on the positive integers, as:

[1] $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$

For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

Now consider the following claim: For any positive integer, $n!$ is even.

We can prove that this claim is false by exhibiting a single value that contradicts it. So our proof is:

$1! = 1$, which is odd (thus not even).

Sometimes we prove that a claim is true by showing that its negation must be false.

Definition: An integer n is even if and only if it is equal to $2 \cdot k$, for some integer k.
Definition: An integer n is odd if and only if it is equal to $(2 \cdot k) + 1$, for some integer k.

Assume that we have already proved (and thus we can take this as a premise) that every integer is either even or odd, but not both.

Now suppose that we want to prove:

[1] For all integers n, if n^2 is even, then n is also even.

We start by assuming that n^2 is even. We want to prove that, therefore, n is also even. Assume, to the contrary, that n is not even. In other words, it is odd and there exists some integer k such that:

[2] $n = 2k + 1$

Squaring both sides, we have:

[3] $n^2 = (2k + 1)(2k + 1)$

Doing some simple algebra, we have:

[4] $$\begin{aligned} n^2 &= 4k^2 + 2k + 2k + 1 \\ &= 4k^2 + 4k + 1 \\ &= 2 \cdot (2k^2 + 2k) + 1 \end{aligned}$$

But an integer is odd just in case it is 1 more than some integer that is divisible by 2. The quantity $2 \cdot (2k^2 + 2k)$ is divisible by 2. So we have that n^2 is odd. But that contradicts our starting premise that n^2 was even. Thus the assumption that n is odd (i.e., not even) must be false. So n is even.

But note that these proofs still possess the structure that is required of all correct proofs: They begin with premises (possibly not stated explicitly, such as the rules of algebra), apply sound inference rules, and finally derive their conclusion.

Problems

1. Consider the claim that there exists some President of the United States who was born in Virginia. Which of the following is true:

a) The claim is true and it can be proved by exhibiting a single example.
b) The claim is false and can be proved false by exhibiting a single example.
c) The claim is true but it's not possible to prove it with a single example.
d) The claim is false but it's not possible to prove it false with a single example.

2. Consider the claim:

For any positive integer greater than 1, *n*! is even.

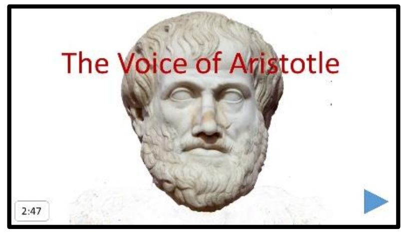
Which of the following is true:

a) The claim is true and it can be proved by exhibiting a single example.
b) The claim is false and can be proved false by exhibiting a single example.
c) The claim is true but it's not possible to prove it with a single example.
d) The claim is false but it's not possible to prove it false with a single example.

Bad Arguments

We hope that, by the end of this class, you'll be able not just to construct good arguments but also to identify bad ones, which come in many flavors. We'll mention just a few of them.

https://www.youtube.com/watch?v=sSMqjmah0nA



Existential vs. Universal Confusion

The claim that there exist some things that possess some particular property is very different from the claim that all things possess that property.

E: Some boys sure are stupid.
A: Wait, are you calling me stupid?

A has jumped to an unjustified conclusion. Notice that *E* has said only that *some* boys are stupid. That's very different from saying that all of them (including *A*) are.

Necessary vs. Sufficient Conditions Confusion

If we claim that P implies Q, we are saying that P is a sufficient condition for Q: P being true is sufficient to guarantee that Q is true, regardless of anything else. If we say that Q implies P, we are saying that P is a necessary condition for Q: Q cannot be true if P isn't. These two claims are different and it's possible for one to be true while the other is false. So when we reason with such claims, we must be careful.

A: I really hate big cities.
E: So we'll have a great time. Oshkosh is really small.

A has said that if a city is big, he'll hate it. He has not said that if he hates a place, it's necessarily big. It's perfectly possible for him also to hate a smaller place.

Running a Proof Backwards

A valid proof must start with premises and reason to conclusions. Sometimes, in *searching* for a proof, we start with our conclusion and look for things that would lead to it. But, in the end, we can't start with the thing we're trying to prove. If we assume some claim P, of course (trivially) we have a proof of it. But that tells us nothing.

Assume the following premises:

[1] If it's raining, the sidewalks are wet.
[2] If the sidewalks are wet, someone will slip and fall.
[3] Jayce just slipped and fell.

Here's a "proof" that it's raining:

If it were raining, the sidewalks would be wet. Then someone would slip and fall. Jayce did just slip and fall. So it must be raining.

Our proof is flawed. We could correctly reason that if it were raining, someone would slip and fall. But that's not what we're trying to do. As we saw above in the big city example, we need different claims if we want to reason in the other direction (from slipping to wet sidewalks and from wet sidewalks to rain). And, in the real world, it's simply not true that slipping implies that the sidewalks are wet. And wet sidewalks don't necessarily imply rain. (Maybe the sprinklers just went off.)

Problems

1. Mark True if *B*'s conclusion logically follows from *A*'s claims. Mark False if *A*'s claims are inadequate to support *B*'s conclusion.

A: It's always hot in the summer.
B: Everyone in this picture is wearing shorts, so it must have been taken in the summer.

