$|x_n| = x_{n-1} + x_{n+1}$ has period 9 .

This morning I heard the at first sight surprising theorem that the sequence of real numbers x_n $(-\infty \leq n \leq +\infty)$ such that $|x_n| = x_{n-1} + x_{n+1}$ has a period of length 9. Here is my proof.

Since $x_{n-1} + x_{n+1} \geq 0$, there exists an element x_i such that $x_i \geq 0$. Since $x_{i-1} + x_{i+1} \geq 0$, $x_{i-1} \geq 0 \vee x_{i+1} \geq 0$, i.e. the sequence contains to successive elements p and q, such that $p \geq 0 \wedge q \geq 0$. Without loss of generality we may choose $p \leq q$. But this means that the sequence contains 3 consecutive nonnegative elements $p \quad q \quad q-p$ or, after renaming $p \quad p+r \quad r$ for nonnegative p and r. Let $r \leq p$ and let us extend the sequence in the direction of r — since the relation for x_n is symmetric in x_{n-1} and x_{n+1}, the direction of indexing is irrelevant —.

$$
\begin{array}{lll}
x_0 & = p & (x_0 \geq 0) \\
x_1 & = p+r & (x_1 \geq 0) \\
x_2 & = r & (x_2 \geq 0) \\
x_3 & = -p & (x_3 \leq 0) \\
x_4 & = p-r & (x_4 \geq 0) \\
x_5 & = 2\cdot p - r & (x_5 \geq 0) \\
x_6 & = p & (x_6 \geq 0) \\
x_7 & = r-p & (x_7 \leq 0) \\
x_8 & = -r & (x_8 \leq 0) \\
x_9 & = p & (x_9 \geq 0) \\
x_{10} & = p+r & (x_{10} \geq 0)
\end{array}
$$

With $x_0 = x_9$ and $x_1 = x_{10}$, the theorem has been proved without case analysis!

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