Some simple theorems on incremental sorting

A standard component of many in situ sorting algorithms is the operation ord , given by

$$\text{ord}.x.y \;=\; \underline{\text{if}}\; x>y \rightarrow x,y := y,x \;[\!]\; x \leq y \rightarrow \text{skip}\; \underline{\text{fi}}\; \{x \leq y\},$$

i.e. swapping two values if and only if they are out of order or, in other words, removing an inversion if present. I shall represent the operation ord.x.y by

$$x \dashrightarrow y \quad ,$$

i.e. a dotted arrow from x to y .

We are interested in other pairs, whose being in order is not destroyed by a collection of ord operations; if, with $u \leq v$, (u,v) is such a pair, we shall denote it by

$$u \longrightarrow v \quad ,$$

i.e. a drawn arrow from u to v .

In the following pictorial representation of some theorems there is no aliasing , i.e. different nodes of the graph correspond to different variables.

<u>Lemma 0</u> $\quad u \rightarrow v \quad x \dashrightarrow y$.

<u>Proof 0</u> The ord operations involve x and y only, in particular u and v are not involved and their order is maintained. (End of Proof 0.)

<u>Lemma 1</u>

$$\begin{tikzcd} u \arrow[r] & y \\ x \arrow[r] \arrow[ur, dashed] & v \end{tikzcd} \quad .$$

<u>Proof 1</u> Operation ord.x.y neither increases x, nor decreases y; from the constancy of u and v the conclusion now follows. (End of Proof 1).

<u>Lemma 2</u>

$$\begin{tikzcd} a \arrow[r, dashed] & A \\ b \arrow[u] \arrow[r, dashed] & B \arrow[u] \end{tikzcd}$$
.

<u>Proof 2</u> Initially, the maximum is an element of {a, A}, hence A finally equals the maximum and hence B → A is not destroyed; similarly, the minimum is initially an element of {b, B} and hence finally equal to b, and therefore b → a is not destroyed (End of Proof 2)

Note that it is essential that both ord.a.A and ord.b.B are executed. Lemma 2 is the most interesting of all. The single arrows b → a and B → A can be extended to arbitrary congruent directed graphs, provided ord.x.X is executed between each pair of corresponding variables.

<u>Lemma 3</u>

$$\begin{tikzcd} \circ \arrow[r, dashed] & \circ \\ b \arrow[u] \arrow[ur] & \end{tikzcd} \quad \& \quad \begin{tikzcd} a & \\ \circ \arrow[u] \arrow[r, dashed] & \circ \arrow[ul] \end{tikzcd}$$
.

<u>Proof 3</u> The value of b is the minimum & the value of a is the maximum. (End of Proof 3.)

As special consequence we mention

<u>Lemma 4</u>

$$\begin{tikzcd} \circ \arrow[r, dashed] & \circ \\ \circ \arrow[u] \arrow[ur] \arrow[r, dashed] & \circ \arrow[u] \arrow[ul] \end{tikzcd}$$
. and

Lemma 5

$$\begin{tikzcd} 2 \arrow[r] & 3 \\ 0 \arrow[u] \arrow[r] & 1 \arrow[u] \arrow[ul, dashed] \end{tikzcd}$$

From Lemmata 2 and 5 it follows that 4 elements can be sorted by a sequence of 5 ord's:

ord.0.2 & ord.1.3 ; ord.0.1 & ord.2.3 ; ord.1.2 .

As $2^4 < 4! < 2^5$, 5 ord's is indeed the minimum.

So much for my simple theorems. Needless to say that my feelings about their pictorial representation are very mixed.

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