# A computing scientist's approach to a once-deep theorem of Sylvester's

Well, actually it wasn't Sylvester's theorem, it was only his conjecture -dating from the year 1893- , and it remained so for more than 40 years until T. Gallai (alias Grünwald) "finally succeeded, using a rather complicated argument" [Coxeter]. We shall derive (essentially) the simple argument of L.M. Kelly (1948).

Theorem Consider a finite number of distinct points in the real Euclidean plane; these points are collinear or there exists a straight line through exactly 2 of them.

To see that this is a truly geometrical theorem and not a combinatorial one, we slightly rephrase the setting. Let us assume for a moment that collinearity of points in the Euclidean plane is fully captured by

(i) any pair of points is collinear, and if two collinear triples have two points in common, their four points are collinear.

Let us rephrase this: replace "points" by "people", "lines" by "clubs" and "collinear" by "club-sharing" (i.e. belonging to one and the same club). Club membership is postulated to satisfy the analogue of (i):

(ii) any pair of people is club-sharing, and if two club-sharing triples have two people in common, their four people are club-sharing.

The analogue of Sylvester's theorem would state for a finite population: all people belong to one and the same club or there exists a club with exactly 2 members. Its falsity is shown by the following counterexample of 7 people – numbered from 0 through 6 – with the following 7 clubs: {013}, {124}, {235}, {346}, {450}, {561}, and {602}. Postulate (ii) is satisfied because each of the 21 pairs of people occurs in exactly 1 club. Thus we have established that Sylvester's theorem is truly a geometrical one; let us now try to prove it.

Being computing scientists, we love constructive arguments, i.e. we like to show that something exists by designing an algorithm that computes such a thing. We therefore propose to design an algorithm that computes a line that passes through exactly 2 of the points from a given finite, non-collinear set of distinct points. (Legenda: from here on we no longer repeat that the points are distinct, nor that they belong to the given, non-collinear set.)

More precisely, we have to design an algorithm that operates on a variable q of type: line and that establishes the post-condition R, given by

R: q passes through exactly 2 of the points .

The simplest idea is to initialize q by the line through 2 arbitrary points. (This is always possible because, the given set being non-collinear, there are at least 3 points.) If q goes through 3 or more points, it has to be changed, otherwise it can be accepted as final value. That is, with invariant P given by

P: q passes through $\geq 2$ points

we propose as first approximation of our algorithm

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    establish P by initialization of q
  ; do q passes through ≥3 points →
        change q under invariance of P
    od
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Because

$$P \wedge \neg(q \text{ passes through} \geq 3 \text{ points}) \Rightarrow R$$

we are done when the algorithm terminates.

Our remaining task is to ensure that it does terminate. To that end we have to exploit the finiteness of the given set and its non-collinearity. Because the exploitation of finiteness is absolutely standard, we first focus our attention on what we can conclude from the non-collinearity. From the latter we can draw <u>only one</u> conclusion in connection with $q$, viz. the existence of a point through which $q$ does not pass. That is, we propose to introduce a variable $E$ of type: point, and to strengthen $P$ to $P1$

P1: $q$ passes through $\geq 2$ points $\wedge$
$q$ does not pass through $E$ .

The new approximation of our algorithm is

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   establish P1 by initializing q and E
 ; do q passes through ≥3 points →
   {?} change q and E under invariance of P1
   od
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(Ignore for a moment the assertion "{?}"; the important thing to realize is that with the feasibility of maintaining the stronger $P1$, the non-collinearity of the given set has been exhausted.)

In the current stage of program design, our only option is a further refinement of the as yet rather nondeterministic

(0) "change $q$ and $E$ under invariance of $P1$".

Because we may have to reduce its nondeterminism lest the algorithm fails to terminate, let us investigate its freedom: what precondition {?} can we guarantee? We know of the existence of 4 points, viz. $E$ and the three points on the current $q$. Because the new $q$ has to pass through $\geq 2$ points and has to differ from the old $q$, the new $q$ passes through the old $E$ and one of the 3 points on the old $q$; in each case, one of the remaining 2 points on the old $q$ has to be chosen as the new $E$. In summary: for the new pair $(q, E)$ we have 6 possibilities.

For the termination argument we need a variant function of the pair $(q, E)$; because the number of points is finite, the number of pairs $(q, E)$ satisfying $P1$ is finite, and any function of the pair $(q, E)$ that decreases at each change will do.

