A short note on universal quantification

With the dummy x ranging over some anonymous range and using

(0) $[(\underline{A}x::p.x) \wedge (\underline{A}x::q.x) \equiv (\underline{A}x::p.x \wedge q.x)]$

we shall show

(1) $[(\underline{A}x::p.x \equiv q.x) \Rightarrow ((\underline{A}x::p.x) \equiv (\underline{A}x::q.x))]$

(2) $[(\underline{A}x::p.x \Rightarrow q.x) \Rightarrow ((\underline{A}x::p.x) \Rightarrow (\underline{A}x::q.x))]$.

From propositional calculus we shall use

(3) $[X \Rightarrow (P \equiv Q) \equiv X \wedge P \equiv X \wedge Q]$

(4) $[(P \equiv Q) \wedge P \equiv P \wedge Q]$

Proof of (3)
$X \Rightarrow (P \equiv Q)$
= {pred. calc}
$\neg X \vee (\neg P \equiv \neg Q)$
= {distribution}
$\neg X \vee \neg P \equiv \neg X \vee \neg Q$
= {de Morgan}
$X \wedge P \equiv X \wedge Q$. (End of Proof)

Proof of (4)
$(P \equiv Q) \wedge P$
= {p.c.}
$(\neg P \not\equiv Q) \wedge P$
= {distribution}
$\neg P \wedge P \not\equiv Q \wedge P$
= {p.c.}
$P \wedge Q$. (End of Proof)

Proof of (1) Denoting for brevity's sake universal quantification over x by "$[\![\ \ ]\!]$", we observe

$$
\begin{aligned}
& [\![p \equiv q]\!] \ \Rightarrow \ ([\![p]\!] \equiv [\![q]\!]) \\
= \ & \{(3)\} \\
& [\![p \equiv q]\!] \wedge [\![p]\!] \ \equiv \ [\![p \equiv q]\!] \wedge [\![q]\!] \\
= \ & \{(0) \text{ twice}\} \\
& [\![(p \equiv q) \wedge p]\!] \ \equiv \ [\![(p \equiv q) \wedge q]\!] \\
= \ & \{(4) \text{ twice}\} \\
& [\![p \wedge q]\!] \equiv [\![q \wedge p]\!] \\
= \ & \{\text{pc}\} \\
& \text{true}
\end{aligned}
$$

(End of Proof)

Proof of (2) With the same notational convention

$$
\begin{aligned}
& [\![p \Rightarrow q]\!] \\
= \ & \{\text{p.c}\} \\
& [\![p \wedge q \equiv p]\!] \\
\Rightarrow \ & \{(1)\} \\
& [\![p \wedge q]\!] \equiv [\![p]\!] \\
= \ & \{(0)\} \\
& [\![p]\!] \wedge [\![q]\!] \equiv [\![p]\!] \\
= \ & \{\text{p.c}\} \\
& [\![p]\!] \Rightarrow [\![q]\!]
\end{aligned}
$$

(End of Proof.)

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