The problem of the maximum length of an ascending subsequence.

We consider a sequence of N elements A(1) through A(N) . The order of increasing subscript value will be denoted by "the order from left to right". From such a sequence we can take so-called "subsequences of length s " by the removal of an arbitrary collection of N-s elements and retaining the remaining s elements in the order in which they occurred in the original sequence. When, in addition, each element has an integer value, we call a subsequence "ascending" iff it contains no element with a right-hand neighbour smaller than itself.

Note. According to this definition, all N subsequences of length 1 and even the empty subsequence are ascending ones. (End of note.)

We look for an algorithm that determines for any given sequence (with $N > 0$) the maximum length of an ascending subsequence that can be taken from it.

Note. Although there need not be a unique longest ascending subsequence, the maximum length is unique, e.g. 3 1 1 2 4 3 gives 4 for the maximum length, realized either by 1 1 2 4 or by 1 1 2 3 . (End of note.)

If k represents the value we are looking for, we seek to establish the relation

R: k = the maximum length of an ascending subsequence taken from A(1) through A(N) .

Because R contains the parameter N , it is strongly suggested to take as invariant relation --or, as we shall see in a moment: as part of the invariant relation--

P1(k, n): k = the maximum length of an ascending subsequence taken from A(1) through A(n) .

It has the virtues that it would do the job in the sense that (P1(k, n) and n = N) => R and is easily established, e.g. by k, n := 1, 1 . These observations suggest to establish P1(k, n) for n = 1 and then to increase n under invariance of P1(k, n) until n = N , more precisely: to increase n repeatedly by 1 and to restore each time, when destroyed, the truth of P1(k, n) by adjusting the value of k . Because extension with a next element can never decrease the maximum length of an ascending subsequence and can increase it by at most 1 , the adjustment of k , when needed, will have the form k:= k + 1 . More precisely: because

$$P1(k, n) = wp(\text{"n:= n + 1"}, P1(k, n - 1))$$

we have to investigate after "n:= n + 1" under which circumstances no adjustment of k is needed, i.e. when

$$P1(k, n - 1) \Rightarrow P1(k, n) \quad , \qquad (1)$$

and under what circumstances adjustment of k is needed, i.e. when

$$P1(k, n - 1) \Rightarrow P1(k + 1, n) \quad . \qquad (2)$$

Relation (2) holds iff A(n) can be used to extend an ascending subsequence of maximum length (= k) taken from A(1) through A(n - 1) ; this is true iff

A(n) $\geq$ the smallest rightmost value of an ascending subsequence of length k taken from A(1) through A(n - 1) .

This last inequality shows us, that besides k -as defined by P1(k, n)-- we would also like to store the minimum rightmost value --let us call it m for a moment-- of an ascending subsequence of maximal length. If (2) holds, k is obviously to be adjusted by k:= k + 1 , and the assignment m:= A(n) would make m again equal to the minimum rightmost value of an ascending subsequence of maximal length (because, in this case, <u>all</u> ascending subsequences of maximal length taken from A(1) through A(n) will have A(n) as their rightmost element.)

The introduction of m as the minimum value of the rightmost value of an ascending subsequence of length k , presents, however, a problem in case (1). In that case, the extension with A(n) , although not leading to an increase of k , may require adjustment of m as it may lead to a decrease of the minimum rightmost value of an ascending subsequence of that unchanged maximal length. This would be the case if the value A(n) --which now satisfies A(n) $<$ m -- could be used to extend an ascending subsequence of length k - 1 , taken from A(1) through A(n - 1) . In order to decide that, we would also need the minimum rightmost value of an ascending subsequence of length k - 1 . Repeating the argument, we conclude that, instead of a scalar m , we need in addition to k an array variable m satisfying

P2(k, n, m): for all j satisfying $1 \leq j \leq k$
m(j) = the minimum rightmost value of an ascending subsequence of length j and taken from A(1) through A(n) .

Our total invariant relation will be P1 <u>and</u> P2 .

Again, for $n = 1$, P2 is easily initialized --with $m(1) = A(1)$--; we have to investigate, however, what updating obligations for the array variable m are implied by our duty to keep P2 invariant. The crucial discovery in the analysis of our updating obligations for the array variable m is that the elements of m itself are ascending, more precisely:

$$(1 \leq i < j \leq k) \Rightarrow (m(i) \leq m(j)) \quad .$$

This follows from the fact that $1 \leq i < j \leq k$ <u>and</u> $m(i) > m(j)$ leads to a contradiction: by removing from an ascending sequence of length j and with $m(j)$ as its rightmost value the leftmost $j-i$ elements, an ascending sequence of length i with $m(j)$ as rightmost value remains, and $m(i) > m(j)$ then contradicts P2 .

Again we investigate the situation as reached after $n:= n + 1$, i.e. when $P1(k, n - 1)$ <u>and</u> $P2(k, n - 1, m)$ holds. Relation (2) holds iff $A(n) \geq m($ The new element $A(n)$ can be used to form a longer ascending sequence, k has to be increased and the sequence of values is extended with $A(n)$ by

m:hiext(A(n)) ;

it is correct to leave the values $m(i)$ with $1 \leq i < k$ unchanged, for the new element $A(n) \geq m(k)$ and can never give rise to a smaller rightmost value for any of the ascending sequences shorter than the new maximum length k . Relation (1) holds iff $A(n) < m(k)$. Remembering that after the increase $n:= n + 1$ the relation $P2(k, n - 1, m)$ holds, we have to answer the question for which value(s) of j is the minimum rightmost value of an ascending sequence of length j take from $A(1)$ through $A(n)$ smaller than taken from $A(1)$ through $A(n-1)$? This can only be the case if $A(n)$ is its new rightmost element, which must be smaller than its old value $m(j)$. So we have

$$A(n) < m(j) \qquad (3)$$

But $A(n)$ can only be the rightmost element of an ascending sequence of length j if

either $j = 1$ or $j > 1$ <u>and</u> $m(j-1) \leq A(n)$ (4)

Combining (3) and (4) we find

$j = 1$ iff $A(n) < m(1)$ and otherwise $j =$ the only(!) solution of $m(j-1) \leq A(n) < m(j)$. This last solution is found in the program with a binary search; the invariant relation for the inner loop is $m(i) \leq A(n) < m(j)$.

Observing that k = m.hib , the current higher bound for the index, we can use for m(k) = m(m.hib) the usual abbrevation m.high and conclude that we don't need the variable k after all. For reasons of symmetry we denote m(1) by m.low , as m.lob = 1 . Omitting all declarations we get the following program.

```
n:= 1; m:= (1, A(1));
do n ≠ N →
      n:= n + 1;
      if A(n) ≥ m.high →
            m:hiext(A(n))
       [] A(n) < m.high →
            if m.low > A(n) →
                  j:= 1
             [] m.low ≤ A(n) →
                  i, j := m.lob, m.hib;
                  do i ≠ j - 1 →
                        h:= (i + j)div 2;
                        if m(h) ≤ A(n) → i:= h
                         [] A(n) < m(h) → j:= h
                        fi
                  od
            fi;
            m:(j)= A(n)
      fi
od;
print(m.hib)
```

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