What is the simplest function of a line and a point (not on that line) that we can think of? The Euclidean distance between the two!

Let us investigate whether we can refine (0) so as to decrease the distance between q and E. Let us name the three points on the old q : A, B, C, so that A becomes the new E. With that convention, the refinement of (0) that decreases the distance of the pair (q, E) as much as possible is

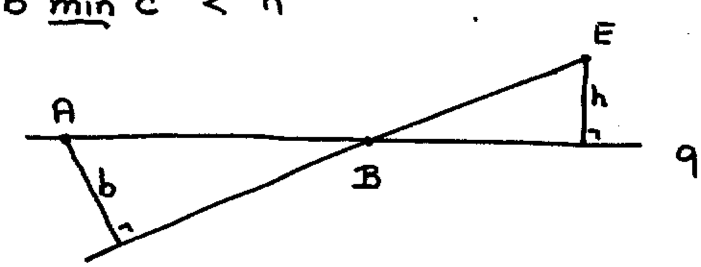
(1) q, E := of BE and CE the nearest to A , A .

Finally we derive a condition on A as our choice for the new E from the requirement that the variant function decreases. With

h = distance between E and q
b = distance between A and BE
c = distance between A and CE

the required decrease of the variant function is expressed by

(2) $b \; \underline{\min} \; c < h$



In order to derive (2), we proceed as follows

$$
\begin{array}{ll}
 & b \; \underline{\min} \; c < h \\
= & \quad \{\text{definition of } \underline{\min}\} \\
 & b < h \;\vee\; c < h \\
= & \quad \{\text{similar triangles, see figure}\} \\
 & AB < EB \;\vee\; AC < EC \\
\Leftarrow & \quad \{\text{monotonicity of } +\} \quad (\text{See Note}) \\
 & AB + AC < EB + EC \\
\Leftarrow & \quad \{P1 \Rightarrow BC < EB + EC, \text{ i.e. the strict} \\
 & \quad\quad \text{triangular inequality}\} \; (\text{See Note}) \\
 & AB + AC \leq BC \\
= & \quad \{AB + AC \geq BC, \text{ i.e. triangular inequality}\} \\
 & AB + AC = BC \\
= & \quad \{AB, AC \text{ and } BC \text{ denote unsigned lengths}\} \\
 & \text{on } q, \; A \text{ lies between } B \text{ and } C \; .
\end{array}
$$

Hence, with for A the point between the two others, (1) does the job. And this concludes our proof of Sylvester's theorem.

<u>Note</u> Since steps that express equivalence don't destroy information, the others need some more justification. We all know the monotonicity of $\geq$, i.e. no one doubts

$$x \geq x' \;\wedge\; y \geq y' \;\Rightarrow\; x+y \geq x'+y' \quad ;$$

its counterpositive yields the equivalent

$$x < x' \;\vee\; y < y' \;\Leftarrow\; x+y < x'+y'$$

and that is what we used.

To justify the next step, a second look at our demonstrandum (2) suffices: since it is impossible to demonstrate (2) for $h=0$, we have to use once more that $q$ does not pass through $E$. Since $E$ occurs in this calculation in the combinations $EB$ and $EC$, we translate this into the non-degeneracy of triangle $BEC$. The step eliminates $E$ from the rest of the calculation. (End of Note.)

For the sake of completeness we point out that, after the choice of the variant function, we have made two silent choices. We have chosen -as usual- to decrease the variant function; because of the finiteness, successfully increasing it would also have yielded a valid termination argument. With a huge $h$ and $ABC$ close together, however, the distance between $E$ and $q$ cannot be increased, which settles this silent choice. Moreover, we could have grouped our 6 cases differently, viz. by common new $q$ instead of by common new $E$. We could have said "Let us name the three points so that $AE$ becomes the new $q$" and instead of (1) we would have come up with

$q, E := AE$, of B and C the nearest to AE .
It does not work.

* * *

A few methodological remarks are in order, because the theorem certainly deserves the name of a once-deep theorem: after 1933, when Karamata and Erdös revived interest in the problem [Coxeter], it took another fifteen years before L.M. Kelly made essentially the above use of the Euclidean distance and found the simple argument.

It was very gratifying to see that, once the decision had been taken to tackle this problem as a programming task, the job of designing the program was all but standard. I have used this problem in oral examinations for a course on "Mathematical Methodology" for Computing Science graduates. Some needed more prompting or more time than others, but none of them needed any prompting to come up with the Euclidean distance between q and E as candidate for the variant function. They knew that the argument required a variant function and they all suggested the Euclidean distance without hesitation. And that was Kelly's great invention!

