# A sequel to EWD874

Lemma 0 of EWD874 "A miscellany of results" can be strengthened and its proof can be simplified.

<u>Lemma 0</u> For monotonic $f$ and any natural $n$, the equations

$$X: [(\underline{E}\, i: 0 \le i: f^{i+1} X) \equiv X] \qquad (0)$$

$$X: [\, f^{n+1} X \equiv X] \qquad (1)$$

$$X: [(\underline{A}\, i: 0 \le i: f^{i+1} X) \equiv X] \qquad (2)$$

have the same extreme solutions.

<u>Proof</u> Since the conjugate of $f$ is monotonic as well, it suffices to give the proof for the strongest solutions.

Let $P$, $Q$, and $R$ be the strongest solutions of (0), (1), and (2) respectively. Since the left-hand sides of (0), (1), and (2) are strengthening in that order, we have — see Lemma 1, below —

$$[R \Rightarrow Q] \ \wedge\ [Q \Rightarrow P] \quad .$$

The proof is concluded by demonstrating $[P \Rightarrow R]$ as follows:

$\quad$ true

$= \{$ monotonicity of $f\}$

$\quad [\, f(\underline{A}\, i :: f^i R) \Rightarrow (\underline{A}\, i :: f(f^i R))]$

$= \{$ pred.calc, definition of functional iteration and renaming the left dummy$\}$

$$[f(R \wedge (\underline{A} i :: f^{i+1} R)) \Rightarrow (\underline{A} i :: f^{i+1} R)]$$
$= \{R$ is a solution of (2) and pred.calc.$\}$
$$[fR \Rightarrow R]$$
$= \{$monotonicity of $f$ and induction over $i\}$
$$(\underline{A} i :: [f^{i+1} R \Rightarrow R])$$
$= \{$pred. calc.$\}$
$$[(\underline{E} i :: f^{i+1} R) \Rightarrow R]$$
$\Rightarrow \{P$ is the strongest solution of (0) and Knaster-T.$\}$
$$[P \Rightarrow R] \quad .$$

(End of Proof.)

<u>Lemma 1</u> Let $f0$ and $f1$ be two monotonic predicate transformers such that

$$[f0\, X \Rightarrow f1\, X] \quad \text{for all } X \quad . \qquad (3)$$

Let the equations

$$X: [f0\, X \equiv X] \quad \text{and} \quad X: [f1\, X \equiv X]$$

have $G0$ and $G1$ as their strongest, and $H0$ and $H1$ as their weakest solutions. Then we have

$$[G0 \Rightarrow G1] \quad \text{and} \quad [H0 \Rightarrow H1] \quad .$$

<u>Proof</u>

true
$= \{$definition of $G1\}$
$[f1\, G1 \equiv G1]$
$\Rightarrow \{(3)\}$
$[f0\, G1 \Rightarrow G1]$
$\Rightarrow \{$def. of $G0$ and K.-T.$\}$
$[G0 \Rightarrow G1]$

true
$= \{$def. of $H0\}$
$[H0 \equiv f0\, H0]$
$\Rightarrow \{(3)\}$
$[H0 \Rightarrow f1\, H0]$
$\Rightarrow \{$def. of $H1$ and K.-T.$\}$
$[H0 \Rightarrow H1]$

(End of Proof.)

<u>Remark</u> Consider for any nonempty bag V of natural numbers the predicates

$$(\underline{E}i{:}\ i\ \underline{in}\ V{:}\ f^{i+1}X) \quad \text{and} \quad (\underline{A}i{:}\ i\ \underline{in}\ V{:}\ f^{i+1}X) \quad .$$

Bag V being nonempty, these predicates are at least as strong as the left-hand side of (0) and at least as weak as the left-hand side of (2). From Lemma 0 and Lemma 1 we now conclude that

$$X{:}\ [(\underline{E}i{:}\ i\ \underline{in}\ V{:}\ f^{i+1}X) \equiv X] \qquad \text{and}$$
$$X{:}\ [(\underline{A}i{:}\ i\ \underline{in}\ V{:}\ f^{i+1}X) \equiv X]$$

have the same extreme solutions as (0) and (2).

(End of Remark.)

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