A supplement on EWD591 "The problem of the maximum length of an ascending subsequence."

In the algorithm described in EWD591 occur the array elements $m(1)$ through $m(k)$ where k = the maximum length of an ascending subsequence taken from $A(1)$ through $A(n)$. In the computational step that increases n by 1 , either k is increased by 1 and the m-sequence is extended by a value (= the new $A(n)$, to be precise), or for some value j $(1 \leq j \leq k)$ $m(j)$, that was larger than the new $A(n)$ is made equal to the new $A(n)$. In short: after the adjustment the new $A(n)$ occurs in the m-sequence and existing m-values never increase.

Suppose that in parallel we determine h = the maximum length of a descending subsequence taken from $A(1)$ through $A(n)$. That computation would comprise a corresponding array $p(1)$ through $p(h)$, that after each adjustment would contain the new $A(n)$ and whose elements never decrease. If, for a given i $(1 \leq i \leq h)$ and j $(1 \leq j \leq k)$ we have $m(j) \leq p(i)$ we call this "an inversion"; because $m(j)$ never increases and $p(i)$ never decreases, an inversion, once introduced, remains in existence.

If we mentally extend the m-sequence at the high end with values = + infinity, and extend the p-sequence at the high end with values = - infinity, each step effectively decreases an m-value by making it equal to the new $A(n)$ and effectively increases an p-value by making it equal to the new $A(n)$. Hence each step increases the total number of inversions by at least one, and we conclude that the total number of inversions $\geq$ the length n of the sequence. Furthermore $h*k \geq$ the total number of inversions, hence

$$h * k \geq n \quad .$$

From this we deduce that for $n \geq N^2 + 1$, we have $h > N$ or $k > N$, and that was the theorem we wanted to prove: a sequence of length greater than N^2 has an ascending or descending subsequence of length greater than N .

Plataanstraat 5
5671 AL Nuenen
The Netherlands

prof.dr.Edsger W.Dijkstra
Burroughs Research Fellow