2. Mark True if *B*'s conclusion logically follows from *A*'s claims. Mark False if *A*'s claims are inadequate to support *B*'s conclusion.

A: Some really crazy people go to Alaska in January.
B: I guess Jo is crazy because she went to Alaska over New Years last year.

3. Mark True if *B*'s conclusion logically follows from *A*'s claims. Mark False if *A*'s claims are inadequate to support *B*'s conclusion.

A: Anyone who plays tennis in August is crazy.
B: Well, since I play tennis every day, I guess I must be crazy.

Remember the Critical Role of the Premises

The tools that we are about to describe, powerful as they are, cannot pull truth out of thin air. What they are designed to do is to *preserve* truth. If we begin with true statements and then reason logically, we will conclude with statements that must also be true.

But if we start with junk, we can easily create more junk.

Suppose that we start with the following premises:

[1] If there's a ladder that reaches from *A* to *B*, then it is possible to go from *A* to *B* by climbing the ladder one rung at a time.
[2] The Fountain of Youth is on the moon.
[3] There is a ladder from Austin to the moon.
[4] There are people in Austin.

Then there is a perfectly logical argument that it is possible for someone to reach the Fountain of Youth.

Unfortunately, that doesn't appear to accord with the facts. The problem is that premises [2] and [3] haven't been chosen well if we want to describe the world in which we live.

Logic can't insulate us from bad premise choices and the junk that can result.

Sometimes though it's not a question of junk. It's simply the case that different premises make sense in different situations. Because this can happen, it's important that we clearly articulate all of our possibly controversial premises.

Suppose that I want to reason about going to the beach. I'll take [1] and [2] as premises:

[1] People like going to the beach when it's warm, not when it's cold.
[2] The beaches will be crowded at times that people like to go.

Then I can argue: The beaches will be crowded in August because people like to go then.

My argument makes sense to me. I live in North America. It's summer (and thus warm) in August. But our friends Down Under may laugh at me. Their beaches are crowded in December. The issue is that I have one additional premise (as yet unstated), while they have another:

[3-mine] It's warm in June, July and August.
[3-theirs] It's warm in December, January and February.

So I can prove that the beaches will be crowded in August. They can prove that the beaches will be crowded in January.

When we want to reason about the real world, we typically choose premises that correspond to our observations and that lead us to conclusions that also accord with the facts. So there's some notion of reasonable premises and unreasonable ones.

In mathematics, however, there is no single "reasonable" set of premises. Each set of premises leads to a theory: a collection of provable claims. Some theories may be more "interesting" than others. But it can easily happen that two theories that start with competing premises can both be useful, although in different contexts.

Probably the most well-known example of this is in geometry.

In plane geometry, we generally accept all five of Euclid's postulates (premises). The fifth of these is often called the ***parallel postulate***:

> Given a line L_1 and a point P not on L_1, there is exactly one line L_2 that is parallel to L_1 and contains P.

Using all five of the postulates, we can prove claims such as, "Any two lines, if they intersect at all, must intersect at exactly one point." Thus we get claims that correspond to what we observe on a plane.

But now consider geometry on a sphere (interesting because, for example, we can approximately model the Earth as a sphere). Now our observations are different. We want a theory that predicts them. So spherical geometry starts with a different notion of a line. Rather than straight lines, we define a line as the shortest distance (on the surface of the sphere) between two points. And we begin with different premises. So, for example, the sum of the interior angles of a triangle is not 180°.

Premise switching also plays a key role in many kinds of fiction. For example, in alternative history, it is assumed that one thing (or maybe a small number of things) happened differently. From then on, everything that happens follows the usual rules of causality and event sequencing. But outcomes can be completely different. Fantasy worlds and games do the same thing. Logic stays the same. Just a few premises change.

Problems

1. Consider the following argument, which will begin with four premises, [1] – [4]. It will conclude [5]:

> [1] Morgan is in Pflugerville without a car.
> [2] Morgan has a class at UT in an hour.
> [3] Morgan can walk 20 miles in an hour.
> [4] The distance between UT and Pflugerville is 15 miles.
>
> [5] Morgan can get to class on time.

The logic of this argument is good but the conclusion is crazy. There's one premise that is in blatant contradiction to the facts of the world in which we live. We should scrap it if we want to make it impossible to conclude [5] while still letting us reason sensibly about the world. Which premise should we scrap?

a) Premise [1]
b) Premise [2]
c) Premise [3]
d) Premise [4]

2. Consider the following "proof" that 2 = 0. We'll take [1] – [5] as premises.

[1] For any numbers a, b, and c, if $a = b$ and $b = c$ then $a = c$.
[2] For any numbers a and b, if $a < b$ is false, then $a = b$.
[3] For any numbers a and b, if $a > b$, then $a < b$ is false.
[4] For any number a, $a + 1 > a$.
[5] For any number a, $a = a$.

Since $2 = 1 + 1$, $2 > 1$. So it's false that $2 < 1$. Then $2 = 1$. By a similar argument, since $1 = 0 + 1$, $1 > 0$. So it's false that $1 < 0$. So $1 = 0$. By premise [1] we then have that $2 = 0$.

The logic in this proof is sound. But one of our premises has made it possible to conclude nonsense. Which one:

a) Premise [1]
b) Premise [2]
c) Premise [3]
d) Premise [4]
e) Premise [5]