

Some simple theorems on incremental sorting

A standard component of many in situ sorting algorithms is the operation ord , given by

$$\text{ord}.x.y = \text{if } x > y \rightarrow x, y := y, x \quad \square \quad x \leq y \rightarrow \text{skip} \quad \text{fi } \{x \leq y\},$$

i.e. swapping two values if and only if they are out of order or, in other words, removing an inversion if present. I shall represent the operation $\text{ord}.x.y$ by

$$x \dashrightarrow y,$$

i.e. a dotted arrow from x to y .

We are interested in other pairs, whose being in order is not destroyed by a collection of ord operations; if, with $u \leq v$, (u, v) is such a pair, we shall denote it by

$$u \rightarrow v,$$

i.e. a drawn arrow from u to v .

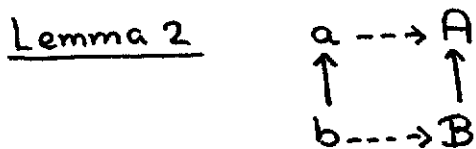
In the following pictorial representation of some theorems there is no aliasing, i.e. different nodes of the graph correspond to different variables.

Lemma 0 $u \rightarrow v \quad x \dashrightarrow y$.

Proof 0 The ord operations involve x and y only, in particular u and v are not involved and their order is maintained. (End of Proof 0.)

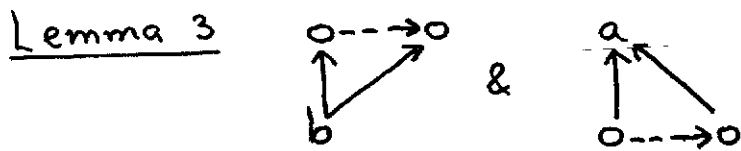
Lemma 1 $u \rightarrow y$
 $x \dashrightarrow v$

Proof 1 Operation $\text{ord}.x.y$ neither increases x , nor decreases y ; from the constancy of u and v the conclusion now follows. (End of Proof 1).



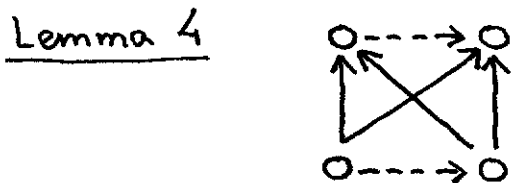
Proof 2 Initially, the maximum is an element of $\{a, A\}$, hence A finally equals the maximum and hence $B \rightarrow A$ is not destroyed; similarly, the minimum is initially an element of $\{b, B\}$ and hence finally equal to b , and therefore $b \rightarrow a$ is not destroyed (End of Proof 2)

Note that it is essential that both $\text{ord}.a.A$ and $\text{ord}.b.B$ are executed. Lemma 2 is the most interesting of all. The single arrows $b \rightarrow a$ and $B \rightarrow A$ can be extended to arbitrary congruent directed graphs, provided $\text{ord}.x.X$ is executed between each pair of corresponding variables.

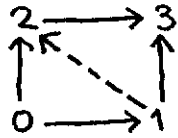


Proof 3 The value of b is the minimum & the value of a is the maximum. (End of Proof 3.)

As special consequence we mention



and

Lemma 5

From Lemmata 2 and 5 it follows that 4 elements can be sorted by a sequence of 5 ord's:

ord.0.2 & ord.1.3 ; ord.0.1 & ord.2.3 ; ord.1.2 .

As $2^4 < 4! < 2^5$, 5 ord's is indeed the minimum.

So much for my simple theorems. Needless to say that my feelings about their pictorial representation are very mixed.

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prof. dr. Edsger W. Dijkstra
 Department of Computer Sciences
 The University of Texas at Austin
 Austin, TX 78712-1188
 United States of America