A fringe benefit of proving the theorem by designing a program is that it takes away the surprise that in such a non-metric context a metric concept such as the Euclidean distance enters the picture. We all know that a monotonic function of an acceptable variant is again an acceptable variant and that the challenge always is to find a nice one. It is very much like the freedom to choose the most convenient coordinate system.

Acknowledgement That the distance from points to lines through other points could be used in the proof was told to me by Bernhard von Stengel; he told be to look at the shortest such distance. (End of Acknowledgement.)

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For Sylvester's theorem to be true it is essential that the points are distinct. (Consider a non-degenerate triangle with each vertex coinciding with a triple of points.) Replace "any 2 points are distinct" by "through any 2 points passes only one straight line". The latter can be generalized to one dimension more "through any 3 points passes only one plane", i.e. "no 3 points are collinear". And now we are ready to generalize Sylvester's theorem to three dimensions:

Theorem Consider in the real three-dimensional Euclidean space a finite number of points such that no 3 of them are collinear; these points are coplanar or there exists a plane through exactly 3 of them.

(In the three-dimensional case, just requiring that the points be distinct is not enough: consider two non-intersecting, nonparallel lines with 4 distinct points on each.)

Proof sketch Select one of the points and a plane outside. Project, with the selected point as centre, the remaining points on the selected plane, to which projection Sylverster's theorem is applied. (End of Proof sketch.)

* * *

Coxeter mentions no more-dimensional generalization of Sylvester's theorem. Maybe he did not think it sufficiently interesting. Maybe he could not comfortably generalize because none of his formulations mentions explicitly that the points have to be distinct.

Coxeter's passage is interesting for historical reasons. He quotes Sylvester's original statement of the problem:

"Prove that it is not possible to arrange any finite number of real points so that a right line through every two of them shall pass [sic] through a third, unless they all lie in the same right line."

Fortunately we don't formulate problems like that anymore. I could not read it and ended up looking up "unless" in the COD, which gave two interpretations "if not" (which boils down to $\vee$) and "except when" (which boils down to $\not\equiv$). I felt excused. (Actually, Sylvester used "unless" in the meaning "$\vee$".)

A few lines further, Coxeter gives credit to T. Motzkin (1951): "Sylvester's 'negative' statement was rephrased 'positively' by Motzkin:

If n points in the real plane are not on one straight line, then there exists a straight line containing exactly two of the points."

It is not quite clear for which achievement Motzkin receives credit. He replaces Sylvester's contorted

$\neg(\underline{A}$ line: line passes $\geq 2$: line passes $\geq 3)$

thanks to the Morgan by

$(\underline{E}$ line: line passes $\geq 2$: line passes $< 3)$

which can be simplified (arithmetically!) to

$(\underline{E}\ line :: line\ passes = 2)$

which in the statement of the theorem is certainly a simplification. (But note that in the proof one immediate uses

$$line\ passes = 2 \ \equiv (line\ passes \geq 2) \wedge \neg(line\ passes \geq 3).)$$

Or did Motzkin get credit for replacing Sylvester's disjunction by an implication? You never know with Coxeter. (I dislike in Motzkin's formulation, besides the dangling "n", the implication: compared to Sylvester's disjunction, I consider that a step backwards. Please note that the implicative formulation introduced - in "not on a straight line" - a negation.)

Coxeter's section opens with a quotation from G.H. Hardy (1940):

"Reductio ad absurdum, which Euclid loved so much, is one of a mathematician's finest weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game.".

No matter how hard I try, almost half a century later I am unable to give even a mildly sensible interpretation to the above quotation of Hardy's, but, in 1961, Coxeter evidently felt he could: his proof gloriously ends with the now infamous "which is absurd".

(Coxeter focusses his attention on the pair (q,E) with minimum distance and derives a contradiction from the assumption that that q passes through at least 3 points. The choice of the pair with minimum distance is over-specific: it is only a device to construct the avoidable contradiction. Tellingly, he concludes

"This completes the proof that there is always a line containing exactly two of the points. Of course, there may be more than one such line [...]".)

Fascinating to analyse mathematical style from such a recent past!

<u>Reference</u>

Coxeter, FRS, H.S.M. "Introduction to Geometry" 2nd Ed., John Wiley & Sons, Inc. New York etc. pp 65-66

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