On finite models for the equivalence

The equivalence, denoted by $\equiv$, is postulated to be reflexive, symmetric, and associative, i.e. we have for all X, Y, and Z

(0) $[X \equiv X]$ (reflexive)

(1) $[(X \equiv Y) \equiv (Y \equiv X)]$ (symmetric)

(2) $[((X \equiv Y) \equiv Z) \equiv (X \equiv (Y \equiv Z))]$ (associative).

On account of the associativity, we feel free to omit the semantically irrelevant parenthesis pairs.

With $[A \equiv B]$ denoting the equality of the expressions A and B, we can derive from the above

(3) $[X \equiv X \equiv Y \equiv Y]$;

parsing (3) as $[X \equiv (X \equiv (Y \equiv Y))]$ we see that $(Y \equiv Y)$ is a right identity element of $\equiv$;
parsing (3) as $[((X \equiv X) \equiv Y) \equiv Y]$ we see that $(X \equiv X)$ is a left identity element of $\equiv$;
parsing (3) as $[(X \equiv X) \equiv (Y \equiv Y)]$ we see that these identity elements are the same. (Not surprisingly so.) In the sequel we shall denote the identity element of the equivalence by 0.

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Theorem The number of distinct values in a finite model for the equivalence is a power of 2.

(The theorem is well-known; this note is written because we like the following argument.)

Since the Theorem holds for the trivial model consisting of only one value — viz. 0 — we concentrate our attention on the nontrivial models, i.e. those with more values. The Theorem now follows from the following

<u>Lemma 0</u> The values of a nontrivial finite model for the equivalence can be partitioned into two disjoint sets E and O such that

(i) E and O are of equal size , and

(ii) the elements of E provide a model for the equivalence.

<u>Proof of Lemma 0</u>. We shall prove Lemma 0 by showing how for a given nontrivial finite model the partitions E and O can be constructed. To begin with, we select an arbitrary element different from 0 and call it 1 . With this choice we introduce a pairing by

$$(X \text{ and } Y \text{ are paired}) \equiv [X \equiv 1 \equiv Y] \quad .$$

Note that

(i) 0 and 1 are paired

(ii) each element Z is paired to $Z \equiv 1$

(iii) no element is paired to itself — $[Z \equiv 1 \equiv Z]$ would lead to $[0 \equiv 1]$, contradicting that 1 has been chosen different from 1 .

From the pairing we conclude that the given model consists of an even number of values.

The idea of the construction is to initialize both E and O to the empty set:

$$E, O := \emptyset, \emptyset$$

and to deal with the pairs in turn, adding one of its elements to E and the other to O, the choice (possibly) being constrained by the invariant. In order to formulate the invariant we introduce the predicate comp, given by

comp.V.Z ≡ "element Z equivales the (continued) equivalence of some elements from V"

from which

(4) $\quad$ $comp.V.X \wedge comp.V.Y \Rightarrow comp.V.(X \equiv Y)$

follows. From (4) follows — with $X, Y, V := Z, Z\equiv 1, E$ —

(5) $\quad$ $\neg comp.E.1 \Rightarrow \neg comp.E.Z \vee \neg comp.E.(Z \equiv 1)$,

and the construction maintains

P: $\quad$ $\neg comp.E.1$,

initially true since $E = \emptyset \Rightarrow P$.

Consider now the next pair to be dealt with: calling one element Z, the other element is $Z \equiv 1$. Under validity of P we have on account of (5)

$$\neg comp.E.Z \vee \neg comp.E.(Z \equiv 1)$$

and hence it is possible to make the choice which of the elements of the pair to call Z in such a way as to ensure

(6) $\neg\, comp.E.(Z \equiv 1)$.

Since

$$\neg\, comp.E.1 \wedge \neg\, comp.E.(Z \equiv 1) \Rightarrow \neg\, comp.(E \cup \{Z\}).1 ,$$

with Z satisfying (6)

$$E, O := E \cup \{Z\}, O \cup \{Z \equiv 1\}$$

maintains P .

Upon completion, we have

(7) $(\underline{A}\, X, Y : X \in E \wedge Y \in E : (X \equiv Y) \in E)$

because we have for any X and Y

$X \in E \wedge Y \in E$
$\Rightarrow$ { definition of comp, in particular $Z \in V \Rightarrow comp.V.Z$ }
$comp.E.X \wedge comp.E.Y$
$\Rightarrow$ { (4) with $V := E$ }
$comp.E.(X \equiv Y)$
$\Rightarrow$ { P and (5), with $Z := X \equiv Y$ }
$\neg\, comp.E.(X \equiv Y \equiv 1)$
$\Rightarrow$ { definition of comp }
$\neg((X \equiv Y \equiv 1) \in E)$
$=$ { $(X \equiv Y \equiv 1) \in E \cup O$ }
$(X \equiv Y \equiv 1) \in O$
$=$ { construction of E and O }

$(X \equiv Y) \in E$.

And (7) says that the elements of E provide a model for the equivalence.

(End of Proof of Lemma 0.)

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Undoubtedly, the above presentation leaves room for editorial improvements (for which we currently lack the opportunity), but we do like the argument.

An alternative argument shows

(i) the existence of a model of size 2^0

(ii) from the existence of a model of size 2^n that the next finite model is of size 2^{n+1} .

Lemma 0, however, enables us to carry out the induction step without any reference to powers of 2.

Furthermore, the argument provides a nice example of how nondeterministic algorithms can be used to avoid case analysis: requirement (6) may or may not prescribe the choice of Z, but this distinction no longer surfaces in our argument.

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