Well-founded sets revisited

In the following

$(C,<)$ is a nonempty partially ordered set,

x, y are elements of C,

S is a subset of C,

P, Q are predicates on C,

where S and P are coupled by

(0) $\quad Px \equiv \neg\, x \in S$, or $S = \{x \mid \neg Px\}$.

As a result we have

(1) $\quad S = \emptyset \equiv (\underline{A}x :: Px)$.

"x is a minimal element of S" means

(2) $\quad x \in S \wedge (\underline{A}y : y < x : \neg\, y \in S)$.

"C is well-founded" means "any nonempty subset of C contains a minimal element".

<u>Theorem 0</u> "C is well-founded" $\equiv$ "mathematical induction over C is valid".

<u>Proof</u>

"C is well-founded"

$= \{$definitions of well-foundedness and minimal element$\}$

$(\underline{A}S :: S \neq \emptyset \equiv$
$\quad (\underline{E}x :: x \in S \wedge (\underline{A}y : y < x : \neg\, y \in S)))$

$= \{$predicate calculus, de Morgan in particular$\}$

$(\underline{A} S :: S = \emptyset \equiv$
$\quad (\underline{A} x :: \neg\, x \in S \ \vee (\underline{E} y : y < x : y \in S)))$

$= \{$renaming the dummy with (0) and (1)$\}$

$(\underline{A} P :: (\underline{A} x :: P x) \equiv$
$\quad (\underline{A} x :: P x \ \vee (\underline{E} y : y < x : \neg P y)))$

$= \{$definition of mathematical induction$\}$

"mathematical induction over C is valid".

(End of Proof.)

Theorem 1 "C is well-founded" $\equiv (\underline{A} x :: Q x)$, where Q is defined as the strongest solution of

(3) $\quad Q : (\underline{A} x :: Q x \ \vee (\underline{E} y : y < x : \neg Q y))$.

Note. The above Q is usually interpreted as:
$Qx \equiv$ "each descending chain starting with x is of finite length".
The following proof does not appeal to this interpretation. (End of Note.)

Proof Rewriting (3) as

$Q : (\underline{A} x :: (\underline{A} y : y < x : Q y) \Rightarrow Q x)$

and recognizing that the antecedent is a monotonic function of Q, we conclude —Knaster-Tarski— that (3) has a strongest solution.

"C is well-founded"

$\Rightarrow \{$Theorem 0$\}$

$(\underline{A}x :: Qx) \equiv (\underline{A}x :: Qx \vee (\underline{E}y : y < x : \neg Qy))$
$= \{Q \text{ is a solution of } (3)\}$
$(\underline{A}x :: Qx)$.

"C is not well-founded"
$\Rightarrow$ {definition of well-foundedness, (2); for some S}
$S \neq \emptyset \wedge (\underline{A}x :: \neg x \in S \vee (\underline{E}y : y < x : y \in S))$
$=$ {for P, as defined by (0), and (1)}
$\neg(\underline{A}x :: Px) \wedge (\underline{A}x :: Px \vee (\underline{E}y : y < x : \neg Py))$
$=$ {(3)}
$\neg(\underline{A}x :: Px) \wedge$ "P solves (3)"
$\Rightarrow$ {Q is the strongest solution of (3)}
$\neg(\underline{A}x :: Qx)$

(End of Proof.)

The above proof was designed in the presence of W.M.Turski and here recorded under supervision of A.J.M. van Gasteren.

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