

On W.H.J. Feijen's solution for the lexicographic minimum of a circular list.

For a given, non-empty - i.e. with $N \geq 1$ - list

$$(C[0], C[1], \dots, C[N-1])$$

we consider the N associated sequences C_i

$$C_i = (C[i], C[i+1], \dots, C[i+N-1])$$

in which all subscripts are reduced modulo N .

We are asked to design a program determining an integer value k satisfying the relation

$$R: \quad 0 \leq k < N \quad \text{and} \quad (\forall i: 0 \leq i < N: C_i \not\prec C_k) \quad ,$$

in which " \prec " should be read as "lexicographically greater than".

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The lexicographic minimum CC is defined by the following two relations

$$(\forall i: 0 \leq i < N: C_i \not\prec CC) \quad (0)$$

$$(\exists i: 0 \leq i < N: C_i = CC) \quad (1)$$

From the transitivity of the lexicographic ordering we conclude

$$(\forall p, q: C_p \not\prec C_q \quad \text{or} \quad C_q \prec CC) \quad (3)$$

Note. When, in a quantification, the range is obvious, we allow ourselves its omission. Hence the " :: ". (End of Note.) *

Let $P(k, n)$ be defined as:

$$P: 0 \leq k < n \text{ and } (\forall i: 0 \leq i < n: C_i \succ CC \text{ or } i = k) \quad (4)$$

We observe immediately

$$(P \text{ and } n \geq N) \Rightarrow R \quad (5)$$

$$(\forall i: 0 \leq i < k: C_i \succ CC) \Rightarrow P(k, k+1) \quad (6)$$

Let $Q(k, n, l)$ be defined as

$$Q: 0 \leq k < n \wedge l \geq 0 \wedge (\forall i: 0 \leq i \leq l: C[k+i] = C[n+i]) \quad (7)$$

Because the lexicographic order of two sequences with equal leading elements does not change when those leading elements are removed from them, we conclude

$$Q(k, n, l) \Rightarrow (Qa \text{ or } Qb \text{ or } Qc) \quad (8)$$

with the mutually exclusive terms

$$Qa: (\forall i: 0 \leq i < N: C_{k+i} = C_{n+i}) \quad (8a)$$

$$Qb: (\forall i: 0 \leq i \leq l: C_{k+i} < C_{n+i}) \wedge l \geq 0 \quad (8b)$$

$$Qc: (\forall i: 0 \leq i \leq l: C_{k+i} \succ C_{n+i}) \wedge l \geq 0 \quad (8c).$$

We furthermore observe that, for $l=0$, the last term of $Q(k, n, l)$ is vacuously true.

We observe that $P \wedge Q$ can be initialized by
 $k, n, \ell := 0, 1, 0$
 and is left invariant by the following three guarded
 commands.

$$1) \quad C[k+\ell] = C[n+\ell] \rightarrow \ell := \ell + 1$$

This follows immediately from (4) — ℓ does not
 occur in P — and (7)

$$2) \quad C[k+\ell] < C[n+\ell] \rightarrow \{P \wedge Q_b\} \quad n, \ell := n+\ell+1, 0$$

Q and the guard imply together Q_b , and
 note that Q_b implies

$$(\forall i: n \leq i < n+\ell+1: C_i \geq CC).$$

Hence, on account of (4), the invariance of P
 is guaranteed. The assignment $\ell := 0$ ensures the
 invariance of Q

$$3) \quad C[k+\ell] > C[n+\ell] \rightarrow \{P \wedge Q_c\} \\
 h := \max(n, k+\ell+1); \\
 k, n, \ell := h, h+1, 0$$

Q and the guard imply together Q_c , and
 note that Q_c implies

$$(\forall i: k \leq i < k+\ell+1: C_i \geq CC) \wedge \ell \geq 0.$$

With $h = \max(n, k+\ell+1)$ we conclude from the
 last result and P

$$(\forall i: 0 \leq i < h: C_i \geq CC)$$

and, thanks to (6), the invariance of P is
 guaranteed. The assignment $\ell := 0$ ensures
 the invariance of Q

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Finally we show that

$$(P \wedge Q \wedge k+l+1 \geq N) \Rightarrow R \quad (9)$$

$$1) P \wedge Qa \wedge k+l+1 \geq N$$

From Qa we conclude that the sequence is periodic, and that the period divides $n-k$; hence

$$(\exists i: k \leq i < n: C_i = CC)$$

Together with - what follows from P -

$$(\forall i: k < i < n: C_i \neq CC)$$

we conclude $C_k = CC$, i.e. R .

$$2) P \wedge Qb \wedge k+l+1 \geq N$$

From Qb we conclude, as before,

$$(\forall i: n \leq i < n+l+1: C_i \neq CC);$$

because $n > k$, $n+l+1 > k+l+1$; because $k+l+1 \geq N$

we conclude, together with P

$$(\forall i: 0 \leq i < N: C_i \neq CC \text{ or } i = k), \text{ i.e. } R$$

$$3) P \wedge Qc \wedge k+l+1 \geq N$$

From Qc we conclude, as before

$$(\forall i: k \leq i < k+l+1: C_i \neq CC);$$

because $k+l+1 \geq N$, we conclude together with P

$$(\forall i: 0 \leq i < N: C_i \neq CC);$$

this, together with (1) implies false, and false $\Rightarrow R$.

(End of Proof of (9).)

Using (5) and (9), we see that the following program establishes R

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k, n, l := 0, 1, 0 {P ∧ Q};
do n < N and k + l + 1 < N →
  if C[k+l] = C[n+l] → l := l + 1 {P ∧ Q}
  [] C[k+l] < C[n+l] → n, l := n + l + 1, 0 {P ∧ Q}
  [] C[k+l] > C[n+l] → h := max(n, k + l + 1);
                        k, n, l := h, h + 1, 0 {P ∧ Q}
  fi
od {R}

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Termination. The function $t = k + n + l$ increases by at least one at each iteration. Before the first iteration $t = 1$, before the last one $t \leq 2N - 3$, hence $2N - 3$ is an upper bound for the number of comparisons.

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The above is - or will be - described much more beautifully by W.H.J. Feyen, including all the heuristics and acknowledgements. I wrote the above as part of my personal correspondence, for the sake of quick dissemination; I owe everything of the above to ir. W.H.J. Feyen.

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