Node degree and the size of disconnected subgraphs

Theorem For a graph of N nodes, we have, with $m \geq 0$ and $k \geq 1$,

(0) $N > (m+1)\cdot(k-1) \Rightarrow P \vee Q$, where

$P \equiv$ (there exists a node connected to more than m other nodes)
$Q \equiv$ (there exists a subset of k mutually disconnected nodes).

The given lower bound for N is sharp. (End of Theorem.)

Proof. The sharpness is demonstrated by constructing a graph with $N = (m+1)\cdot(k-1)$, for which $\neg P \wedge \neg Q$ holds, viz. k-1 copies of the complete $(m+1)$-graph.

Each node being connected to m other nodes, we have $\neg P$.

Each subset of k nodes containing a pair belonging to one of the k-1 complete graphs, we have $\neg Q$.

The sufficiency of the lower bound shown by mathematical induction over k , more precisely the base for $k=1 \vee k=2$, and the step from k to $k+2$.

Base $k=1$

$N > (m+1)\cdot(1-1)$
$= \{\text{arithmetic}\}$
$N > 0$
$\Rightarrow \{\text{a subset of 1 node contains no connected pair}\}$
$Q(k:=1)$
$\Rightarrow \{P \text{ does not depend on } k\}$

$(P \lor Q)(k := 1)$

<u>$k = 2$</u>

$$
\begin{array}{ll}
& N > (m+1)\cdot(2-1) \\
= & \{\text{arithmetic}\} \\
& N-1 > m \geq 0 \\
\Rightarrow & \{\text{a node is connected to } N\text{-}1 \text{ others or to fewer}\} \\
& P \lor Q(k := 2) \\
= & \{P \text{ does not depend on } k\} \\
& (P \lor Q)(k := 2)
\end{array}
$$

<u>Step</u> Using (0) we shall show $Q(k := k+2)$ under the assumption $N > (m+1)\cdot(k+1) \land \neg P$ by constructing a subset of $k+2$ mutually disconnected nodes.

Under the assumption, there is a disconnected pair, (X, Y) say; because of $\neg P$, of the other $N-2$ nodes, at most $2\cdot m$ nodes are connected to X or to Y, hence at least $N-(m+1)\cdot 2$, i.e. more than $(m+1)\cdot(k-1)$, of those other nodes are connected to neither X nor Y. From (0) we conclude among them a subset of k mutually disconnected nodes; together with X and Y this yields a subset of $k+2$ mutually disconnected nodes, hence $Q(k := k+2)$.

(End of Proof.)

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prof. dr. Edsger W. Dijkstra
Department of Computer Sciences
The University of Texas at Austin
Austin, TX 78712-1188
USA