Once more the bichrome 3x7 grid (See EWD764)

This note deals with a minor observation, just big enough to be recorded. It compares three counting arguments. PPP refers to the Purified Pigeonhole Principle, which states that for a finite, non-empty bag of numbers the maximum is at least the average.

Consider a rectangular array of 7 rows, each consisting of 3 grid points, each of which is red or blue. The problem is to show the presence of a rectangle the 4 corners of which have the same colour.

Argument A

- since there are 3 columns and 2 colours, (PPP) each row has at least 2 points of the same colour; call that colour the dominant colour of that row;
- since there are 7 rows and 2 colours, (PPP) at least 4 rows have the same dominant colour; call that the majority colour
- select for each of these 4 rows a pair of columns in what that row has grid points of the majority colour; since there are 4 of these rows and 3 pairs of columns, (PPP) for at least 2 rows the same pair of columns has been selected, and the conclusion follows.

Argument B

- since there are 3 columns and 2 colours, (PPP) we can select for each row a pair of columns in which that row has grid points of the same colour; call that pair of columns the equality pair of that row;
- since there are 7 rows and 3 pairs of columns, (PPP) at least 3 rows have the same equality pair; call that the target pair
- since these 3 rows are each monochrome in the target pair and there are 2 colours, (PPP) there are at least 2 such rows that have the same colour in the target pair, and the conclusion follows.

Argument C

- since there are 3 columns and 2 colours, (PPP) we can select for each row a pattern, i.e. a combination of a pair of columns and a colour, such that that row has in those columns grid points of that colour;
- since there are 2 colours and 3 pairs of columns, there are 6 (= 2×3) patterns;
- since there are 7 rows and 6 patterns, (PPP) there are at least 2 rows with the same pattern, and the conclusion follows.

* * *

The observation to be made is that argument C is better than arguments A and B, which both refer

to PPP once more than C. The only difference between A and B is that, in the last two appeals to PPP, A averages first over 2 and then over 3, whereas B does that the other way round. Argument C combines these two appeals by averaging over 6, thus, firstly, avoiding the choice of that irrelevant order, and, secondly, avoiding the additional nomenclature introduced to describe what is counted in the intermediate stage.

Note that the "patterns" - with their implicit Cartesian product - are very similar to the U·D "labels" from the argument about maximum up/down sequence lengths as presented in EWD980.

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